EXAMINATION

7 April 2005 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

1 An investor holds a portfolio consisting of N assets in equal proportions.

Derive an expression for the variance of the portfolio as N gets very large.

[You may assume that all assets have variance less than a certain level V_{max}. You may also assume that the average covariance is $\bar{c}$.]

[6]

2 An investor wishes to measure the investment risk presented by an asset which has the following distribution:

State	*Return*	*Probability*
1	10%	0.5
2	20%	0.3
3	50%	0.2

(i) Evaluate three different measures of investment risk for this asset. Where necessary, you may assume a benchmark return of 25%. [4]

(ii) (a) State two key properties of Value at Risk (VaR).

(b) VaR is frequently calculated assuming a normal distribution of returns. State an advantage and a disadvantage of this approach.

[2]

[Total 6]

3 An investor has the choice of the following assets that earn rates of return as follows in each of the four possible states of the world:

State	*Probability*	*Asset 1*	*Asset 2*	*Asset 3*
1	0.2	5%	5%	6%
2	0.3	5%	12%	5%
3	0.1	5%	3%	4%
4	0.4	5%	1%	7%
Market capitalisation		10,000	17,546	82,454

Determine the market price of risk assuming CAPM holds.

Define all terms used. [6]

4 Let R_i denote the return on security i given by the following multifactor model

$$R_i = a_i + b_{i,1} I_1 + b_{i,2} I_2 + \ldots + b_{i,L} I_L + c_i$$

a_i and c_i are the constant and random parts respectively of the component of the return unique to security i.

$I_1, \ldots I_L$ are the changes in a set of L indices.

$b_{i,k}$ is the sensitivity of security i to factor k.

(i) State the category of the above model where:

(a) index 1 is a price index
index 2 is the yield on government bonds
index 3 is the annual rate of economic growth

(b) index 1 is the level of Research and Development expenditure
index 2 is the price earnings ratio

[2]

(ii) Determine the number of parameters to be estimated in a single index model and in a multifactor model. [4]

[Total 6]

5 The following unusual model has been proposed for the (real-world) stochastic behaviour of the short term interest rate:

$$dr_t = \mu r_t \, dt + \sigma dZ_t,$$

where $\mu > 0$ and σ are fixed parameters and Z is a standard Brownian motion under the proposed real-world measure P.

Under the same measure P, a (zero coupon) bond with maturity T has price at time t $B(t, T) = \exp(-(T - t)\, r_t + \sigma^2 (T - t)^3/6)$.

(a) Derive the SDE satisfied by $B(t, T)$.

(b) Determine the market price of risk and deduce the corresponding SDE for r, under the risk neutral measure Q.

[7]

PLEASE TURN OVER

6 One particular company over which an investment bank writes European call options has experienced a severe fall in its share price. However, analysts have not revised their expectation that the share price will grow to £4 in six months. The table below shows the share price together with the price of the options.

Date	*Share price*	*Option price*
1 November	£3.00	£0.90
2 November	£2.00	£0.60

You may assume that the basic Black-Scholes framework is used to price the options.

(i) Explain why the option price has fallen even though the expected return has increased according to the analysts. [3]

(ii) State any requirements for the option price to have fallen to its level on 2 November. [3]

[Total 6]

7 (i) Outline the approach adopted by Shiller to test for "excessive volatility" and state the criticisms of his work. [7]

(ii) State one difficulty of testing the strong form of the efficient market hypothesis and state the general conclusion of studies carried out on it. [2]

[Total 9]

8 (i) State the martingale representation theorem, including conditions for its application, defining all terms used. [3]

Let S_t denote the price of an underlying security at time t; r denotes the risk free rate of return expressed in continuously compounded form, B_t represents an accumulated "bank account" at time t that earns the risk free rate of return.

Let X be any derivative payment contingent on F_T, payable at some fixed future time T, where F_T is the sigma algebra generated by S_u for $0 \le u \le T$.

You may assume that, under the equivalent measure Q, the process

$D_t = e^{-rt} S_t$ is a martingale

and that

$$dS_t = B_t(rD_t dt + dD_t)$$

(ii) Show that the value of this derivative at time $t < T$ is

$$V_t = e^{-r(T-t)} E_Q[X \mid F_t]$$ [11]

[Total 14]

9 (i) Describe the role that the inflation model plays within the Wilkie model. [3]

The Wilkie model proposes an AR(1) process for the continuously compounded rate of inflation $I(t)$ that can be written as:

$$I(t) = a + bI(t-1) + \varepsilon(t)$$

Where $\varepsilon(t) \sim N(0, t^2)$ and a and b are constants with $-1 < b < 1$.

(ii) Derive an expression for the long term average rate of inflation in terms of a and b. [1]

(iii) Explain why a model of the form above would not be suitable for share prices. [3]

(iv) Explain why a lognormal model may be used for share prices and state its weaknesses. [8]

[Total 15]

10 An investment bank has issued a derivative on a share (with share price, S, of 100) that provides for the following payoff after two months:

$$F(S) = \ln(S - 90) \quad \text{if } S > 90$$
$$= 0 \quad \text{otherwise}$$

You may assume that:

- There exists a risk free asset that earns 5% per month, continuously compounded.
- The expected effective rate of return on the share is 2% per month.
- The monthly standard deviation of the log share price is 10%.

(i) By using a two period recombining model of future share prices, derive the state price deflators at time 2. The parameters determining the share price after an up-jump and down-jump should be determined by considering the standard deviation of the log share price. [9]

(ii) Using the state price deflators from (i) derive the value at time zero of the option. [3]

The delta of this derivative at time zero is 7% and the gamma is 10%. The bank which issued the derivative wishes to delta hedge its position in the most efficient manner. Assume that the share price can also be modelled in continuous time with a geometric Brownian motion with volatility (diffusion parameter) of 0.1 consistent with a Black-Scholes framework.

(iii) Determine the delta hedging portfolio, as a combination of the risk free asset, the underlying share, and a European Call option on the share with term of 3 months and exercise price of 100. [13]

[Total 25]

END OF PAPER

EXAMINATION

April 2005

Subject CT8 — Financial Economics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M Flaherty
Chairman of the Board of Examiners

15 June 2005

1 Let V_i denote the variance of the ith asset.
C_{ij} denote the covariance of the ith and jth assets ($i \neq j$).

The variance of the investor's portfolio is therefore

$$V = \sum_i \frac{1}{N^2} V_i + \sum_i \sum_{j \neq i} \frac{1}{N^2} C_{ij}$$

Let $\bar{V} = \frac{1}{N} \sum V_i$, and $\bar{c} = \frac{1}{N(N-1)} \sum_i \sum_{j \neq i} C_{ij}$

$$V = \frac{1}{N}\bar{V} + \frac{(N-1)}{N}\bar{c}$$

As $N \to \infty$

$$\frac{1}{N}\bar{V} \to 0 \text{ because } \frac{1}{N}\bar{V} < \frac{1}{N} V_{\max}$$

and $\frac{1}{N} V_{\max} \to 0$ as $N \to \infty$

therefore $V \to \bar{c}$ as $N \to \infty$

2 (i) The mean return μ is $0.1 \times 0.5 + 0.2 \times 0.3 + 0.5 \times 0.2 = 21\%$

Variance of return

$(0.1 - 0.21)^2 \times 0.5 + (0.2 - 0.21)^2 \times 0.3 + (0.5 - 0.21)^2 \times 0.2 = 2.29\%\%$

Semi variance of return

$= (0.1 - 0.21)^2 \times 0.5 + (0.2 - 0.21)^2 \times 0.3 = 0.608\%\%$

Shortfall probability

$50\% + 30\% = 80\%$

(ii) (a)

- It is a statistical measure of downside risk.
- It assesses the potential minimum loss over given time with given degree of confidence.

(b) Advantage: normal distribution is easy to manipulate to calculate VaRs based on only two parameters.
Disadvantage: results may be misleading with skewed or "fat tailed" distribution.

3 The market price of risk is $(E_m - r)/\sigma_m$ where asset 1 is the risk free asset so $r = 5\%$.

E_m = (17,546/100,000) × (0.2 × 5% + 0.3 × 12% + 0.1 × 3% + 0.4 × 1%)

+ (82,454/100,000) × (0.2 × 6% + 0.3 × 5% + 0.1 × 4% + 0.4 × 7%)

= 5.79472%

σ_m^2 = 0.2 × (0.17546 × 5% + 0.82454 × 6% − 5.79472%)^2

+ 0.3 × (0.17546 × 12% + 0.82454 × 5% − 5.79472%)^2

+ 0.1 × (0.17546 × 3% + 0.82454 × 4% − 5.79472%)^2

+ 0.4 × (0.17546 × 1% + 0.82454 × 7% − 5.79472%)^2

= 0.000045402

= (0.674%)^2

$\Rightarrow$ market price of risk is (5.79472% − 5%) / 0.674%

= 1.179 = 118%

4 (i) (a) Macroeconomic.
(b) Fundamental.

(ii) The single index model requires the return on the market, plus for each security: α_i, β_i and σ_i .

Therefore $3N + 1$ data items are required

The multifactor model requires:

L means of indices (Note: some candidates may assume that they have 0 mean, which is acceptable.)

$\frac{L(L+1)}{2}$ covariances

N a_i's

NL sensitivities

N standard deviations of c_i

Therefore $\frac{L(L+3)}{2} + N(L+2)$ data items are required

5 (a) $B(t, T) = f(r_t, t)$, where

$$f(x, t) = \exp(-(T-t)x + \sigma^2(T-t)^3 / 6),$$

so, by Ito's lemma,

$$dB(t, T) = B(t, T)((\sigma^2(T-t)^2 / 2 - (T-t)\mu r_t + r_t - \sigma^2(T-t)^2 / 2)dt - \sigma(T-t)dZ_t)$$

$$= B(t, T)((-(T-t)\mu r_t + r_t)dt - \sigma(T-t)dZ_t)$$

(b) The market price of risk is

$$\gamma(t, T) = (m(t, T) - r_t) / S(t, T),$$

where

$$dB(t, T) = B(t, T)(m(t, T)dt + S(t, T)\, dZ_t), \text{ so } \gamma(t, T) = \mu/\sigma r_t$$

and so under the risk-neutral measure, Q, $dr_t = \sigma dW_t$, where W is a standard BM under Q.

6 (i) Options are priced by "relative valuation" techniques (i.e. risk neutral valuation).

This approach is equivalent to building a hedging strategy for the option and does not take account of the expected return on the share. Since the hedging strategy involves holding some shares, the drop in price will result in a drop of the value of the option even though the expected future share price has remained the same.

(ii) Unless the option is deep in the money, the drop in price of the option will be less than proportional to the share price and hence some combination of the following must also have occurred:

- dividends increased
- share price volatility decreased
- risk free interest rate decreased

7 (i) Shiller used a discounted cashflow model of equities going back to 1870.

A perfect foresight price was determined using actual dividends paid and a terminal value for the stock.

If markets are rational there would be no systematic forecast errors (i.e. error between the perfect foresight price and the actual price).

If markets are efficient, the perfect foresight price matches with share price.

Strong evidence was found that contradicted the EMH.

- Criticisms of terminal stock price.
- Choice of constant discount rate.
- Bias in estimates of variance because of autocorrelation.
- Non-stationarity of the series.

(ii)

- Researchers require access to information that is not in the public domain.
- Studies suggest that it is difficult to out perform with inside information.

8 (i) Suppose X_t is a martingale with respect to a measure P, that is for any $t < s$

$$E_P[X_s|F_t] = X_t$$

and that the volatility of X_t is always non-zero.

Suppose Y_t is another martingale with respect to P.

Then, there exists a unique previsible process ϕ_t such that

$$Y_t = Y_0 + \int_0^t \phi_s \, dX_s$$

Or equivalently $dY_t = \phi_t \, dX_t$

Full credit for either integral or differential form above.

(ii) Let $E_t = e^{-rT}E_Q[X|F_t] = e^{-rt}V_t$, which is a martingale with respect to Q.

Using the martingale representation theorem, there exists a unique previsible process ϕ_t such that

$$dE_t = \phi_t dD_t$$

Let $\psi_t = E_t - \phi_t D_t$

Suppose that at time t we hold

ϕ_t units of asset S_t
ψ_t of cash B_t

The value of the portfolio at time t is

$$\phi_t S_t + \psi_t B_t = V_t$$

over the period t to $t + dt$ the change in the value of the portfolio is

$$\phi_t dS_t + \psi_t dB_t$$

$$\begin{aligned} \phi_t dS_t + \psi_t dB_t &= \phi_t B_t(rD_t dt + dD_t) + \psi_t rB_t dt \\ &= B_t[\phi_t dD_t + r(\phi_t D_t + \psi_t)dt] \\ &= B_t[\phi_t dD_t + rE_t dt] \\ &= B_t[dE_t + rE_t dt] \\ &= B_t dE_t + E_t dB_t + dB_t\, dE_t \\ &= dV_t \end{aligned}$$

Therefore (ϕ_t, ψ_t) is self financing

$V_T = E_Q[X|F_T] = X$, therefore (ϕ_t, ψ_t) is replicating so V_t is the value of the claim.

9 (i) The Wilkie model can be described as a cascade or hierarchical model, with inflation being the key component. Variations of dividend yields, growth and interest rates are affected by shocks in the inflation model and moving averages of past inflation.

(ii) $I_\infty = a + bI_\infty$

$$\Rightarrow I_\infty = \frac{a}{1-b}$$

(iii)

- AR process is stationary, share prices have tended to increase over time.

- AR(1) implies a systematic element to the changes in prices which is inconsistent with high risk and return.

- Non-normality, jumps in share prices.

- Prices can be negative.

(iv) Log-normal distribution makes the maths for option pricing simple (i.e. tractable solutions).

Returns in non-overlappng periods are independent, which is consistent, with the EMH, for example.

It does not allow negative share prices.

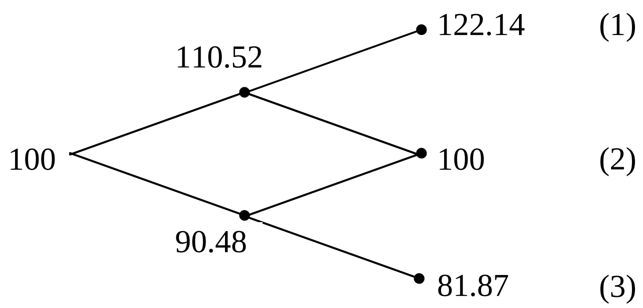
Mean and variance are proportional to time period.

Weaknesses:

- The variance may not be stable over time.
- The mean (drift) may not be constant over time.
- Share prices may be considered to be mean-reverting.
- Share prices exhibit jumps.

10 (i) $u = \exp(0.1) = 1.10517$

$d = 1/1.10517 = 0.904837$



Since the real world expected return is 2% per month, we can derive the real world probability of an up-jump

$$102 = p \times 110.52 + (1 - p) \times 90.48$$

$$\Rightarrow p = \frac{(102 - 90.48)}{(110.52 - 90.48)} = 57.5\%$$

The risk neutral probability of an up-jump is

$$q = \frac{e^{5\%} - 0.904837}{1.10517 - 0.904837} = 73.1\%$$

The state price deflator A_2 at time 2 is defined as follows:

Node

(1) $\quad e^{-0.1}\left(\frac{q}{p}\right)^2 = 1.4624$

(2) $\quad e^{-0.1}\left(\frac{q}{p}\right)\left(\frac{1-q}{1-p}\right) = 0.72809$

(3) $\quad e^{-0.1}\left(\frac{1-q}{1-p}\right)^2 = 0.36249$

(ii) The value is $E_P[A_2 f(S_2)]$ where S_2 is the share price at time 2.

$= (57.5\%)^2 \times 1.4624 \times \log(32.14) + 2 \times 57.5\% \times (1 - 57.5\%) \times \log(10) \times 0.72809$

$= 2.4972$

(iii) The value of the European call option is

$$V = 100\Phi(d_1) - 100e^{-0.05\times 3}\Phi(d_2)$$

Where $d_1 = \dfrac{\log\frac{100}{100} + (0.05 + ½0.1^2)\times 3}{0.1\sqrt{3}} = 0.95263$

$$d_2 = \frac{\log\frac{100}{100} + (0.05 - ½0.1^2)\times 3}{0.1\sqrt{3}} = 0.7794256$$

Therefore value is

$100(0.829611) - 100e^{-0.05 \text{ x } 3}\, 0.7821347$

$= 15.642$

The delta of the European Call option is given by

$$\Delta = \Phi(\text{d1}) \text{ where } d_1 = \frac{\log\frac{S}{K} + (r + ½\sigma^2)t}{\sigma\sqrt{t}}$$

$$= 0.95263$$

$\Delta = 0.82961$

The gamma of the European call option is given by

$$\Gamma = \frac{\phi(d_1)}{s\sigma\sqrt{t}} = \frac{\phi(0.95263)}{100\times 0.1\sqrt{3}} = \frac{1}{\sqrt{2\pi}}.\frac{e^{-½\times 0.95263^2}}{100\times 0.1\sqrt{3}}$$

$$= 1.46\%$$

Clearly the risk free rate has nil delta and gamma.

The underlying share has $\Delta = 1$ and $\Gamma = 0$. Therefore, equating

delta	0.07	=	$1.x_2 + 0.82961x_3$
gamma	0.10	=	$0.0146x_3$
and value	2.4972	=	$1.x_1 + 100x_2 + 15.642x_3$

Solving these three equations in three unknowns gives

$$x_3 = \frac{0.1}{0.0146} = 6.8493$$

$$\Rightarrow \quad 0.07 = x_2 + 0.82961 \times 6.8493$$

$$x_2 = -5.6123$$

$$\Rightarrow \quad 2.4972 = x_1 + 100 \times -5.6123 + 15.642 \times 6.8493$$

$$\Rightarrow \quad x_1 = 456.59$$

Therefore

hold 456.59 in risk free asset
sell 5.6123 of underlying share
hold 6.8493 European Call option

END OF EXAMINERS' REPORT

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1 An investor is contemplating an investment with a return of £R, where:

$$R = 300{,}000 - 500{,}000U$$

where U is a uniform [0, 1] random variable.

Calculate each of the following four measures of risk:

(a) variance of return

(b) downside semi-variance of return

(c) shortfall probability, where the shortfall level is £100,000

(d) Value at Risk at the 5% level [8]

2 A market consists of three securities A, B and C with capitalisations of £22bn, £33bn and £22bn respectively. Annual returns on the three shares (R_A, R_B and R_C) have the following characteristics:

Asset	*Standard deviation*
A	40%
B	20%
C	10%

The expected rate of return on the market portfolio is 22.86% p.a.

The correlation between the returns on each pair of distinct securities is 0.5.

The risk-free rate of return is 3.077% p.a. No adjustments to an investor's portfolio are possible within the year.

(i) Prove that the expected returns on A, B and C are 40%, 20% and 10% respectively if the CAPM is assumed to hold. [9]

(ii) Derive a single index model (with the index equal to R_M, the random return on the market portfolio) with the same expected returns and variances as in the CAPM. You are required to calculate the values of all parameters in the model. [4]

(iii) Prove that this single index model is not completely consistent with the CAPM model. [3]

[Total 16]

3 (i) Define in words Δ, Γ, θ, λ, ρ, and $\mathcal{V}$ for an individual derivative. [6]

(ii) Explain how Γ and $\mathcal{V}$ can be used in the risk management of a portfolio that is delta-hedged. [4]

[Total 10]

4 (i) State the assumptions underlying the Black-Scholes option pricing formula and discuss how realistic they are. [6]

An investment bank has written a number, N, of European call options on a non-dividend paying stock with strike price 200p, current stock price 180p, time to expiry of six months and an assumed continuously-compounded interest rate of 3% p.a. The bank is delta-hedging the option position assuming the Black-Scholes framework holds and currently holds 250,000 shares of the stock and is short £413,057 in cash.

(ii) By using the hedging position and the Black-Scholes formula for the value of the option, derive two equations satisfied by N and σ, the bank's assumed volatility. [3]

(iii) Estimate σ by interpolation. [5]

(iv) Deduce the value of N. [1]

[Total 15]

5 A binomial model for a non-dividend-paying security with price S_t at time t is as follows: the price at time $(t + 1)$ is either αS_t (down-jump) or βS_t (up-jump). £1 held in cash between times t and $t + 1$ receives interest to become £$(1 + r)$ at time $t + 1$. The parameters satisfy $\beta > 1 + r > \alpha$.

A derivative security with price X has the following payoff at time $t + 1$:

$$\begin{aligned} X_{t+1} &= b \text{ if } S_{t+1} = \beta S_t \\ &= a \text{ if } S_{t+1} = \alpha S_t \end{aligned}$$

A portfolio of cash (amount y) and stock (value x) at time t exactly replicates the payoff of the derivative at time $t + 1$.

(i) Derive expressions for x and y in terms of b, β, a, α and r. [3]

(ii) Derive an expression for q in terms of $(x + y)$, a, b and r, where q is the risk-neutral probability of an up-jump. [2]

(iii) Suppose that $r = 0\%$. Two derivatives each have a payoff of a if $S_{t+1} = \alpha S_t$, but the first derivative pays $2a$ and the second $3a$ if $S_{t+1} = \beta S_t$. The price at time t of the first derivative is 10. Derive an expression, in terms of a, for the price at time t of the second derivative. [3]

[Total 8]

6 (i) Explain why, in the absence of arbitrage, the forward price for a forward contract on one share (over a period where no dividends are payable) is S_0e^{rt}, where S_0 is the initial price of the share, r is the continuously-compounded risk-free rate of interest and t is the time to delivery of the contract. [4]

(ii) Determine a fair (forward) price for a forward contract on a share (currently priced at £10) with delivery in 20 months when the share pays a dividend of 3% of the share price every six months, the continuous risk-free rate is 7% p.a. and the next dividend is due in one month's time.
[You may assume that dividends are immediately re-invested.] [5]

[Total 9]

7 Consider the following discrete time models for (log) share prices and (log) dividend yields:

$$\ln S_{t+1} = \ln S_t + \mu + \sigma Z_{t+1}$$
$$\ln D_{t+1} = \ln\delta + \alpha(\ln D_t - \ln \delta) + \eta W_{t+1},$$

where

S_t = share price at time t
D_t = dividend yield at time t,

W_t and Z_t are both serially uncorrelated standard normal random variables but are correlated with each other and μ, σ, δ and η are positive parameters and $0 < \alpha < 1$.

The unit of time in this model is a month.

(i) Explain the magnitude and sign of the correlation coefficient you would expect between Z_t and W_t. You do not have to calculate the correlation coefficient or derive an expression for it. [2]

(ii) State two properties of the dividend yield model and comment on their realism. [2]

(iii) State three properties of the share price model and comment on them relative to empirical evidence and, if relevant, the efficient markets hypothesis. [3]

[Total 7]

8 Suppose that under the unique Equivalent Martingale Measure, Q, for a term structure model, the SDE satisfied by the instantaneous interest rate r is

$$dr_t = \alpha(\mu - r_t)\, dt + \sigma dZ_t,$$

where $\alpha > 0$, μ and σ are fixed parameters and, under Q, Z is a standard Brownian Motion.

The process X is defined by

$$X_t = r_t\, b(T - t) + \int_0^t r_s\, ds,$$

where the function b is given by $b(s) = (1 - e^{-\alpha s}) / \alpha$.

The function f is given by $f(x, t) = \exp(a(T - t) - x)$, where a is a differentiable function.

(i) Apply Ito's formula to $f(X_t, t)$. [6]

[Hint: First show that $dX_t = A_t dt + B_t dZ_t$ where $A_t = \alpha\mu b(T - t)$ and $B_t = \sigma b(T - t)$]

(ii) (a) Find a differential equation which the function a must satisfy for $f(X_t, t)$ to be a martingale.

(b) Determine an additional condition on a which is necessary for a bond with unit payoff at time T to have a price given by the formula

$$B(t, T) = f(X_t, t)\ \exp\left(\int_0^t r_s ds\right).$$ [4]

[Total 10]

9 The following model has been suggested for the short term interest rate at time t, r_t:

$$dr_t = \mu r_t\, dt + \sigma r_t\, dZ_t$$

where σ and μ are fixed parameters and Z_t is a standard Brownian motion.

(i) Outline three properties of this model and comment on their desirability. [3]

(ii) Outline the properties of the following two models for interest rates:

(a) the one-factor Vasicek model
(b) the Cox-Ingersoll-Ross model

[3]
[Total 6]

 PLEASE TURN OVER

10 (i) Describe four examples of tests that have been done to assess informational efficiency in stock markets. [4]

(ii) Explain to what extent the results of such tests should affect the assessment of the validity or otherwise of the efficient markets hypothesis. [3]

(iii) Explain what an efficient portfolio is in the context of modern portfolio theory, being careful to include a description of what is assumed about investors. [4]

[Total 11]

END OF PAPER

EXAMINATION

September 2005

Subject CT8 — Financial Economics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M Flaherty
Chairman of the Board of Examiners

15 November 2005

1 (i) $\text{Var}(R) = 500{,}000^2\ \text{Var}(U) = 2.5 \times 10^{11} \times 1/12 = 2.08333 \times 10^{10}$

(ii) Downside semi-variance of $R = 2.5 \times 10^{11} \times$ upside semi-variance of U; the upside semi-variance of U is by symmetry 1/24 so downside semi-variance of R is 1.04166×10^{10}.

(iii) $P(R < 100{,}000) = P(U > 0.4) = 0.6$

(iv) If $\text{Va}R_{5\%}(R) = t$ then $P(R \le -t) = 0.05$, so

$$P(300{,}000 - 500{,}000U \le -t) = P(U > 0.6 + (t / 500{,}000)) = 5\%,$$

hence (since $P(U > x) = 1 - x$), $0.4 - (t / 500{,}000)) = 0.05$, so

$$t = 500{,}000\ (0.35) = 175{,}000.$$

2 (i) The market portfolio is (2/7, 3/7, 2/7), so

$$R_M = (2R_A + 3R_B + 2R_C) / 7.$$

Thus

$$\text{Cov}(R_i, R_M) = [2\ \text{Cov}(R_i, R_A) + 3\text{Cov}(R_i, R_B) + 2\ \text{Cov}(R_i, R_C)] / 7.$$

So,

$$\text{Cov}(R_A, R_M) = [.32 + .12 + .04] / 7 = .06857$$

$$\text{Cov}(R_B, R_M) = 0.22/7 = .03143,$$

and

$$\text{Cov}(R_C, R_M) = .09/7 = .01286,$$

and

$$\sigma_M^2 = [2\ \text{Cov}(R_M, R_A) + 3\ \text{Cov}(R_M, R_B) + 2\ \text{Cov}(R_M, R_C)] / 7 = .03674.$$

We conclude that $\beta_A = 1.8664$, $\beta_B = 0.8555$ and $\beta_C = 0.3500$.

Finally, solving

$$r_i - r_0 = \beta_i(r_M - r_0), \text{ we get } r_A = 0.4,\ r_B = 0.2 \text{ and } r_C = 0.1.$$

(ii) The corresponding single index model is

$$R_i = (1 - \beta_i)\, r_0 + \beta_i\, R_M + \varepsilon_i$$

where the ε_i s are uncorrelated with each other and with R_M, and ε_i has variance equal to

$$\text{Var}(R_i) - \beta_i^2 \sigma_M^2 ,$$

so that, setting

$$\text{Var}(\varepsilon_i) = \sigma_i^2, \;\; \sigma_A^2 = 0.0320, \;\; \sigma_B^2 = 0.0131 \text{ and } \sigma_C^2 = 0.0055.$$

(iii) The single index model is not the same as the CAPM model because the covariances of asset returns are different in the two models: in the single index model

$$\text{Cov}(R_i, R_j) = \beta_i\, \beta_j\, \sigma_M^2 ,$$

so for example we would obtain

$$\text{Cov}(R_A, R_B) = 0.0587,$$

whereas in the CAPM model

$$\text{Cov}(R_A, R_B) = 0.04.$$

3 (i) Delta: the rate of change in derivative price with respect to change in the price of underlying asset.

Gamma: the rate of change of delta with respect to change in the price of underlying asset.

Theta: the rate of change in the value of the derivative with respect to change in time to expiration.

lambda: the rate of change in the value of the derivative with respect to change in the assumed continuous dividend yield on the underlying asset.

rho: the rate of change in the value of the derivative with respect to change in the risk-free rate of interest.

vega: the rate of change in the value of the derivative with respect to the (assumed) volatility of the underlying asset.

(ii) Assuming that the portfolio under management is delta hedged at discrete times, the two most important Greeks are gamma and vega. Between rebalancing at the trading times, delta will drift away from zero as the underlying asset prices move. If the portfolio is gamma-hedged at the discrete trading times then the amount of such drift will be small (comparable to the square of the change in underlying price).

The underlying volatilities used in hedging calculations are all estimates. If these are incorrect then delta hedging may be incorrect, consequently it is appropriate to attempt to immunise a portfolio against (small) errors in volatility estimates. Just as in delta hedging, achieving a portfolio vega of zero achieves this. Consequently, good risk managers will seek to achieve a (close to) zero vega for the bank's portfolio.

4 (i) No frictions; short-selling permitted; small investor (i.e. does not "move the market"); market is arbitrage-free; stock price is given by

$$dS_t = \mu_t S_t \, dt + \sigma S_t \, dZ_t$$

where Z is a standard Brownian motion.

All are, in some sense, implausible. Friction (spreads and commission) is present; short-selling is available but on very different terms; "small investor" not true for an investment bank; stock-market returns are not compatible with normality (fat tails, jumps); arbitrages occur (for short periods).

(ii) Let N be the number of options written, then $N\Phi(d_1) = 250{,}000$. Now the value of the bank's portfolio is

$$1.8 \, N\Phi(d_1) - 2e^{-.015} \, N\Phi(d_2) = 1.8 \times 250{,}000 - 413{,}057 = 36{,}943$$

(iii) So $\Phi(d_2) / \Phi(d_1) = 413{,}057 / (250{,}000 \times 2e^{-.015}) = 0.8386$.

With $\sigma = 10\%$: $d_1 = -1.2425$, $d_2 = -1.3132$, $\Phi(d_1) = .1070$, $\Phi(d_2) = .0946$

so $\Phi(d_2) / \Phi(d_1) = 0.8841$

With $\sigma = 30\%$: $d_1 = -.3199$, $d_2 = -.5320$, $\Phi(d_1) = .3745$, $\Phi(d_2) = .2974$

so $\Phi(d_2) / \Phi(d_1) = 0.7941$

Linear interpolation gives an estimate of

$$10 + 20(.8841 - .8386) / (.8841 - .7941) = 20.1\%$$

for the implied volatility.

(iv) Consequently,

$$N = 250{,}000 / \Phi(d_1) = 874126 \text{ contracts.}$$

5 (i) Consider an investment of x in the stock and y in cash at time t: the value of the holding at time $t + 1$ is

$$x\beta + y(1 + r),$$

if there is an up-jump and is

$$x\alpha + y(1 + r),$$

if there is an down-jump. The value (with $t = 0$) is supposed to be b in the first case and a in the second. So, we need to solve:

$$x\beta + y(1 + r) = b, (1)$$

$$x\alpha + y(1 + r) = a, (2).$$

Subtracting (2) from (1) we get:

$$x = (\mathrm{b} - a) / (\beta - \alpha) \text{ and}$$

$$y = (a\beta - b\alpha) / (\beta - \alpha)(1 + r)$$

(ii) $x + y = X_t = [qb + (1 - q)a](1 + r)^{-1}$

$\Rightarrow q(b - a) + a = (x + y)(1 + r)$
$\Rightarrow q = [(x + y)(1 + r) - a] / (b - a)$ (3)

(iii) Using (3), we see from the first derivative that $q = (10/a) - 1$, while, from the second we see that $q = (\mathrm{c}_2/2a) - 1/2$, so we deduce that $c_2 = 20 - a$.

6 (i) Consider a portfolio which is, initially, short one forward contract, holds 1 share and is short c in cash. At the delivery date for the forward contract, the portfolio contains 1 share and is short ce^{rt}, where r is the risk free rate and t is the duration of the contract.

So, immediately after delivery the portfolio contains zero shares and is short $ce^{rt} - p$ in cash, where p is the forward price.

Setting $c = pe^{-rt}$, the portfolio contains nothing. It follows that the portfolio should have a zero set-up cost so, so $p = S_0e^{rt}$.

(ii) Consider a portfolio which is, initially, short one forward contract, holds s shares and is short c in cash. At each dividend date the dividend is used to buy more shares. At the delivery date for the forward contract, the portfolio contains 1.03^4s shares and is short ce^{rt}, where r is the risk free rate and t is the duration of the contract.

So, immediately after delivery the portfolio contains $1.03^4s - 1$ shares and is short $ce^{rt} - p$ in cash, where p is the forward price.

Setting $s = 1 / 1.03^4$, and $c = pe^{-rt}$, the portfolio contains nothing. It follows that the portfolio should have a zero set-up cost so $0 = c - 10s$, so $p = 10se^{rt} = £9.98$.

7 (i) We expect a strong negative correlation.

Dividend yield = dividend/price so if there is a strong price rise it's likely to be accompanied by a decrease in yield.

(ii) Mean-reverting: this is in line with historical evidence in most markets.
Non-negative: dividend yield cannot be negative.

(iii) Not mean-reverting: consistent with weak form EMH. Empirical evidence is mixed.

Constant volatility: this is inconsistent with empirical evidence

Normal distribution: markets jump and returns have fat tails, so inconsistent with empirical evidence.

8 (i) $dr_t = \alpha(\mu - r_t)\,dt + \sigma dZ_t$

$f(X_t, t) = e^{a(T-t)-X_t}$

Now $X_t = b(T-t)r_t + \int_0^t r_s ds$

$$\Rightarrow dX_t = b'(T-t)r_t dt + b(T-t)dr_t + r_t dt$$

$$= -e^{-\alpha(T-t)} r_t dt + \left[1 - e^{-\alpha(T-t)}\right](\mu - r_t)dt$$

$$+ r_t dt + \left[1 - e^{-\alpha(r-t)}\right]\frac{\sigma}{\alpha} dZ_t$$

$$= \left\{\mu\left(1 - e^{-\alpha(T-t)}\right)\right\} dt + \left\{\frac{\sigma}{\alpha}\left(1 - e^{-\alpha(T-t)}\right)\right\} dZ_t$$

$$= A_t dt + B_t dZ_t$$

Using Ito

$$df(X_t, t) = \frac{\partial f}{\partial x} B_t dZ_t + \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} A_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} B_t^2\right] dt$$

$$= -f(X_t, t) B_t dZ_t - f(X_t, t)\, a'(T-t) dt$$

$$- f(X_t, t) A_t dt + \frac{1}{2} f(X_t, t) B_t^2 dt$$

$$= f(X_t, t)\left\{\left[a'(T-t) - A_t + \frac{1}{2}B_t^2\right] dt - B_t dZ_t\right\}$$

(ii) (a) To be a martingale the […] term in (i) must be zero:

$$a'(T-t) = A_t - \frac{1}{2}B_t^2$$

$$= \alpha\mu b(T-t) - \frac{1}{2}\sigma^2 b^2 (T-t)$$

(b) $B(T,T) = 1 \Rightarrow e^{a(0) - r_T b(0)} = 1$

$$\Rightarrow a(0) = 0$$

9 (i) This means short term interest rates would in the long term increase if $\mu > 0$ geometrically. This is not desirable as it does not reflect reality.
The model also has the following properties:

The change in rate is dependent on the current rate. This is undesirable as typical rates mean revert.

The model requires constant volatility over time. This is not desirable as volatility of short term interest rates changes over time.

(ii) (a) Incorporates mean reversion.
Arbitrage free.
Allows negative interest rates.

(b) Incorporates mean reversion.
Arbitrage free.
Volatility high/low when rates high/low.
Does not allow negative interest rates.
More difficult to implement than Vasicek model

10 (i) **Over-reaction tests**

- past winners tend to be future losers (or vice versa)

- certain accounting ratios appear to have predictive power (e.g. BV/MV or E/P)

- IPOs and other new offerings have poor subsequent performance

Under-reaction

- stock prices react slowly to earnings announcements
- abnormal excess returns for parent/subsidiary following a demerger
- abnormal negative returns following mergers

(ii) Over-reaction or under-reaction to the arrival of public information would appear to contradict
the semi-strong form of the EMH since excess returns could be earned.

However, some of the tests (such as accounting ratios) may not allow properly for risk and the results are therefore not incompatible with the EMH.

Many of these tests appear to be time-period specific.

(iii) Assume that investors are non satiated (always prefer more expected return to less) and risk averse (in the sense of wanting to avoid volatility of returns).

An efficient portfolio is one with the highest expected return for a given level of volatility **and** the lowest volatility for the expected return.

END OF EXAMINERS' REPORT

EXAMINATION

5 April 2006 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 9 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

1 An investor can construct a portfolio using only two assets A and B with the following properties:

	A	*B*
Variance of return	24%%	12%%
Correlation coefficient between assets	0.25	

(i) Derive a formula for and determine the composition of the investor's minimum variance portfolio. [3]

(ii) Explain in general terms the benefits of diversification. [3]
[Total 6]

2

			Asset		
State	*A*	*B*	*C*	*D*	*Probability of state*
1	5%	5%	5%	2%	0.3
2	4%	7%	5%	6%	0.2
3	7%	3%	5%	9%	0.5
Value of asset	10,000	20,000	n/a	10,000	

The table above gives the returns on all four assets in an investment market under the three possible states of the world.

(i) Calculate the market price of risk under CAPM. [4]

(ii) Outline the limitations of CAPM. [3]
[Total 7]

3 (i) State five key defining properties of a standard Brownian motion. [5]

(ii) Outline the advantages and disadvantages of using the continuous time Lognormal model for stock prices by considering both the theoretical features of the model and its consistency with empirical evidence. [6]
[Total 11]

4 (i) List the desirable characteristics of a term structure model. [4]

Under the real world probability measure, P, the price of a zero coupon bond with maturity T is given by:

$$B(t, T) = \exp(T(T - t)\, r_t + \sigma^2 (T - t)^3/6)$$

where r_t is the risk free rate of interest at time t.

r_t satisfies the following SDE under the real world measure P:

$$dr_t = \mu r_t dt + \sigma dz_t$$

where $\mu > 0$ and dz_t is a standard Brownian motion under P.

(ii) Derive:

(a) the market price of risk and
(b) the SDE for r_t under the risk neutral measure Q

[7]
[Total 11]

5 (i) Comment on the implications for an assessment of stock market efficiency of the necessity for a ban on the management of a company trading in their company stock particularly during takeover talks. [2]

(ii) Explain why it is not straight forward to identify when the semi-strong form of stock market efficiency holds. [3]

(iii) Comment on the implications of stock market efficiency for passive and active fund managers. [2]
[Total 7]

6 In the Wilkie model, the force of inflation from time $t - 1$ to t, $I(t)$, is modelled as:

$$I(t) = Qmu + QA[I(t-1) - Qmu] + QSD \,.\, QZ(t)$$

where $QZ(t) \sim N(0, 1)$

and Qmu, QA and QSD are fixed parameters as follows:

$Qmu = 0.03$
$QA = 0.55$
$QSD = 0.45$

(i) Calculate the 95% confidence interval for the force of inflation over the following year given inflation over the past year was 2.75%. [4]

(ii) Explain an economic justification for using an $AR(1)$ process for inflation. [1]

(iii) Explain whether a model of the form that is used for inflation is also suitable for share prices. [3]

[Total 8]

7 (i) Describe the advantages of the martingale approach to derivative valuation compared with an approach based on deriving an appropriate partial differential equation. [3]

(ii) State and compare the risk-neutral and state price deflator approaches to valuing derivatives. [4]

[Total 7]

8 An employer contracts with his staff to give each of them 1,000 shares in one year's time provided the share price increased from its current level of £1 to at least £1.50 at the end of the year.

You may assume the following parameters:

- risk free interest rate: 4% p.a. continuously compounded
- stock price volatility: 30% p.a.
- dividend yield: nil

(i) Calculate the value of the contract with each employee by considering the terms of the Black-Scholes formula, [6]

The employer now wishes to limit the gain to each employee to £2,000.

(ii) Calculate the value of this revised contract. [6]

(iii) An employee has said that he believes the original uncapped contract is worth £300. He has determined this by saying that he believes there is a 30% chance of the share price being at least £1.50 therefore 30% × £1 = 30p.

(a) Compare the approach taken by the employee and the approach used in (i).

(b) Comment on the implications of the differences in (iii) (a) if there were a market in such contracts.

[4]

[Total 16]

9 Consider a recombining binomial model for the price of a share where:

risk free interest rate $r = 4\%$ p.a. (equivalent to 0.016% per trading day) continuously compounded

volatility $\sigma = 20\%$ p.a.

initial share price $S_0 = 100$

the ratio of the share price after an "up jump" compared with the share price before the jump is given by $u = \exp(\sigma \,.\, 250^{-½})$

There are 250 trading days per year and ignore dividends except where specifically mentioned.

(i) Calculate the price at time 0 of a European-style put option with a strike price of 101p that expires in 2 days time. [6]

(ii) (a) Sketch a graph of the delta of a put option against share price with exercise price 100.

(b) Explain the key features of the graph. [6]

(iii) Derive, using the binomial lattice, the price of a European-style call option with exercise price 101p expiring in 2 days time. [1]

(iv) (a) State the relationship known as "put-call parity".
(b) Prove it from first principles.
[4]

(v) Compare the result of (iii) to the result of (i) using put-call parity. [2]

Assume now that an investor holds a call option and a put option, both with exercise prices of 101p, that can be exercised at the investor's option at the end of day 1 or the end of day 2. At the end of day 1, just before the investor is allowed to exercise the options, the company announces unexpectedly that a dividend of 3p per share will be paid at the end of day 2 immediately prior to the expiry of the options.

(vi) (a) Construct the binomial lattice of share prices allowing for the dividend payment.

(b) Explain the conditions under which the holders of the put and call options will exercise at the end of day 1 after the announcement of the dividend. [8]
[Total 27]

END OF PAPER

EXAMINATION

April 2006

Subject CT8 — Financial Economics
Core Technical

EXAMINERS' REPORT

Introduction

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M Flaherty
Chairman of the Board of Examiners

June 2006

Comments

Please see individual comments on questions 4 and 6. No other comments given.

1 (i) Let $V = V_A\alpha_A^2 + (1-\alpha_A)^2 V_B + 2\alpha_A(1-\alpha_A)C_{AB}$

$$\frac{\partial V}{\partial \alpha_A} = V_A 2\alpha_A - 2(1-\alpha_A)V_B + (2(1-\alpha_A) - 2\alpha_A)C_{AB}$$

Set equal to zero gives

$$0 = V_A 2\alpha_A - (2 - 2\alpha_A)V_B + 2(1-2\alpha_A)C_{AB}$$

$$0 = \alpha_A(V_A + V_B - 2C_{AB}) - V_B + C_{AB}$$

$$\Rightarrow \alpha_A = \frac{V_B - C_{AB}}{V_A + V_B - 2C_{AB}}$$

In this case

$$\alpha_A = \frac{12\%\% - 0.25 \times (24\%\% \times 12\%\%)^{1/2}}{24\%\% + 12\%\% - 2 \times 0.25 \times (24\%\% \times 12\%\%)^{1/2}}$$

= 28.2%

$\Rightarrow$ 71.8% of asset B

(ii) As a portfolio is diversified, the return on the portfolio is less exposed to the specific risk of any one component.

This means that as portfolios are diversified the correlation components become less important, therefore variance of return is minimised.

2 (i) The market price of risk is $(E_m - r)/\sigma_m$.

Asset C is the risk free asset therefore r = 5%.

E_m = (10,000 × (5% × 0.3 + 4% × 0.2 + 7% × 0.5) +

20,000 × (5% × 0.3 + 7% × 0.2 + 3% × 0.5) +

10,000 × (2% × 0.3 + 6% × 0.2 + 9% × 0.5)) ÷ 40,000

= 5.225%

$$\sigma_m^2 = \left(\frac{10{,}000\times 5\% + 20{,}000\times 5\% + 10{,}000\times 2\%}{40{,}000} - 5.225\%\right)^2 \times 0.3 +$$

$$\left(\frac{10{,}000\times 4\% + 20{,}000\times 7\% + 10{,}000\times 6\%}{40{,}000} - 5.225\%\right)^2 \times 0.2 +$$

$$\left(\frac{10{,}000\times 7\% + 20{,}000\times 3\% + 10{,}000\times 9\%}{40{,}000} - 5.225\%\right)^2 \times 0.5$$

$= 4.4312 \times 10^{-5} = 0.66567\%^2$

$\Rightarrow$ The market price of risk is (5.225% –5%) / 0.66567%

= 33.8%.

(ii) Empirical studies do not provide strong support for the model.

It does not account for taxes, inflation or where there is no riskless asset.

It does not consider multiple time periods or optimisation of consumption over time.

3 (i) Credit should be awarded for any five from the following:

(a) B_t has <u>independent increments</u>, i.e. $B_t - B_s$ is independent of $\{B_r, r \le s\}$ whenever $s < t$.

(b) B_t has <u>stationary increments</u>, i.e. the distribution of $B_t - B_s$ depends only on $t - s$.

(c) B_t has <u>Gaussian increments</u>, i.e. the distribution of $B_t - B_s$ is $N(0, t - s)$.

(d) B_t has <u>continuous sample paths</u> $t \to B_t$.

(e) $B_0 = 0$.

(f) $\text{Cov}(B_s, B_t) = \min(s, t)$ since, for $s > t$, $\text{Cov}(B_t, B_t) = t$ and $\text{Cov}(B_s - B_t, B_t) = 0$.

(g) $\{B_t, t \ge 0\}$ is a Markov process: this follows directly from the independent increment property.

(h) $\{B_t, t \ge 0\}$ is a martingale: 0.1 demonstrates that $E(B_s \mid F_t) = B_t$.

(i) $\{B_t, t \geq 0\}$ returns infinitely often to 0, or indeed to any other level.

(j) If $\{B_1(t), t \geq 0\}$ is defined by

$$B_1(t) = \frac{1}{\sqrt{c}} B_{ct}$$

then $\{B_1(t), t \geq 0\}$ is also a standard Brownian motion. (This is the *scaling property* of Brownian motion.)

(k) If $\{B_2(t), t \geq 0\}$ is defined by

$$B_2(t) = tB_{1/t}$$

then $\{B_2(t), t \geq 0\}$ is also a standard Brownian motion. (This is the *time inversion property* of Brownian motion.)

(ii) **Advantages**

- The mean and variance of return are proportional to the length of the time interval considered.

- Returns over non-overlapping time intervals are independent of each other.

- Cannot use past history to identify whether prices are cheap or dear implying weak form market efficiency consistent with empirical observations.

- Does not permit negative share prices.

Disadvantages

- Estimates of volatility vary widely over time periods. This is supported by implied volatility from option prices.

- The model is not mean reverting, which is contradicted by some evidence of momentum effects and reversion after market crashes.

- Does not reflect jumps and discontinuities observed in the market.

4 (i)

- The model should be arbitrage free.
- Interest rates should be positive.
- Interest rates should exhibit some element of mean reversion.
- The model should be computationally tractable.
- Gives a reasonable range of possible yield curves.
- Fits historical data.
- Can be calibrated to current market data.
- Flexible to cope with a range of derivatives.

Comments on question 4(ii):

Please note that there was a typographical error in this question where the first "T" in the expression should have been a "minus" sign.

Candidates who identified this and proceeded on the assumption that there had been a typographical error were given appropriate credit, as were candidates who noted that the expression led to inconsistencies. The "solution" below shows the correct technical approach applied to the question as it stood. All candidates' scripts were assessed to take into account any additional impact of this error.

(ii) Using Ito's lemma:

$$dB(t, T) = B(t, T)\,(-Tr_t dt - \tfrac{1}{2}\sigma^2(T-t)^2 dt + T(T-t)\,dr_t + \tfrac{1}{2}T^2(T-t)^2\sigma^2 dt)$$

(a) Therefore as the market price of risk is

$$\gamma(t, T) = (m(t, T) - r_t) / s(t, T)$$

where $dB(t, T) = B(t, T)\,(m(t, T)\,dt + s(t, T)\,dz_t)$

$$m(t, T) = T((T-t)\,\mu - 1)\,r_t + \tfrac{1}{2}(T-t)^2\,\sigma^2(T^2 - 1)$$

$$s(t, T) = \sigma(T-t)\,.\,T$$

Therefore $\gamma(t) = \dfrac{(r_t(T(T-t)\mu - 2) + \tfrac{1}{2}(T-t)^2\sigma^2(T^2-1))}{T(T-t)\sigma}$

(b) SDE for r_t is $dr_t = \sigma d\tilde{z}_t$ under the risk neutral measure Q

$$dr_t = \frac{r_t(\mu - (T(T-t)\mu - 2) + \tfrac{1}{2}(T-t)^2\sigma^2(T^2-1))dt}{T(T-t)} + \sigma d\tilde{z}_t$$

5 (i) If a market is strong form efficient then senior managers could not make abnormal profits. The existence of a ban suggests abnormal profits could be made, which suggests markets are not strong form efficient.

(ii) Different stock exchanges have different disclosure levels therefore different markets have different levels of efficiency.

Even if information is publicly available, there is a cost and possibly a time delay to obtain it. This erodes the advantages of obtaining the information.

Individuals do not have access to the management of companies that fund managers do. They spend time and money interviewing senior management.

(iii) In aggregate, active managers hold the market, which is identical to the passive manager. Therefore the average active manager should perform in line with the passive manager.

If markets are inefficient, we would expect active managers with above average skill to perform better than passive managers.

The existence of active managers suggests a belief that markets are inefficient.

6 (i) Let $Q(t)$ be the inflation index at time t

$$\frac{Q(t-1)}{Q(t-2)} = 1.0275 = e^{I(t-1)}$$

therefore $I(t-1) = \ln(1.0275)$

The 95% confidence interval for $I(t)$ is therefore at the upper level

$$I(t) = 0.03 + 0.55(\ln(1.0275) - 0.03) + 0.45 \times 1.96$$

$$= 0.910$$

at the lower level

$$I(t) = 0.03 + 0.55(\ln(1.0275) - 0.03) - 0.45 \times 1.96$$

$$= -0.854$$

Comments on question 6 (i)

Candidates who used 0.045 as in the original Wilkie calibration (instead of 0.45) were given appropriate credit.

(ii) Inflation in many countries tends to be mean-reverting because central banks and governments attempt to manage it close to target ranges

(iii) The model used for inflation is not suitable for share prices for the following reasons:

(1) Strong mean reversion implies prices are 'predictable' so high returns are possible with little risk through active market timing. This runs counter to much empirical evidence.

(2) Lots of evidence for share price 'jumps' in market prices that are not reflected in the model.

(3) Share prices tend to increase rather than mean revert therefore a stationary process is not suitable.

(4) The model permits negative share prices, which is highly unrealistic.

7 (i) The martingale approach gives much more clarity in the valuation process. By providing an explicit expectation to evaluate.

It gives the replicating strategy for the derivative.

It can be applied to exotic options where the PDE approach cannot.

(ii)

- Risk neutral pricing approach is the same as the martingale approach, i.e. values are derived from the risk neutral world.

- Deflators values the derivative in a real world probability measure with a stochastic adjustment factor.

  The approach is the same as risk-neutral pricing, the only difference is that calculations are presented using the real world measure and a stochastic adjustment factor versus a risk neutral measure. Intuitively the deflator approach can also give information about real world expected outcomes.

8 (i) The value of the promise can be thought as part of a call option contract. A call option consists of a contract to deliver a share in return for the payment of an exercise, where the share price exceeds the exercise price.

The promise made to the employees is the first part of the call option, the promise to deliver a share provided the price exceeds a certain level. Therefore, the first component of the Black Scholes formula gives the value

Value = $S.N(d_1)$ where $d_1 = \left(\dfrac{\ln s/k + r + ½\sigma^2}{\sigma\sqrt{T}}\right)T$

$$= 1{,}000 \times N\left(\frac{\ln\left(\frac{1}{1.5}\right) + 0.04 + ½0.3^2}{0.3}\right)$$

$$= N(-1.0682) \times 1{,}000$$

$$= £142.70$$

(ii) Limiting the gain under the contract can be represented by a portfolio of the above promise less a call option with exercise price of £2.

The value of the call option is therefore

$1{,}000(N(d_1) - e^{-0.04} N(d_2) * 2)$ d_1 as above
$d_2 = d_1 - \sigma\sqrt{T}$

where

$d_1 = -2.027$
$d_2 = -2.327$

The value of the call option is therefore

$$1{,}000 \times 0.00215 = £2.15$$

The value of the revised promise is therefore

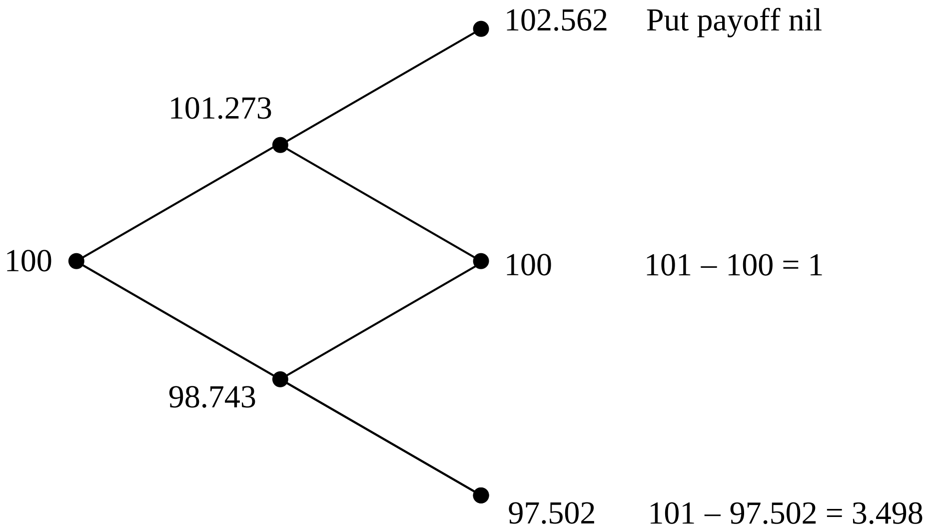
$$£142.70 - £2.15 = £140.53$$

(iii) (a) The employee has taken a view about the expected growth in share price. The result is a value that is not consistent with risk neutral pricing.

This means that, if there were a market in these contracts the price the employee has derived is not equal to the price the market would place on it.

(b) If the employer were willing to buy such promises at their suggested price, an arbitrageur would sell of £300 and hedge their position at £142.70. Resulting in substantial risk free profits.

9 (i)



$u = \exp(\sigma \,.\, 250^{-½}) = 1.01273$

The risk neutral probability of an upstep is

$$q = \frac{e^{0.016\%} - 0.98743}{1.01273 - 0.98743}$$

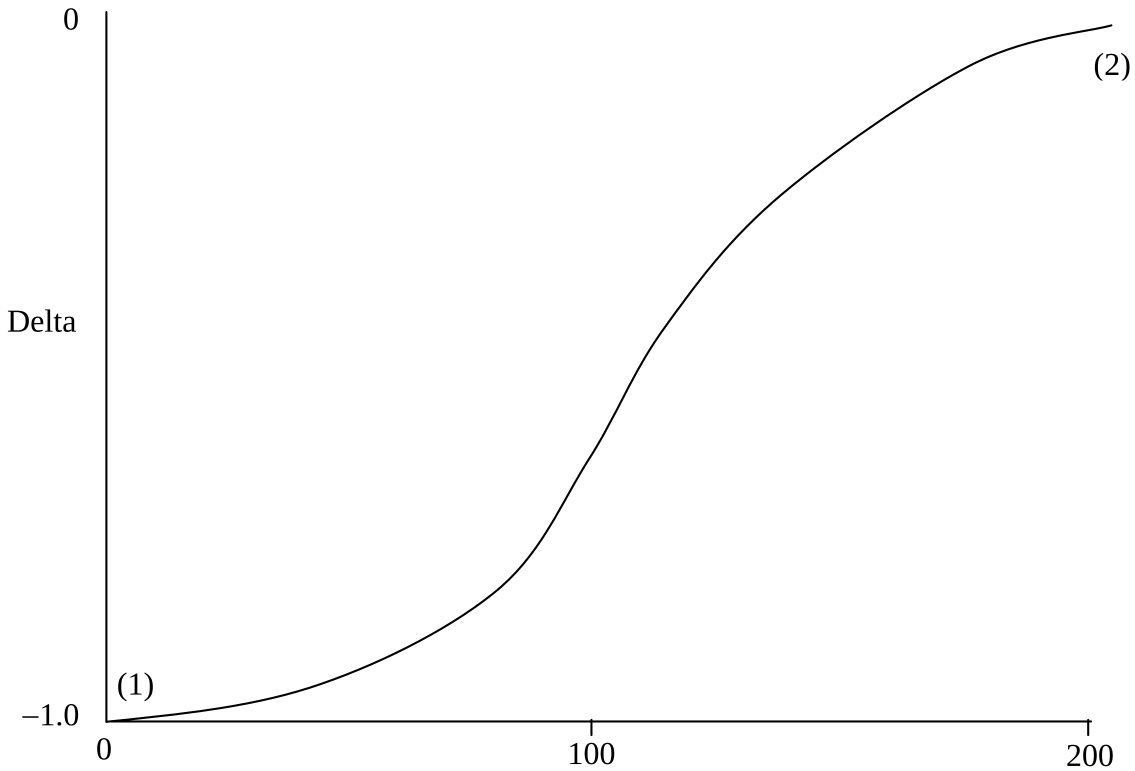
$$= 0.50316$$

$\Rightarrow$ The value of the put option is therefore

$$e^{-2\times 0.016\%}(0.50316(1 - 0.50316) \times 2 \times 1 + (1 - 0.50316)^2 \times 3.498)$$

$$= 1.3635$$

(ii) (a)



(b) Key feature:

Delta is negative as the value of a put option falls when share price rises.

(1) When the share price is very low the value is almost 100 therefore delta is almost − 1.

(2) When share price is high the value is almost nil therefore delta is almost nil.

(iii) Price of call option

$$= e^{-2\times 0.016\%}(0.50316^2 \times 1.562) = 0.3953$$

(iv) Put call parity means for all time $t < T$, then

$$c_t + k\mathrm{e}^{-r(T-t)} = p_t + s_t$$

c_t = call option with exercise price k and expiry T
p_t = put option with exercise price k and expiry T

Consider

A one call plus cash of $ke^{-\sigma(T-t)}$
B one put plus share

At expiry if $S_T > K$ then portfolio A is worth S_T as is portfolio B.

If $S_T \leq K$ then portfolio A is worth k as is portfolio B.

By the principle of no arbitrage since the payoffs are identical at time T the value of the portfolios must be identical at time $t < T$.

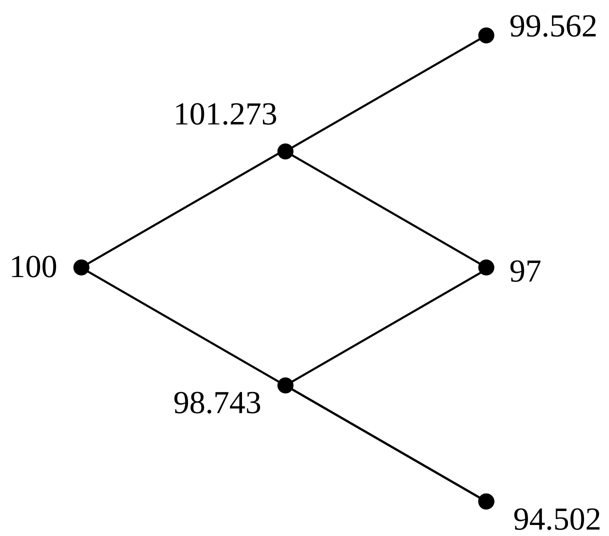
(v) Put call parity gives

$$c_t = 1.374 + 100 - 101e^{-0.016\% \times 2}$$

$$= 0.4063$$

This is not equal to the value above because the binomial model is a discrete time approximation to a continuous model. Therefore, the difference in value is due to discretisation error.

(vi) Share price



Call option

If the call option is held to expiry it will be worthless as it will expire underwater.

If the option holder exercise at day 1 when the share price has risen they will have a positive gain. Therefore, exercising early, if the share price has risen will be advantageous.

Put option

It is clear that it is not advantageous for the option holder to exercise early as follows:

If share price is 101.273 then gain = nil.
If the option is held to expiry there is a positive gain.
If the share price is 98.743 the gain is 2.257.
If held to expiry the gain must be at least 101 – 97 = 4.

END OF EXAMINERS' REPORT

EXAMINATION

13 September 2006 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 8 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

1 An investor is contemplating an investment with a return of £R, where:

$$R = 250{,}000 - 100{,}000N,$$

and N is a Normal [1, 1] random variable.

Calculate each of the following four measures of risk:

(a) variance of return
(b) downside semi-variance of return
(c) shortfall probability, where the shortfall level is £50,000
(d) Value at Risk at the 5% level

[8]

2 A non-dividend-paying stock has a current price of 800p. In any unit of time $(t, t + 1)$ the price of the stock either increases by 25% or decreases by 20%. £1 held in cash between times t and $t + 1$ receives interest to become £1.04 at time $t + 1$. The stock price after t time units is denoted by S_t.

(i) Calculate the risk-neutral probability measure for the model. [4]

(ii) Calculate the price (at $t = 0$) of a derivative contract written on the stock with expiry date $t = 2$ which pays 1,000p if and only if S_2 is not 800p (and otherwise pays 0). [4]

[Total 8]

3 (i) Explain what is meant by "self-financing" in the context of continuous-time derivative pricing, defining all notation used. [4]

(ii) Define the delta of a derivative, defining all notation and terms used other than those already defined in your answer to (i). [2]

(iii) Explain, in general terms, how delta and self-financing are used in the martingale approach to valuing derivatives. [5]

[Total 11]

4 The Wilkie model has been used to produce stochastic simulations of inflation rates.

The following runs were made:

- 1,000 simulations of one-year
- one simulation of 1,000 years

and the standard deviations were calculated.

(i) Explain why you would expect the standard deviations calculated in each run to be different. [4]

(ii) State the conditions under which the standard deviations in the two runs would be expected to be the same. [2]

(iii) Discuss the advantages and disadvantages of using economic theory rather than statistical models to construct and calibrate a stochastic model. [5]
[Total 11]

5 (i) List five desirable characteristics of a model for the term structure of interest rates. [5]

(ii) State the Stochastic Differential Equation satisfied by the short rate in the Vasicek model for the term structure of interest rates. [1]

(iii) Comment on the appropriateness of the Vasicek model in the light of your answer to part (i). [5]
[Total 11]

6 An investor can invest in only two assets with the following characteristics (annualised):

Asset	*Expected rate of return*	*Standard deviation*
A	10%	20%
B	5%	0%

(i) Show that the efficient frontier for the investor is a straight line passing through the points (0, 0.05) and (0.1, 0.075) in (standard deviation, expected return) space. [5]

A third security C becomes available to the investor. It has an annualised expected return of 6% and an annualised standard deviation of 10%. It is uncorrelated with A and B.

(ii) Determine the portfolio using only A and C that maximises:

$$\frac{\text{expected return} - 5\%}{\text{standard deviation}}.$$ [6]

(iii) Using (ii), or otherwise, show that the new efficient frontier using A, B and C passes through the point (0.1, 0.0769). [6]
[Total 17]

7 Consider the following two-factor model of security returns:

$$R_i = \alpha_i + \beta_{i1}I_1 + \beta_{i2}I_2 + \varepsilon_i$$

where:

- R_i = return on security i
- α_i, β_{i1}, β_{i2} are security-specific parameters
- I_1 and I_2 are the changes in the 2 factors on which the model is based
- ε_i is an independent random normal variate with variance σ_i^2.

(i) Describe briefly three categories of model that could help in choosing the factors, I_1 and I_2. [6]

Suppose the factors I_1 and I_2 are chosen to be total return indices with I_1 based on the whole market and I_2 based on the 50 stocks with the highest dividend yield.

(ii) Explain in detail how the two factors can be transformed into two orthogonal factors, one of which is the same as the index on which I_1 is based. [3]

(iii) Derive an expression for the variance of the returns on the security in terms of the variances of the changes of the orthogonal factors and σ_i^2. [3]

(iv) Explain in words the expression in (iii). [2]

[Total 14]

8 (i) State the SDE of a non-dividend paying stock price in the Black-Scholes model, under the EMM defining all symbols used. [2]

(ii) Give the general formula for the price of a derivative security which has a terminal value of C at time T. [2]

(iii) A special option on a share pays £1 at time T if (and only if) the share price at time T lies in the interval $[a, b]$.

Prove that the price of such an option is given by:

$$e^{-rT}[\Phi(d(b)) - \Phi(d(a)] \text{ where } d(x) = \frac{\ln\left(\frac{x}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sqrt{T}\sigma}$$

where S_0 = price of underlying stock, r = continuously compounded rate of return on the risk free asset and σ = volatility parameter of stock price process. [5]

A fund manager currently charges an annual management fee of 0.5% of the value of the funds under management at the end of a one-year contract.

The value of the funds under management are governed by the following SDE:

$$dS_t = S_t(\mu dt + \sigma dZ_t)$$

where S_t = value of funds under management
Z_t = standard Brownian motion
$\mu = 0.08$
$\sigma = 0.25$

The funds generate no income during the year.

The continuously compounded risk-free rate is 5% per annum.

The owner of the funds wishes to change the management fee to be performance-related.

Specifically the fee, KS_1 is set so that:

$$K = \begin{cases} 0.1\% & \text{if } S_1 < S_0 \\ 1\% & \text{if } S_1 > U \\ 0.5\% & \text{otherwise} \end{cases}$$

(iv) Calculate the value at time 0 of the management fee under the original fee structure if $S_0 = 100$. [1]

(v) Calculate U so that the management fee under the performance-related fee structure has the same value at time 0 as the fixed fee in (iv). [10]

Hint: the fee can be written as a basic fee plus two call options plus two options of the form in (iii). [Total 20]

END OF PAPER

EXAMINATIONS

September 2006

Subject CT8 — Financial Economics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

November 2006

Comments

No comments are given

1 (a) $\text{Var}(R) = 100{,}000^2 \, \text{Var}(N) = 10^{10}$

(b) Downside semi-variance of $R = 10^{10} \times$ upside semi-variance of N; the upside semi-variance of N is ½, so downside semi-variance of R is 5×10^9

(c) $P(R < 50{,}000) = P(N > 2) = 1 - \Phi(1) = 1 - .8413 = .1587$

(d) If $\text{Va}R_{5\%}(R) = t$ then $P(R \leq - t) = 0.05$, so $P(250{,}000 - 100{,}000N \leq -t) =$ $P(N > 2.5 + (t/100{,}000)) = 5\%$, hence (since $N - 1$ is a standard normal r.v) $\Phi(1.5 + (t/100{,}000) = .95$, so $t = 100{,}000(1.645\text{-}1.5) = £14{,}500$.

2 (i) The pricing measure **Q** must satisfy:

$$\mathbf{E_Q}\left[\frac{1}{1+r}S_{t+1}|\mathbf{F}_t\right] = S_t;$$

so, if we set

$$q_t = \mathbf{Q}(S_{t+1} = 1.25S_t\,|\mathbf{F}_t),$$

then

$$1.04 = 1.25q_t + 0.8(1 - q_t) \Leftrightarrow q_t = q = 8/15$$

Thus the unique pricing measure makes S a multiplicative random walk with up-jump probability of 8/15.

(ii) The price of the derivative is $P = \mathbf{E}_Q[X/(1 + r)^2]$, where X is the terminal value of the derivative.

Thus,

$$\begin{aligned} P &= 1{,}000Q(S_2 \neq 800)/1.04^2 \\ &= 1{,}000 \times ((8/15)^2 + (7/15)^2)/1.04^2 \\ &= 464.33\text{p} \end{aligned}$$

3 (i) Suppose that at time t we hold the portfolio (φ_t, ψ_t) where φ_t represents the number of units of S_t held at time t and ψ_t is the number of units of the cash bond held at time t.

We denote the value of the portfolio at time t by $V(t)$.

The portfolio strategy is described as self-financing if $dV(t)$ is equal to $\varphi_t\, dSt + \psi_t\, dB_t$: that is, at time $t + dt$ there is no inflow or outflow of money necessary to make the value of the portfolio back up to $V(t + dt)$.

(ii) Let F_t be the discounted value of a derivative (priced using the EMM) then since it's martingale, there is (by martingale representation) a φ_t such that $dF_t = \varphi_t dD_t$, where D is the discounted price of the underlying. This φ_t is the derivative's delta.

(iii) It follows from the above that if we hold φ_t in the underlying asset and $\psi_t = F_t - \varphi_t D_t$ in the bond, then the discounted value of our holding is F_t.

The holding is self-financing, since $dV(t) = d(e^{rt}F_t) = re^{rt}\, F_t dt + e^{rt}\, dF_t$
$= re^{rt}F_t dt + e^{rt}\, \varphi_t dS_t + re^{rt}\,(F_t - \varphi_t D_t)\, dt = \varphi_t dS_t + \psi_t\, dB_t$.
The final discounted value of our holding is F_T, and so we have hedged the derivative with terminal value of V_T.

4 (i) In the Wilkie model, the force of inflation, $I(t)$, over the period $t - 1$ to t is an autoregressive model of order 1, $AR1$: $I_{t+1} = (1 - \alpha)\, m + \alpha I_t + e_t$, where the e_t are iid normal errors.

It follows that it is mean-reverting and the longitudinal distribution from the 1,000 year simulation will converge to stationarity.

Consequently we will get the unconstrained s.d., whereas the s.d. from repeated one year simulations (cross-sectional) will depend strongly on initial conditions.

(ii) In a pure random walk environment, the force of inflation would be independent across the years and (as for any model) across simulations. As a result, cross-sectional and longitudinal quantities would coincide. This happens if $\alpha = 0$.

(iii) In a statistical model, the model structure is derived from past time series, together with some intuition regarding what model formulae look reasonable. However, these statistical models can produce some odd results. It can be useful to impose additional economic constraints on model behaviour. The advantage of using more economic theory is that it gives us a more concrete way of interpreting model output. For example, if we model a market which is broadly governed by rational pricing rules, we can apply those same pricing rules to simulated output from a model. This gives us a market-based way of comparing strategies, and deciding which strategy is most valuable. The difficulty with this approach is that the model's optimal strategy may not be the strategy that managers wish to follow. In this context, a more flexible judgmental approach may better meet the client's needs.

5 (i) Arbitrage free
Positive rates
Instantaneous and other rates mean reverting
Ease of computation/pricing of derivatives and bonds
Realistic dynamics/yield curves
Historical fit (with suitable parameter values)
Ease of calibration
Flexibility (to cope with range of derivatives)

(ii) The stochastic differential equation for the short rate r is:

$$dr_t = \sigma dB_t + \alpha(\mu - r_t)\, dt.$$

(iii) Arbitrage free - yes
Positive rates - no
Instantaneous and other rates mean reverting - yes
Ease of computation/pricing of derivatives and bonds - yes
Realistic dynamics/yield curves - no
Historical fit (with suitable parameter values) - no
Ease of calibration - no
Flexibility (to cope with range of derivatives) - no
-not very good as a model.

6 (i) Since the efficient frontier consists of pairs of points (in (s.d., return) coordinates) such that no higher return is available for the same or lower s.d. we see that to get a return of r, greater than or equal to .05, we need a portfolio of $((r - .05)/.05,\ 1 - ((r - .05)/.05) = (20r - 1,\ 2 - 20r)$, this portfolio has a standard deviation $0.2\ (20r - 1)$, hence the efficient frontier is the straight line $(4r - 0.2,\ r)$ which does indeed pass through the two specified points.

(ii) A portfolio with x invested in A and $(1 - x)$ invested in C has an expected return of $.06 + .04x$ and s.d. of $\sqrt{(.04x^2 + .01(1 - x)^2)}$. Thus we seek x to maximize $(.01 + .04x) / \sqrt{(.04x^2 + .01(1 - x)^2)}$. Taking logs and differentiating we see (after a lot of algebra) that the optimal x is 5/9, so the optimal portfolio is (5/9, 4/9).

(iii) The efficient frontier in the presence of a risk free asset is the tangent to the efficient frontier (without a risk free asset) which passes through the point in (s.d., return)-space corresponding to the risk free asset.

Clearly this is the line through (0, .05) with maximal gradient which passes through some point of the efficient frontier.

Consider the point corresponding to the portfolio in part (ii): it is on the efficient frontier for the pair A and C, and the line from (0, .05) to it has gradient
$(.01 + .04x) / \sqrt{(.04x^2 + .01(1 - x)^2)}$

Hence the new efficient frontier is a straight line which passes through (0, .05) and $(\sqrt{(.04(5/9)^2 + .01(1 - 5/9)^2)}, .06 + .04 \times 5/9)$.

This is the line $y = .05 + .2692x$, which clearly passes through (.1, .076926).

7 (i) The three types are:

Macroeconomic factor models

These use observable economic time series as the factors. They could include factors such as the annual rates of inflation and economic growth, short term interest rates, the yields on long term government bonds, and the yield margin on corporate bonds over government bonds. Once the set of factors has been decided on, a time series regression is performed to determine the sensitivities for each security in the sample.

Fundamental factor models

Fundamental factory models are closely related to macroeconomic models but instead of (or in addition to) macroeconomic variables the factors used are company specific variables. These may include such fundamental factors as:

- the level of gearing
- the price earnings ratio
- the level of R&D spending
- the industry group to which the company belongs

Again, the models are constructed using regression techniques.

Statistical factor models

Statistical factor models do not rely on specifying the factors independently of the historical returns data. Instead a technique called principal components analysis can be used to determine a set of indices which explain as much as possible of the observed variance.

(ii) Denoting the changes in the two indices by I_t and J_t, let $K_t = J_t - c\, I_t$, where $c = \mathrm{Cov}(I_t, J_t)/\mathrm{Var}(I_t)$, then the two factors I and K are orthogonal. We can check: $\mathrm{Cov}(I_t, K_t) = \mathrm{Cov}(I_t, J_t) - c\,\mathrm{Var}(I_t) = 0$. Alternatively, we may regress index J on index I to obtain $J = a + bI + d_2$, and set $K = d_2$, where a is a constant and d_2 is uncorrelated with I.

(iii) Suppose that $R_i = \alpha_i + \beta_{i,1}I + \beta_{i,2}K + \varepsilon_i$, then $\mathrm{Var}(R_i) = \beta_{i,1}^2\,\mathrm{Var}(I) + \beta_{i,2}^2\,\mathrm{Var}(K) + \sigma_i^2$.

(iv) The interpretation is (as in principal components analysis) that we have a decomposition of the variance into the portion explained by the behaviour of the first index, that explained by the second and the residual or unexplained error or variance.

8 (i) Under the Black Scholes assumptions, the unique risk-neutral measure is Q, where, under Q,

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t,$$

with B a standard Brownian motion.

(ii) The unique fair price for a derivative security which pays C at time T is

$$V_0 = E_Q\,[e^{-rT}C].$$

(iii) For the special option, $C = 1$ if S_T is in $[a, b]$, otherwise 0, so

$$V_0 = E_Q\,[e^{-rT}1_{[a,b]}(S_T)]$$

$$= e^{-rT}Q(S_T \text{ in } [a, b])$$

Because B is a Brownian motion $\ln\frac{S_T}{S_0}$ is normally distributed (under Q) with mean $\left(r - \frac{\sigma^2}{2}\right)T$ and standard deviation $\sqrt{T}\sigma$.

Hence $Q(S_T < x) = \Phi(d(x))$ where $d(x) = \dfrac{\ln\frac{x}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sqrt{T}\sigma}$

Hence $V_0 = e^{-rT}\,[\Phi(d(b)) - \Phi(d(a))]$

(iv) The value is .5% of the holding, so is $.005S_0$.

(v) Payoff = $.001S_1 + .004(S_1 - S_0)^+ + .005(S_1 - U)^+$
$+ .004S_0 1_{(S_1 - S_0)} + .005U1_{(S_1 > U)}$

Denoting the prices of the four options in the decomposition immediately above as $c_1, c_2, c_3,$ and c_4:

$$c_1 = S_0\Phi(d_1) - S_0e^{-r}\Phi(d_1 - \sigma) \text{ where } d_1 = \frac{r + ½\sigma^2}{\sigma} = \frac{r}{\sigma} + ½\sigma$$

$$= 100\Phi(0.325) - 100e^{-0.05}\Phi(0.075)$$

$$= 12.33599$$

$$c_2 = S_0\Phi(d_3) - Ue^{-0.05}\,\Phi(d_3 - \sigma) \text{ where } d_3 = \frac{\ln\left(\frac{S_0}{U}\right) + (r + ½\sigma^2)}{\sigma}$$

$$c_3 = 100[e^{-0.05}(1 - \Phi(d_4))] \text{ where } d_4 = \frac{-\left(r - \frac{\sigma^2}{2}\right)}{\sigma} = \frac{-r}{\sigma} + \frac{\sigma}{2}$$

$$= 100[0.5040495] = 50.40495$$

$$c_4 = e^{-0.05}(1 - \Phi(d_5)) \text{ where } d_5 = \frac{\ln\left(\frac{U}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)}{\sigma}$$

$$= \frac{-\ln\left(\frac{S_0}{U}\right) - \left(r - \frac{\sigma^2}{2}\right)}{\sigma}$$

$$= -(d_3 - \sigma)$$

Value = 0.1

$+ 12.33599 \times .004$

$+ [100\Phi(d_3) - Ue^{-0.05}\,\Phi(d_3 - \sigma)] \times 0.005$

$+ 50.40495 \times .004$

$+ U \times e^{-0.05}(1 - \Phi\{-(d_3 - \sigma)\}) \times .005$

$$\text{Value} = 0.1 + 12.33599 \times 0.04 + 50.40495 \times .004$$

$$+ 0.5\Phi(d_3) - Ue^{-0.05}\,\Phi(d_3 - \sigma)\,0.005$$

$$+ Ue^{-0.05} \times 0.005 \times (1 - (1 - \Phi(d_3 - \sigma)))$$

$$= 0.35096376 + 0.5\Phi(d_3) = 0.5$$

$$\Rightarrow d_3 = \Phi^{-1}\left(1 - \frac{0.35096376}{0.5}\right)$$

$$= -0.52995$$

$$\Rightarrow \ln\frac{100}{U} = -0.52995 \times 0.25 - 0.05 - \tfrac{1}{2}0.25^2$$

$$\Rightarrow U = 123.83$$

END OF EXAMINERS' REPORT

EXAMINATION

13 April 2007 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

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4. *Attempt all 9 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

1 (i) Derive an expression for the theta of an option under the Black-Scholes model involving delta and gamma. [4]

(ii) Explain why a deep out of the money call option in the Black-Scholes world will experience a rate of return close to the risk-free rate of return. [2]
[Total 6]

2 Consider a set of risky assets in a mean-variance framework where:

V_i = variance of the return for asset i
C_{ij} = covariance between the returns of assets i and j where $i \neq j$

(i) Derive an expression for the variance of a portfolio of N such assets where x_i is the relative weight of asset i in the portfolio. Assume that the weights sum to unity and that short selling is prohibited. [3]

(ii) Show that the variance of the returns of a very large portfolio of equally weighted allocations to the assets depends mainly on the average covariance between the asset returns. [5]
[Total 8]

3 Consider a two period recombining binomial model for S_t, the price of a non-dividend paying security at times $t = 0$, 1 and 2, with real world dynamics:

$$\begin{aligned} S_{t+1} &= S_t\, u \quad \text{with probability } p \\ &= S_t\, d \quad \text{with probability } 1-p \end{aligned}$$

and $u > d > 0$.

There also exists a risk-free instrument that offers a continuously compounded rate of return of 5% per period.

The state price deflator in this model after one period is:

$$\begin{aligned} A_1 &= 0.7610 \quad \text{when } S_1 = S_0 u \\ &= 1.5220 \quad \text{when } S_1 = S_0 d \end{aligned}$$

The price of a derivative at time 0 that pays 1 at time 2 if $S_2 < S_0$ is 0.1448.

(i) Calculate the value of p. [5]

(ii) Calculate the risk-neutral probability measure. [2]

(iii) Calculate the price at time 0 of a derivative that pays 1 at time 2 if $S_2 > S_0$ using the risk-neutral probability measure derived in (ii). [2]
[Total 9]

4 An investor is contemplating an investment with a payoff of R, where R has the probability density function f, given by:

$$f(t) = 0 : t < 0.5$$
$$f(t) = c/t^4 : t \geq 0.5,$$

for $c = 0.375$. All amounts are in units of £ million.

Calculate the following two measures of risk for the net return when the cost of the investment is 0.7:

(a) downside semi-variance of return
(b) Value at Risk at the 5% level

[9]

5 Consider a single period multifactor model of security returns where:

$$R_i = \alpha_i + \sum_{j=1}^{K} \beta_{ij} I_j + \varepsilon_i$$

where:

R_i = return on security i

α_i, β_{ij} are security specific parameters

ε_i = cross-sectionally independent random component of return that is also independent of all I_j

I_j = cross-sectionally independent rate of change in factor j

(i) Derive an expression for the covariance between the returns of two securities in terms of the statistical properties of the factors using the model above. [4]

(ii) Explain the implications of your expression in (i) for constructing a diversified portfolio. [3]

(iii) Explain how the multifactor model above can be used to form an asset pricing theory when combined with the principle of no-arbitrage. [5]

[Total 12]

6 (i) State the SDEs under the risk-neutral measure for $r(t)$, the default-free instantaneous rate of interest at time t, under the following two models, defining all notation used:

(a) Hull & White
(b) 2-factor Vasicek

[4]

(ii) State the advantages of the Hull & White model over the single factor Vasicek model. [4]

(iii) Explain the limitations of using a model with only one factor, taking into account both theoretical and empirical considerations. [4]
[Total 12]

7 Consider a special type of scheme that has an obligation to pay £10,000 in exactly one year. There is currently £9,000 in the fund created to meet this obligation and it is invested in the shares of an infinitely divisible non-dividend paying security with price S_t governed by the SDE:

$$dS_t = S_t\left(\mu\, dt + \sigma\, dZ_t\right)$$

where:

Z_t is a standard Brownian motion
$\mu = 0.10$
$\sigma = 0.20$
t is the time since the start of the year
$S_0 = 1$
a risk free rate of return of 5% p.a. compounded continuously is available.

(i) Derive the distribution of S_1. [4]

(ii) Calculate the following risk measures applied to the surplus of the scheme where the surplus of the scheme is defined as the difference between the value of the fund and the obligation at the end of the year:

(a) variance
(b) shortfall probability relative to a benchmark surplus of £0

[6]

(iii) Calculate the cost of a put option to protect against the surplus being negative at the end of the year. [3]
[Total 13]

8 (i) Explain the difference between an efficient market and an arbitrage-free market. [4]

Empirical investigations of stock market returns have revealed a fractal dimension of 1.4.

(ii) Explain what this means about the distribution of returns. [2]

(iii) Explain how mean-reversion in the stock market can be consistent with an efficient market. [2]

(iv) Outline the claim and test of excessive volatility in stock markets made by Shiller, along with four criticisms made of the test. [7]
[Total 15]

9 (i) Describe three types of credit risk model. [6]

Assume that a company has fixed debt of £40m with term 10 years, the value of the equity in the company is £20m and the Merton model for credit risk holds true. The risk free rate of return is 5% p.a. and there are no other dividends or interest payments.

(ii) Explain how to calculate the (risk neutral) probability of default. You do not have to calculate the probability, but should state how each value would be calculated. [6]

In a particular two state model for credit rating with deterministic transition intensity, the risk free rate is a constant, r, the recovery rate is δ and the zero coupon bond price is given by:

$$B(t,T) = e^{-r(T-t)}\left[1-(1-\delta)\left(1-e^{-\frac{(T^2-t^2)}{4}}\right)\right].$$

(iii) (a) State the general formula for the zero coupon bond prices in a two state model for credit ratings.

(b) Deduce the risk-neutral default intensity for the particular two state model above.

[4]
[Total 16]

END OF PAPER

EXAMINATION

April 2007

Subject CT8 — Financial Economics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

June 2007

Comments

No comments given.

1 (i) $\Theta = \dfrac{\partial f}{\partial t}$

From the Black-Scholes PDE we have

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} = rf$$

or $\Theta + rs\Delta + \dfrac{1}{2}\sigma^2 s^2 \Gamma = rf$

(ii) Deep out of the money, delta and gamma will be close to zero which implies that theta will equal the risk free rate of return.

2 (i) The expression for the variance of the portfolio can be rewritten as:

$$V = \Sigma_i \; x_i^2 \, V_i + \sum_{i \neq j} x_i . x_j . C_{ij}$$

(ii) If we assume that equal amounts are invested in each asset, then with N assets the proportion invested in each is $1/N$. Thus:

$$V = \Sigma_i \; (1/N)^2 \, V_i + \sum_{i \neq j} (1/N)(1/N).C_{ij}$$

Factoring out $1/N$ from the first summation and $(N - 1)/N$ from the second yields:

$$V = 1/N \, \Sigma_i \, V_i /N + (N - 1)/N \sum_{i \neq j} C_{ij}/N(N - 1)$$

Replacing the summation by averages we have

$$V = 1/N \, V_i + (N - 1)/N. \overline{C}_{ij}$$

The contribution to the portfolio variance of the variances of the individual securities goes to zero as N gets very large. However, the contribution of the covariance terms approaches the average covariance as N gets large. The individual risk of securities can be diversified away, but the contribution to the total risk caused by the covariance terms cannot be diversified away.

3 (i) Derivative pays off after two down moves

Derivative price = 0.1448 = $A(2,dd) \times (1 - p)^2 \times 1$
$A(2,dd) = A(1,d)^2 = 2.316$
Hence $p = 0.75$

(ii) $A(1,u) = 0.7610 = A(0) \times \exp(-0.05) \times q/p$
Knowing p, we can get $q = 0.6$

The alternative approach shown below is possible for (i) and (ii), students were given full credit for either approach.

(i) and (ii) p and q, the risk neutral probability measure, can be obtained by solving the equation for the state price deflator (Unit 11 Page 9 of Core Reading)

$$A_1 = e^{-r}q/p \quad \text{if } S_1 = S_0u$$
$$= e^{-r}(1 - q)/(1 - p) \quad \text{if } S_1 = S_0d$$

This gives $p = 0.75$ and $q = 0.6$.

Note if this approach is used it is not necessary to know that the price of the derivative is 0.1448.

(iii) Price = $0.6^2 \times \exp(-.05 \times 2) \times 1 = 0.3257$

The solution to (iii) given above assumes that $ud = 1$, students who worked on this basis were given full credit as this is a common presumption in this type of work. However, some students realized that the strict definition of a recombining model is that $ud = du$. In this case

If $ud = 1$ the solution given above holds i.e. price = 0.3257 (only upper node pays off)

There is another possible case where $ud > 1$ (upper and middle nodes both pay off)

In this case price $= 0.6^2 e^{-0.05\times 2} + 2 \times 0.6 \times 0.4 e^{-0.05\times 2}$
$= 0.3257 + 0.4344$
$= 0.7601$

4 (a) SV = Downside semi-variance = $\int_{0.5}^{\mu}(t-\mu)^2 f(t)dt$

$$\mu = \int_{0.5}^{\infty} tf(t)dt = -\frac{c}{2t^2}\Bigg]_{0.5}^{\infty} = 0.75$$

$$SV = c\left\{\int_{0.5}^{0.75} t^{-2}dt - 2\mu\int_{0.5}^{0.75} t^{-3}dt + \mu^2\int_{0.5}^{0.75} t^{-4}dt\right\}$$

$$= c\left\{-t\Big]_{0.5}^{0.75} - 1.5\left[-\frac{t^{-2}}{2}\right]_{0.5}^{0.75} + 0.75^2\left[-\frac{t^{-3}}{3}\right]_{0.5}^{0.75}\right\}$$

$= 0.02083$
(in units of $(£m)^2$)

<u>OR</u>

$$SV = \int_{0.5}^{\infty}\frac{(t-\mu)^2}{t^4}dt - \int_{0.75}^{\infty}\frac{(t-\mu)^2}{t^4}dt$$

then, using the same integration steps as above,

$SV = 0.1875 - 0.16666$

$= 0.02083$

(b) $P(R > x) = \int_{x}^{\infty} f(t)dt = \frac{c}{3}x^{-3}$

$Pr[S \le -t] = Pr[R \le 0.7 - t]$

$= 1 - Pr[R > 0.7 - t]$

$= 1 - \frac{c}{3}(0.7-t)^{-3}$

For 5% VaR:

$Pr[S \le -t] = 1 - \frac{c}{3}(0.7 - t)^{-3} = 0.05$

$\Rightarrow t = 0.1914$

5 (i)

$$C_{ij} = \sum_k \sum_l \beta_{ik}\beta_{jl}\text{Cov}(I_k, I_l)$$

$$+\sum_k \beta_{ik}\text{Cov}(I_k, \varepsilon_j)$$

$$+\sum_k \beta_{jk}\text{Cov}(I_k, \varepsilon_i)$$

$= \sum_k \beta_{ik}\beta_{jk}\text{Var}(I_k)$ because of independence of all the other terms.

(ii) low covariance if betas are low, i.e. pick stocks with different sensitivities to the factors

(iii) This is exactly the same as the multi-index model for returns on individual securities. The contribution of APT is to describe how we can go from a multi-index model for individual security returns to a equilibrium market model. Non-mathematically, the argument can be made as follows. Consider a two index model. The return on the *i*th security is given by

$$R_i = a_i + b_{i,1} I_1 + b_{i,2} I_2 + c_i .$$

For investors who hold well diversified portfolios the specific risk of each security, represented by c_i can be diversified away so an investor need only be concerned with expected return, $b_{i,1}$ and $b_{i,2}$ in choosing his portfolio. Suppose we hypothesize the existence of three widely diversified portfolios, represented by the points $(E_i, b_{i,1}, b_{i,2})$ in $E - b_1 - b_2$ space where $i =$ 1, 2, 3. These three portfolios define a plane in $E - b_1 - b_2$ space with equation

$$E[R_i] = \lambda_0 + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} .$$

A portfolio having any combination of b_1 and b_2 can be formed by combining portfolios 1, 2 and 3 in the correct proportions. For example the portfolio *P*, obtained by taking one-third each of each of 1, 2 and 3 would have

$$b_{P,1} = (b_{1,1} + b_{2,1} + b_{3,1})/3 ,$$

$$b_{P,2} = (b_{1,2} + b_{2,2} + b_{3,2})/3 ,$$

and $\quad E[R_P] = \lambda_0 + \lambda_1 b_{P,1} + \lambda_2 b_{P,2} .$

Now, consider what would happen if another portfolio *Q* existed, with exactly the same values of b_1 and b_2 but a higher expected return. Both portfolios would have the same degree of systematic risk but *Q* would have a higher expected return than *P*. Rational investors would therefore sell *P* and buy *Q*, and this would continue until the forces of supply and demand had brought portfolio Q onto the same plane as portfolios 1, 2 and 3. Thus, in equilibrium, all securities and portfolios must lie on a plane in $E - b_1 - b_2$ space.

The more general result of APT, that all securities and portfolios have expected returns described by the L-dimensional hyperplane

$$E_i = \lambda_0 + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} + \ldots + \lambda_L b_{i,L},$$

can be derived by a more rigorous mathematical argument.

6 (i) The Hull & White (HW) model does this by extending the Vasicek model in a simple way. We define the SDE for $r(t)$ under Q as follows

$$dr(t) = \alpha(\mu(t) - r(t))dt + \sigma d\tilde{W}(t)$$

where $\mu(t)$ is a deterministic function of t. $\mu(t)$ has the natural interpretation of being the local mean-reversion level for $r(t)$.

A simple example of a multifactor model is the 2-factor Vasicek model. This models two processes: $r(t)$, as before, and $m(t)$, the local mean-reversion level for $r(t)$. Thus

$$dr(t) = \alpha_r(m(t) - r(t))dt + \sigma_{r1} d\tilde{W}_1(t) + \sigma_{r2} d\tilde{W}_2(t)$$

$$dm(t) = \alpha_m(\mu - m(t))dt + \sigma_{m1} d\tilde{W}_1(t)$$

where $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ are independent, standard Brownian motions under the risk-neutral measure Q. This looks superficially like the Hull & White model, but the HW model has a deterministic mean-reversion level, whereas here $m(t)$ is stochastic.

(ii) We will now look at a simple extension of the Vasicek model. Recall the SDEs for both the Vasicek and CIR models gave us time-homogeneous models. This means that bond prices at t depend only on $r(t)$ and on the term to maturity. This results in a lack of flexibility when it comes to pricing related contracts. For example, on any given date theoretical bond prices will probably not match exactly observed market prices. We can re-estimate $r(t)$ to improve the match and even re-estimate the constant parameters α, μ and σ but we will still, normally, be unable to get a precise match.

A simple way to get theoretical prices to match observed market prices is to introduce some elements of *time-inhomogeneity* into the model. The Hull & White (HW) model does this by extending the Vasicek model in a simple way.

(iii) One factor models have certain limitations which it is important to be familiar with. First, if we look at historical interest rate data we can see that changes in the prices of bonds with different terms to maturity are not perfectly correlated as one would expect to see if a one-factor model was correct. Sometimes we even see, for example, that short-dated bonds fall in price while long-dated bonds go up. Recent research has suggested that around three factors, rather than one, are required to capture most of the randomness in bonds of different durations.

Second, if we look at the long run of historical data we find that there have been sustained periods of both high and low interest rates with periods of both high and low volatility. Again these are features which are difficult to capture without introducing more random factors into a model. This issue is especially important for two types of problem in insurance: the pricing and hedging of long-dated insurance contracts with interest-rate guarantees; and asset-liability modelling and long-term risk-management.

Third, we need more complex models to deal effectively with derivative contracts which are more complex than, say, standard European call options. For example, any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated.

7 (i) If we were dealing with an ordinary differential equation, integration would lead to the expression $\mu t + \sigma Z_t$ for $\log(S_t/S_0)$ and thus to $S_0 \exp(\mu t + \sigma Z_t)$ for S_t. To solve the problem within stochastic calculus, use Itô's Lemma to calculate $d \log S_t$:

$$d \log S_t = \frac{1}{S_t} dS_t + \tfrac{1}{2}\left(-\frac{1}{S_t^2}\right)\left(dS_t\right)^2$$

$$= \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma\, dZ_t\,.$$

Written in integral form, this reads

$$\log S_t = \log S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma Z_t$$

or, finally,

$$S_t = S_0 \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma Z_t\right].$$

We see that the process S satisfying the equation above is a geometric Brownian motion with parameter $\mu - \tfrac{1}{2}\sigma^2$. Since $\log S_t$ is normally distributed, it follows that S_t has a lognormal distribution with parameters $\left(\mu - \tfrac{1}{2}\sigma^2\right) t$ and $\sigma^2 t$.

Should insert the parameters given in the question, i.e. $\mu = 0.1$ and $\sigma = 0.2$.

(ii) The properties of the lognormal distribution give us the expectation and variance of S_t:

$$E(S_t) = \exp\left(\left(\mu - ½\sigma^2\right)t + ½\sigma^2 t\right) = e^{\mu t},$$

$$\mathrm{Var}(S_t) = e^{2\mu t}(e^{\sigma^2 t} - 1)$$

Need to look at

$X = 9000\ S_1 - 10{,}000$

which will also have a lognormal distribution

(a) $0.049846 \times 9{,}000^2 = 4{,}037{,}526$

(b) $\Pr[X < 0] = \Pr[S_1 < 10/9] = \Pr[\log S_1 < \log 10/9]$, then use normal distribution to get 0.55

(iii) 1021.42 based on Black Scholes

$S = 9{,}000$, $K = 10{,}000$, other parameters as in question

$d_1 = -0.1768$, $d_2 = 0.3768$, use Black Scholes to get price of call option as 509.12.

Use put call parity (or Black Scholes formula for put option directly) to get price of put = price of call + 10,000*exp(-0.05) - 9,000 = 1021.42.

8 (i) Attempts to explain this phenomenon gave rise to the *efficient markets hypothesis*, which claims that market prices already incorporate the relevant information. The market price mechanism is such that the trading pattern of a small number of informed analysts can have a large impact on the market price. Lazy (or cost conscious) investors can then take a free ride, in the knowledge that the research of others is keeping the market efficient.

If we assume that there are no arbitrage opportunities in a market, then it follows that any two securities or combinations of securities that give exactly the same payments must have the same price. This is sometimes called the "Law of One Price".

Arbitrage-free markets can be inefficient.

(ii) One measure of these non-normal features is the *Hausdorff fractal dimension* of the price process. A pure jump process (such as a Poisson process) has a fractal dimension of 1. Random walks have a fractal dimension of 1½. Empirical investigations of market returns often reveal a fractal dimension around 1.4.

(iii) Even mean reversion can be consistent with efficient markets. After a crash, many investors may have lost a significant proportion of their total wealth; it is not irrational for them to be more averse to the risk of losing what remains. As a result, the prospective equity risk premium could be expected to rise.

(iv) Several observers have commented that stock prices are "excessively volatile". By this they mean that the change in market value of stocks (observed volatility), could not be justified by the news arriving. This was claimed to be evidence of market over-reaction which was not compatible with efficiency.

The claim of "excessive volatility" was first formulated into a testable proposition by Shiller in 1981. He considered a discounted cashflow model of equities going back to 1870. By using the actual dividends that were paid and some terminal value for the stock he was able to calculate the perfect foresight price, the "correct equity" price if market participants had been able to predict future dividends correctly. The difference between the perfect foresight price and the actual price arise from the forecast errors of future dividends. If market participants are rational we would expect no systematic forecast errors. Also if markets are efficient broad movements in the perfect foresight price should be correlated with moves in the actual price as both react to the same news.

Shiller found strong evidence that the observed level of volatility contradicted the EMH. However, subsequent studies using different formulations of the problem found that the violation of the EMH only had borderline statistical significance. Numerous criticisms were subsequently made of Shiller's methodology, these criticisms covered

- the choice of terminal value for the stock price
- the use of a constant discount rate
- bias in estimates of the variances because of autocorrelation
- possible non-stationarity of the series, i.e. the series may have stochastic trends which invalidate the measurements obtained for the variance of the stock price

Although subsequent studies by many authors have attempted to overcome the shortcomings in Shiller's original work there still remains the problem that a model for dividends and distributional assumptions are required. Some equilibrium models now exist which calibrate both to observed price volatility and also observed dividend behaviour. However, the vast literature on volatility tests can at best be described as inconclusive.

9 (i) The three types of credit risk model are:

structural models: these are explicit models of a corporate entity issuing both debt and equity. They aim to link default events explicitly to the fortunes of the issuer.

reduced-form models: these are statistical models which use market statistics (such as credit ratings) rather than specific data relating to the issuer, a nd give statistical models for their movement.

intensity-based models: these model the factors influencing the credit events which lead to default and typically do not consider what triggers these events.

(ii) In the Merton model, the company is modelled as having a fixed debt, 40 with term 10 years and variable assets S_t. The equity holders can be regarded as holding a European call on the assets with a strike of 40.

In the current question the value of the option is 20.

Using Black Scholes formula, with $(T - t) = 10$, $K = 40$, $S_0 = 60$, $r = 0.05$, solve for σ, the implied volatility.
[Candidates need not actually do this calculation]

The assets of the company therefore follow a geometric Brownian motion under the risk neutral measure with drift $r = 0.05$ and volatility σ.

Therefore $\log(S_{10}/ S_0)$ follows a normal distribution with mean $10*(0.05 – \sigma_2/2)$ and variance $10* \sigma^2$.

The risk neutral probability of default is obtained by calculating the probability that $\log(S_{10}/ S_0)$ is less than $\log(40/60)$.

(iii) In the two state model for credit rating with deterministic transition intensity, the formula for the Zero Coupon Bond price is

$$B(t, T) = e^{-r(T-t)} (1 - (1 - \delta) (1 - e^{-\int_t^T \tilde{\lambda}(s)ds})).$$

It follows that the risk-neutral default intensity is given by

$$\tilde{\lambda}(s) = s/2.$$

END OF EXAMINERS' REPORT

EXAMINATION

26 September 2007 (am)

Subject CT8 — Financial Economics Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 8 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

1 (i) Define the Δ, Γ, θ, λ, ρ and ν for an individual derivative security. [6]

(ii) Explain put-call parity and use it to calculate the delta and gamma of a European call option on a non-dividend paying stock with the same strike and maturity as the put option. [4]

A portfolio with a delta of zero consists of cash, European put options, P, on a stock and 1 million shares of the underlying (non-dividend paying) stock . The delta of a single put option is –0.212, while the gamma is 0.377.

Two further derivatives on the stock are also traded: a European call option, C, with the same strike and maturity as P and another derivative security, D, with a delta of 0.222 and a gamma of 0.111.

(iii) Calculate n, the number of put options in the original portfolio of cash, stock and P. [2]

(iv) Calculate the numbers of derivatives D and C that have to be purchased and added to the portfolio so that both the delta and gamma of the expanded portfolio are zero. [4]

[Total 16]

2 A binomial model for a non-dividend-paying security with price S_t at time t is as follows: the price at time $(t + 1)$ is either $1.25S_t$ (up-jump) or $0.8S_t$ (down-jump). Cash receives interest of 10% per time unit.

(i) Calculate the risk neutral probability measure for this model. [3]

The value of S_0 is 100. A derivative security with price D_t at time t pays the following returns at time 2:

$D_2 = 1$: if $S_2 = 156.25$
$D_2 = 2$: if $S_2 = 100$
$D_2 = 0$: if $S_2 = 64$.

(ii) Determine D_1 when $S_1 = 125$ and when $S_1 = 80$ and hence calculate the value of D_0. [5]

(iii) Derive the corresponding hedging strategy, i.e. the combination of the underlying security and the risk free asset required to hedge an investment in the derivative security. [4]

(iv) Comment on your answer to (iii) in the light of your answer to part (ii). [1]

[Total 13]

3 (i) State the assumptions underlying the Black-Scholes option pricing formula and discuss how realistic they are. [6]

A discounted stock price can be written as:

$$\tilde{S}_t = \cosh(\sigma Z_t)\exp(-\sigma^2 t),$$

where Z_t is a standard Brownian motion under the real world measure $\mathbb{P}$.

Hint: $\cosh(x) = (e^x + e^{-x})/2$.

(ii) Apply Ito's formula to derive an SDE satisfied by $\tilde{S}_t$. [5]

(iii) Explain why the discounted stock price (under $\mathbb{P}$) is not a martingale. [1]

(iv) State the SDE satisfied by $\tilde{S}_t$ under the equivalent martingale measure. [2]

[Total 14]

4 (i) State the general zero-coupon bond pricing formula in terms of random short rates. Define all notation used. [2]

(ii) State the SDEs defining the dynamics of the short rates under the risk-neutral measure for each of the Vasicek, Cox-Ingersoll-Ross and Hull & White models. Define all notation used. [6]

(iii) State the zero-coupon bond price formula for the Vasicek model. [3]

(iv) Comment on the limitations of one-factor interest rate models. [3]

[Total 14]

5 Consider a special company that has just issued a zero-coupon bond of nominal value £10m with maturity 10 years. The value of the assets of the company is £20m and this value is expected to grow at an average of 10% per annum compound with an annual volatility of 40%. The company is expected to be wound up after 10 years when the assets will be used to pay off the bond holders with the remainder being distributed to the equity holders.

A constant risk-free rate of return of 5% p.a. compounded continuously is available in the market.

Calculate the credit spread on the debt for the zero-coupon bond. State any additional assumptions that you make.

Hint: use the Merton model. [8]

6 A market consists of three assets A, B and C. Annual returns on the three assets (R_A, R_B and R_C) have the following characteristics:

Asset	*Expected return %*	*Standard deviation %*
A	9	20
B	6	20
C	3	10

The correlation between the returns are as follows: $Corr(R_A, R_B) = -¼$, $Corr(R_B, R_C) = -½$ and $Corr(R_A, R_C) = -½$.

(i) Calculate the variance of the returns of each asset and the covariances between the returns of each pair of assets. [5]

(ii) Define an efficient portfolio for the corresponding mean-variance portfolio model. [2]

Efficient portfolios for this model are of the form $(\frac{2}{9}, \frac{2}{9}, \frac{5}{9})^T + c(4, 1, -5)^T$, for a suitable choice of c, where the vector represents proportions of the investor's capital invested in assets A, B and C respectively.

(iii) (a) Determine an expression in terms of c for the variance of an efficient portfolio.

(b) Deduce the global minimum variance and the portfolio that attains it. [4]

Assume that the risk-free rate of interest is 4% p.a.

(iv) (a) Determine the tangent that passes through (0%, 4%) to the original efficient frontier in (standard deviation, mean return) space.

(b) Deduce the market capitalisations of the three assets consistent with the Capital Asset Pricing Model if the total market capitalisation is £180 bn. [6]

[Total 17]

7 Consider a corporate bond that will return £1 per bond to an investor at the end of a year provided the borrower does not default during the year. The constant annual probability of default is 4%.

Investor 1 holds one thousand such bonds that depend on the same borrower.

Investor 2 holds one thousand such bonds, each of which depends on a different borrower. Each borrower defaults (or not) independently of the other borrowers, but with the same probability of 4%.

All bonds were purchased at par.

(i) For each investor calculate (using suitable approximations if necessary):

(a) the variance of the investment
(b) the 95% value at risk
(c) the 90% value at risk
(d) the probability of shortfall relative to a target level of shortfall of 0 [10]

(ii) Comment on your answers to (i). [3]
[Total 13]

8 (i) Outline the three forms of the Efficient Markets Hypothesis. [3]

(ii) State two reasons why it is hard to test whether any of the three forms hold in practice. [2]
[Total 5]

END OF PAPER

EXAMINATIONS

September 2007

Subject CT8 — Financial Economics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

December 2007

1 (i) The Greeks are the derivatives of the price of a derivative security with respect to the different parameters needed to calculate the price. Thus:

$$\Delta = \frac{\partial f}{\partial s}; \Gamma = \frac{\partial^2 f}{\partial s^2}; \theta = \frac{\partial f}{\partial t}; \lambda = \frac{\partial f}{\partial q}; \rho = \frac{\partial f}{\partial r}; \nu = \frac{\partial f}{\partial \sigma};$$

where f is the value of the derivative, s is the price of the underlying security, q is the continuous dividend yield on the security, σ is the volatility, r is the interest rate and t is time. In each case the relevant Greek measures sensitivity (rate of change) of the option price to change in that variable.

(ii) Suppose we hold one European call option with strike k and maturity T and are short one put with the same strike and maturity. Our payoff at maturity is exactly the same as if we held one share of the underlying and were short k zero coupon bonds with maturity T. The value at time 0 of such a payoff, under the assumption of no arbitrage, is $S_0 - ke^{-rT}$, where S_t is the price of the stock and r is the risk-free interest rate. Thus is P and C are the price at time 0 of the put and call, we have:

$$C = P + S_0 - ke^{-rT}$$

Hence $\Delta_C = \Delta_P + 1$ and $\Gamma_C = \Gamma_P$.

(iii) Since the original portfolio is delta-hedged, and the delta of a share is 1, we must have

$$-.212n + 1M = 0 \Rightarrow n = 4716981.$$

(iv) Using the formulae in part (ii), the delta and gamma of the call are .788 and .377 respectively. The Γ of the original portfolio is

$$1M \times .377 = 377{,}000,$$

so we need:

$$.222d + .788c = 0$$

and

$$.111d + .377c + .377 \times 4716981 = 0.$$

Solving these two simultaneous equations, we get

$$c = 104, 605, 990, d = -371, 304, 146.$$

2 (i) We need to equate the expected average return on the stock and the return on a bond: so we solve solve $1.25p + 0.8q = 1.1 \Rightarrow p = \frac{2}{3}$. Thus, under Q, the risk-neutral measure, S is a multiplicative random walk with up-jump probability $\frac{2}{3}$.

(ii) The fair price at time t for a claim of X payable at time T is $E_Q[e^{-r(T-t)}X \mid \mathcal{F}_t]$, so, if $S_1 = 125$,

$$D_1 = (p \times 1 + q \times 2)/1.1 = 4/3.3 = 1.2121,$$

while if $S_1 = 80$ the fair price

$$D_1 = (p \times 2 + q \times 0)/1.1 = 4/3.3 = 1.2121.$$

Hence, the value $D_0 = 1.2121/1.1 = 1.1019$.

(iii) to hedge at time 1 if $S_1 = 125$ we let the amount invested in the stock be ϕ and the amount invested in cash be ψ and solve:

$$1.25\phi + 1.1\psi = 1$$

$$0.8\phi + 1.1\psi = 2$$

which gives

$\phi = \dfrac{20}{9} = -2.22222$ (the equivalent shareholding is $-2.2222/125 = -.017778$)
and $\psi = \dfrac{340}{9} = 3.4343$.

If $S_1 = 80$ then we solve

$$1.25\phi + 1.1\psi = 2$$

$$0.8\phi + 1.1\psi = 0$$

which gives

$\phi = \dfrac{40}{9} = 4.4444$ (the equivalent shareholding is $4.4444/80 = .0556$),
and $\psi = -\dfrac{320}{99} = -3.2323$.

Finally, we solve

$$1.25\phi + 1.1\psi = 1.2121$$

$$0.8\phi + 1.1\psi = 1.2121$$

which gives

$$\phi = 0 \text{ and } \psi = \frac{1.2121}{11} = 1.1019.$$

(iv) This last is obvious since the value of D_1 doesn't depend on S_1 and hence we must hedge using the risk-free asset only.

3 (i) No frictions; short-selling permitted; small investor (i.e. does not "move market"); market is arbitrage-free; stock price is given by $dS_t = \mu_t S_t dt + \sigma S_t dB_t$ for some process μ_t, where B is a Brownian motion.

All are, in some sense, implausible. Friction (spreads and commission) is present; short-selling is available but on very different terms; "small investor" not true for an investment bank; stock-market returns are not compatible with normality (fat tails); arbitrages occur (for short periods).

(ii) (a) Since

$$\tilde{S}_t = \cosh(\sigma Z_t)\exp(-\sigma^2 t),$$

$\tilde{S}$ can be written as

$$\tilde{S}_t = f(Z_t, t),$$

where

$$f(x, t) = \cosh(\sigma x)e^{-\sigma^2 t}.$$

Applying Ito's lemma we get that

$$d\tilde{S}_t = f_x dZ_t + (f_t + \frac{1}{2} f_{xx})dt$$

$$= \sigma \sinh(\sigma Z_t)e^{-\sigma^2 t} dZ_t + (\frac{1}{2}\sigma^2 \cosh(\sigma Z_t) - \sigma^2 \cosh(\sigma Z_t))e^{-\sigma^2 t} dt$$

$$= \sigma e^{-\sigma^2 t} \sinh(\sigma Z_t) dZ_t - \frac{1}{2}\sigma^2 e^{-\sigma^2 t} \cosh(\sigma Z_t) dt$$

$$= \sigma\sqrt{e^{-2\sigma^2 t} - \tilde{S}_t^2}\,dZ_t - \frac{1}{2}\sigma^2\tilde{S}_t dt.$$

(b) The drift (the *dt*) term is not 0.

(c) The drift term would be zero under the risk-neutral measure: the *dZ* term would be unchanged, therefore we'd get:

$$d\tilde{S}_t = \sigma\sqrt{e^{-2\sigma^2 t} - \tilde{S}_t^2}\,d\tilde{Z}_t,$$

where $\tilde{Z}$ is a Brownian Motion under the risk-neutral measure.

4 (i) The general (zero coupon) bond pricing formula is

$$B(t, T) = E[\exp(-\int_t^T r_s ds) \mid \mathcal{F}_t],$$

where $B(t, T)$ is the price at time t of the ZCB with maturity T, and r is the random short rate.

(ii) Under the Vasicek model we have

$$dr_t = \alpha(\mu - r_t)\,dt + \sigma dZ_t,$$

where Z is Brownian Motion under the risk-neutral measure, Q, and α, μ and σ are positive constants.

Under the Cox-Ingersoll-Ross (CIR) model,

$$dr_t = \alpha(\mu - r_t)\,dt + \sigma\sqrt{r_t}\,dZ_t.$$

Finally, in the Hull and White model,

$$dr_t = \alpha(\mu_t - r_t)\,dt + \sigma dZ_t,$$

for a deterministic function of t, μ_t.

(iii) The corresponding bond-pricing formula for the Vasicek model is

$$B(t, T) = \exp(a(T - t) - b(T - t)r(t)),$$

where $b(\tau) = \frac{1-e^{-\alpha\tau}}{\alpha}$ and $a(\tau) = (b(\tau) - \tau)\,(\mu - \frac{\sigma^2}{2\alpha^2}) - \frac{\sigma^2}{4\alpha}b(\tau)^2$.

(iv) Historical data suggests that three factors are necessary;

historically, there have been sustained periods of both high and low interest rates with periods of both high and low volatility — features which are difficult to capture without more random factors;

we need more complex models to deal with more complex derivative contracts, for example those which depend on more than one interest rate should allow for these rates to be less than perfectly correlated.

5 Assume no other debt, frictionless markets, perfect information, black scholes type model for the movement of the assets

Use black scholes to calculate the value of a call option based on the value of the assets exceeding a strike of 10 after 10 years. Equal to equity of firm = E = 15.07631

Value of bond is $B = 20 - E = 4.923688$

Solve 10*exp(‑10 rb) = B $\Rightarrow$ rb = $-\frac{1}{10}\ln\frac{8}{10} = 7.085\%$

rb – 5% = credit spread in continuously compounded form = 2.085%

6 (i) $Var(R_A) = Var(R_B) = .2^2 = .04$, $Var(R_C) = .1^2 = .01$.
$Cov(R_A, R_C) = Cov(R_B, R_C) = -.5 \times .2 \times .1 = -.01$,
$Cov(R_A, R_B) = -.25 \times .2 \times .2 = -.01$.

(ii) An efficient portfolio is one with minimum variance of return for the given expected return, or maximum expected return for the given variance.

(iii) (a) The variance is given by

$$\left(\frac{2}{9}+4c\right)^2 \times .04 + \left(\frac{2}{9}+c\right)^2 \times .04 + \left(\frac{5}{9}-5c\right)^2 \times .01$$

$$-2\times.01\left(\frac{2}{9}+4c\right)\left(\frac{2}{9}+c\right) - 2\times.01\left(\frac{2}{9}+c\right)\left(\frac{5}{9}-5c\right) - 2\times.01\left(\frac{2}{9}+4c\right)\left(\frac{5}{9}-5c\right)$$

$= .01/9 + 1.35c^2 = .001111 + 1.35c^2$

(b) Thus the minimum variance portfolio is when $c = 0$ with variance .001111 and portfolio $(\frac{2}{9}, \frac{2}{9}, \frac{5}{9})^T$.

(iv) It follows from previous answers that the efficient frontier consists of points of the form (σ_c, r_c) with $c \geq 0$, $r_c = .05 + .27c$ and $\sigma_c = \sqrt{.001111 + 1.35c^2}$. Now the tangent at the point parametrised by c has gradient

$$g_c = \frac{r'_c}{\sigma'_c} = \frac{.27\sqrt{.001111+1.35c^2}}{1.35c}$$

and this intercepts the vertical axis at the point

$$p = r_c - g_c\sigma_c = .05 + .27c - \frac{.27(.001111+1.35c^2)}{1.35c}.$$

Setting $p = 0.04$ we obtain

$$.04 \times 1.35c = (.05 + .27c)\ 1.35c - .27(.001111 + 1.35c^2).$$

Rearranging this we get

$$c = .27 \times .001111/.0135 = .02222 = \frac{2}{90}.$$

The corresponding portfolio is $(\frac{14}{45}, \frac{11}{45}, \frac{4}{9})^T$. If the CAPM holds then these should be the proportions corresponding to the market capitalisation of the companies so they should be (56bn,44bn,80bn) respectively.

7 (i) Investor 1:

(a) variance = 1000^2*0.04*0.96 = 38,400
(b) value at risk = 0 as there is a less than 5% chance of any loss
(c) value at risk = 0
(d) prob of shortfall = 4%

Investor 2:

(a) variance = 1000*.04*.96 = 38.4

(b) normal approximation has mean 0.04*1000 = 40 and std deviation of 6.20 (sqrt of above).

(c) VaR = 40 + 1.645*6.20 = 50.2

(d) VaR = 40 + 1.282*6.20 = 47.9

(e) probability of shortfall = *c*.100% (mean loss is over 6 standard deviations from 0)

(ii) Investor 2 is clearly holding a more diversified portfolio, but two of four measures of risk would suggest the diversified portfolio was riskier.

Value at risk is highly sensitive to the confidence level chosen with 90% level suggesting investor 2 is riskier than investor 1 and 95% level vice versa.

8 (i) The three forms are:

Strong — stock prices reflect all current information relevant to the stock, including information which is not public.

Semi-strong — stock prices reflect all current, publicly available information relevant to the stock.

Weak — stock prices reflect all information available in the past history of the stock price.

(ii) Tests need to make assumptions (which may be invalid) such as normality of returns or stationarity.

Transaction costs may prevent the exploitation of anomalies, so that the EMH might hold net of transaction costs.

Allowance for risk: the EMH does not preclude higher returns as a reward for risk; however the EMH does not tell us how to price such risks.

END OF EXAMINERS' REPORT

EXAMINATION

16 April 2008 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Define Gamma and Vega for a derivative written on a portfolio of assets in a market where the assumptions underpinning the Black-Scholes model hold. [3]

2 State the stochastic differential equation for geometric Brownian motion and its solution. (No proof is required.) [4]

3 Consider a two-period Binomial model of a stock whose current price $S_0 = 100$. Suppose that:

- over each of the next two periods, the stock price can either move up by 10% or move down by 10%
- the continuously compounded risk-free rate is r = 8% per period

(i) Show that there is no arbitrage in the market. [1]

(ii) Calculate the price of a one-year European call option with a strike price $K = 100$. [4]
[Total 5]

4 (i) Outline the two-state model for credit ratings assuming constant transition intensity. [5]

(ii) State the formula for the zero-coupon bond price in terms of the risk-neutral default rate λ, when this rate is deterministic. [3]
[Total 8]

5 An investor is considering investing in one of two assets. The distribution of returns from each asset is shown below:

Asset 1		*Asset 2*	
Return (%)	*Probability (%)*	*Return (%)*	*Probability (%)*
-1	$8^1/_3$	0	50
11	$91^2/_3$	20	50

(i) Calculate for each asset:

(a) the variance
(b) semi-variance
(c) and shortfall probability

Where necessary assume a benchmark return of 0%. [4]

(ii) Explain which asset an investor with a quadratic utility function would choose. [2]

(iii) State the reasons why variance of return is frequently used as a measure of risk. [3]
[Total 9]

6 (i) Outline the assumptions used in modern portfolio theory regarding investor behaviour that are necessary to specify efficient portfolios. [3]

(ii) An investor can construct a portfolio using only two assets X and Y. The statistical properties of the two assets are shown below:

	X	*Y*
Expected return	12%	8%
Variance of return	30%	15%
Correlation coefficient between assets *X* and *Y*	0.5	

Assuming that the investor cannot borrow to invest:

(a) Determine the composition of the portfolio which will give the investor the highest expected return.

(b) Calculate the composition of the portfolio which will give the investor the minimum variance.

[3]

(iii) Explain and sketch how the investor would choose a utility maximising portfolio. [3]

[Total 9]

7 (i) State the assumptions, additional to those used in modern portfolio theory, that allow the capital asset pricing model (CAPM) to be consistent with an equilibrium model of prices in the whole market. [5]

(ii) Explain why in the CAPM all investors should hold all risky assets in proportion to the market capitalisation of those assets. [2]

In an investment market there are three risky assets available. The table below shows the returns each of the assets will earn in the three possible states of the world and the current market capitalisation of the assets. Assume a risk free rate of return of 4% is available.

States	*Probability*	*Asset 1*	*Asset 2*	*Asset 3*
1	0.4	5%	6%	7%
2	0.1	8%	2%	1%
3	0.5	3%	5%	4%
Market Capitalisation		30,000	50,000	30,000

(iii) Calculate the market price of risk under the CAPM. [4]

[Total 11]

8 Consider a continuous time log-normal model for a security price, S, with parameters μ and σ.

(i) Write down formulae for:

(a) the log-return of the process
(b) the expected value of an investment at a specified future time
(c) the variance of the value of an investment at a specified future time

[3]

(ii) Explain what the model implies about market efficiency. [2]

(iii) Outline the empirical evidence for and against the model. [5]

[Total 10]

9 Consider two call options, which are identical (same maturity, same underlying asset) except for the strike price. Denote by $C(K)$ the price at time 0 of the call option with strike price K. Stating the key arguments required, prove that, if there are no arbitrage opportunities, the following relation holds true, for $K_1 \le K_2$.

$$\forall \lambda \in [0,1] \quad \lambda C(K_1) + (1-\lambda) C(K_2) \ge C(\lambda K_1 + (1-\lambda) K_2)$$ [10]

10 In a situation where the zero-coupon bond market is arbitrage-free and complete, consider the following Vasicek model for the short-rate process:

$$dr(t) = a(b - r(t))dt + \sigma dW_t$$

where $(W_t; t \ge 0)$ is a standard Brownian motion with respect to the risk-neutral probability measure **Q**.

(i) State the general expression $r(t)$ of the solution of this stochastic differential equation. [2]

(ii) Derive an expression for $\int_t^T r(u)du$, where t and T are given.

Hint: consider the stochastic differential equation of $r(u)$, for $u \ge t$.

[6]

(iii) State the distribution of $\int_t^T r(u)du$. [1]

(iv) Derive the price of a zero-coupon bond at time t with maturity $T \ge t$ related to the distribution of $\int_t^T r(u)du$. [6]

[Total 15]

11 A stock is currently priced at €8.20. A writer of 100,000 units of a one year European call option on this stock with an exercise price of €8 has hedged the option with a portfolio of 75,000 shares and a loan. The annual risk-free interest rate (continuously compounded) is 7% and no dividends are payable during the life of the option.

Assume the Black-Scholes pricing formula applies.

(i) (a) Derive an expression for the Delta of the option.
(b) State the value of the Delta in this case.
[4]

(ii) Calculate the implied volatility of the stock to within 0.1% p.a., assuming that it is below 100%. [5]

(iii) Calculate:

(a) the value of the loan
(b) the price of the option [4]

(iv) (a) Calculate the current price of a one year European put option with the same exercise price.

(b) State any assumptions you make in your calculation in (iv)(a).
[3]
[Total 16]

END OF PAPER

Subject CT8 — Financial Economics

EXAMINERS' REPORT

April 2008

Introduction

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M A Stocker
Chairman of the Board of Examiners

June 2008

1 *Gamma*

$$\Gamma = \frac{\partial^2 f}{\partial s^2}$$

Vega

$$\nu = \frac{\partial f}{\partial \sigma}$$

f is the price of the derivative; s is the price of the underlying asset; σ is the volatility of the stochastic process of the price of the underlying

2 Consider the stochastic differential equation

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dB_t .$$

$$\log S_t = \log S_0 + \left(\alpha - \tfrac{1}{2}\sigma^2\right) t + \sigma B_t$$

or, finally,

$$S_t = S_0 \exp\left[\left(\alpha - \tfrac{1}{2}\sigma^2\right) t + \sigma B_t\right].$$

3 (i) There is no arbitrage in the market since

$$d < \exp(r) < u \qquad \text{with } r = 8\%.$$

(ii) To price the call option, we use the risk-neutral pricing formula. The risk-neutral probability of an upward move is

$$q = \frac{\exp(r) - d}{u - d} = 0.9164.$$

The price of the call is determined by a backward procedure:

$$\begin{cases} Cuu = (u^2 S_0 - K)^+ = 21 \\ Cud = (udS_0 - K)^+ = 0 \\ Cdd = (d^2 S_0 - K)^+ = 0 \end{cases} \Rightarrow \begin{cases} C_1(u) = \exp(-r)(qCuu + (1-q)Cud) = 17.7655 \\ C_1(d) = \exp(-r)(qCud + (1-q)Cdd) = 0 \end{cases}$$

$$\Rightarrow \quad C_0 = \exp(-r)(qC_1(u) + (1-q)(C_1(d)) = 15.0292$$

4 (i) **The two-state model for credit ratings with a constant transition intensity**.

A model can be set up, in continuous time, with two states N (not previously defaulted) and D (previously defaulted). Under this simple model it is assumed that the default-free interest rate term structure is deterministic with $r(t) = r$ for all t. If the transition intensity, under the real-world measure P, from N to D at time t is denoted by $\lambda(t)$, this model can be represented as:

$$\boxed{\text{No default, } N} \xrightarrow{\lambda(t)} \boxed{\text{Default, } D}$$

and D is an absorbing state.

(ii) $B(t,T) = e^{-r(T-t)}\,[1 - (1 - \delta)(1 - \exp(-\int_t^T \tilde{\lambda}(s)ds))]$

5 (i) **Mean Return**

Asset 1 $-1 \times 8\tfrac{1}{3}\% + 11 \times 91\tfrac{2}{3}\% = 10\%$

Asset 2 $0 \times 50\% + 20 \times 50\% = 10\%$

Variance of Return

Asset 1 $(10 - (-1))^2 \times 8\tfrac{1}{3}\% + (10 - 11)^2 \times 91\tfrac{2}{3}\% = 11\%\%$

Asset 2 $(10 - 0)^2 \times 50\% + (10 - 20)^2 \times 50\% = 100\%\%$

Semi-Variance of Return

Asset 1 $(10 - (-1))^2 \times 8\tfrac{1}{3}\% = 10.08333\%\%$

Asset 2 $(10 - 0)^2 \times 50\% = 50\%\%$

Shortfall Probability

Asset 1 $8\tfrac{1}{3}\%$

Asset 2 0%

(ii) Both have same expected return. The variance is appropriate risk measure in this case.

=> Choose Asset 1

(iii)

- Mathematically tractable.
- Leads to elegant solutions for optimal portfolios.

- Often a good approximation to the other possible methodologies.
- Gives optimum portfolios if returns are normally distributed or investors have quadratic utility functions

6 (i) Investors select their portfolios on the basis of the expected return and the variance of the return over a single time horizon.

Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

Investors dislike risk. For a given level of return they will always prefer a portfolio with lower variance to one with higher variance.

(ii) (a) 100% in X – expected return of 12%

(b) Proportion in $X = (V_Y + C_{XY})/(V_X + V_Y + C_{XY})$

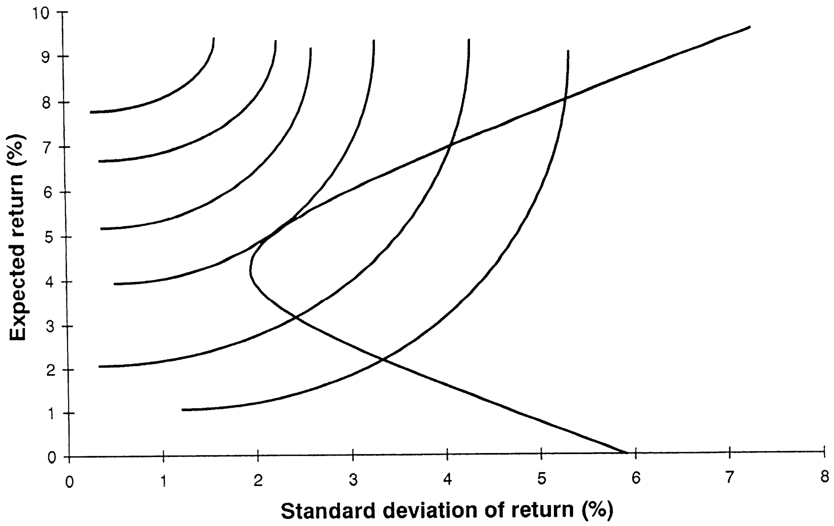
$= (15\%\% - 0.5 \times (30\%\% - 15\%\%)^{0.5})/(30\%\% + 15\%\% + 0.5 \times (30\%\% - 15\%\%)^{0.5})$

$= 18.47\%$

(iii) Plot indifference curves in return-standard deviation space.

Utility is maximised by choosing the portfolio on the efficient frontier where the frontier is at a tangent to the indifference curve.

Graphically candidates should reproduce a diagram similar to Figure 3 from the core reading.



7 (i) The extra assumptions of CAPM are:

- All investors have the same one-period horizon.
- All investors can borrow or lend unlimited amounts at the same risk-free rate.
- The markets for risky assets are perfect. Information is freely and instantly available to all investors and no investor believes that they can affect the price of a security by their own actions.
- Investors have the same estimates of the expected returns, standard deviations and covariances of securities over the one-period horizon.
- All investors measure in the same "currency" e.g. pounds or dollars or in "real" or "money" terms.

(ii) If investors have homogeneous expectations, then they are all faced by the same efficient frontier of risky securities. If in addition they are all subject to the same risk-free rate of interest, the efficient frontier collapses to the straight line in $E - \sigma$ space which passes through the risk-free rate of return on the E-axis and is tangential to the efficient frontier for risky securities.

All rational investors will hold a combination of the risk-free asset and the portfolio of risky assets at the point where the straight line through the risk-free return touches the original efficient frontier. Because this is the portfolio held in different quantities by all investors it must consist of all risk assets in proportion to their market capitalisation. It is commonly called the "market portfolio". The proportion of a particular investor's portfolio consisting of the market portfolio will be determined by their risk-return preference.

(iii) The market price of risk is $(E_m - r)/\sigma_m$, where

$$\begin{aligned} E_m = {} & (30{,}000 \times (5\% \times 0.4 + 8\% \times 0.1 + 3\% \times 0.5) + \\ & 50{,}000 \times (6\% \times 0.4 + 2\% \times 0.1 + 5\% \times 0.5) + \\ & 30{,}000 \times (7\% \times 0.4 + 1\% \times 0.1 + 4\% \times 0.5)) \div 110{,}000 \\ = {} & 4.8273\% \end{aligned}$$

$$\begin{aligned} \sigma_{m\cdot} = {} & [(30{,}000 \times 5\% + 50{,}000 \times 6\% + 30{,}000 \times 7\%) \\ & \qquad \div 110{,}000 - 4.8273\%]^2 \times 0.4 + \\ & [(30{,}000 \text{ x } 8\% + 50{,}000 \times 2\% + 30{,}000 \times 1\%) \\ & \qquad \div 110{,}000 - 4.8273\%]^2 \times 0.1 + \\ & [(30{,}000 \times 3\% + 50{,}000 \times 5\% + 30{,}000 \times 4\%) \\ & \qquad \div 110{,}000 - 4.8273\%]^2 \times 0.5 \end{aligned}$$

$$= 9.7264 \times 10^{-5} = 0.9862\%^2$$

Thus the market price of risk is (4.8273% – 4%)/0.9862%

= 83.89%

8 (i) (a) $\log(S_u) - \log(S_t) \sim N[\mu(u - t), \sigma^2(u - t)]$

(b) $E[S_u] = S_t \exp(\mu(u - t) + \frac{1}{2}\sigma^2(u - t))$

(c) $\text{Var}[S_u] = (S_t)^2 \exp(2\mu(u - t) + \sigma^2(u - t)).\ [\exp(\sigma^2(u - t)) - 1]$

(ii) As the model incorporates independent returns over disjoint intervals, it is impossible to use past history to deduce that prices are cheap or dear at any time.

(iii) Technical analysis does not lead to excess performance.

Estimates of σ vary widely according to what time period is considered, and how frequently the samples are taken.

Examination of historic option prices suggests that implied volatility based on the Black Scholes model fluctuate markedly over time.

There appears to be some evidence for some mean reversion in markets, but the evidence rests heavily on the aftermath of a small number of dramatic crashes. Furthermore, there also appears to be some evidence of momentum effects, which imply that a rise one day is more likely to be followed by another rise the next day.

In particular, market crashes appear more often than one would expect from a normal distribution.

While the random walk produces continuous price paths, jumps or discontinuities seem to be an important feature of real markets. Furthermore, days with no change, or very small change, also happen more often than the normal distribution suggests.

One measure of these non-normal features is the *Hausdorff fractal dimension* of the price process. A pure jump process (such as a Poisson process) has a fractal dimension of 1. Random walks have a fractal dimension of 1½. Empirical investigations of market returns often reveal a fractal dimension around 1.4.

9 The idea is to assume that the converse inequality holds true and show that this leads to an arbitrage opportunity. More precisely, let us assume that

$$\forall\lambda \in [0,1] \quad \lambda C(K_1) + (1-\lambda)C(K_2) < C(\lambda K_1 + (1-\lambda)K_2).$$

Then we construct the following (self-financing) portfolio:

- At time 0: we sell one call with strike price $K_3 = \lambda K_1 + (1-\lambda)K_2$ and buy λ calls with strike K_1 and $(1-\lambda)$ calls with strike K_2. We lend the difference. The total value of the portfolio at time 0 is equal to 0.

At time 0	
sell $C(K_3)$	$C(K_3)$
buy $\lambda C(K_1)$	$-\lambda C(K_1)$
buy $(1-\lambda)C(K_2)$	$-(1-\lambda)C(K_2)$
lend the difference	$M \equiv \lambda C(K_1) + (1-\lambda)C(K_2) - C(K_3)$
Total	0

- At time T: we look at the various possibilities depending on the value S_T of the underlying asset at that time. In all situations, the terminal value of the portfolio is either >0 or ≥ 0 .

	$S_T < K_1$	$K_1 < S_T < K_3$	$K_3 < S_T < K_2$	$K_2 < S_T$
$\lambda C(K_1)$	0	$\lambda(S_T - K_1)$	$\lambda(S_T - K_1)$	$\lambda(S_T - K_1)$
$(1-\lambda)C(K_2)$	0	0	0	$(1-\lambda)(S_T - K_2)$
$C(K_3)$	0	0	$-(S_T - K_3)$	$-(S_T - K_3)$
lending	$M\exp(rT)$	$M\exp(rT)$	$M\exp(rT)$	$M\exp(rT)$
Total	$M\exp(rT) > 0$	$M\exp(rT) + \lambda(S_T - K_1) > 0$	$M\exp(rT) + (1-\lambda)(K_2 - S_T) > 0$	$M\exp(rT) > 0$

This is an arbitrage opportunity. Hence the result.

10 (i) This is an Ornstein-Uhlenbeck process. The solution is given by:

$$r(t) = r_0 \exp(-at) + b(1-\exp(-at)) + \sigma\exp(-at)\int_0^t \exp(as)\,dW_s.$$

(ii) Using similar arguments, we can get for $u \geq t$:

$$r(u) = r(t)\exp(-a(u-t)) + b(1-\exp(-a(u-t))) + \sigma\exp(-au)\int_t^u \exp(as)\,dW_s.$$

Hence

$$\int_t^T r(u)du = r(t)\int_t^T \exp\left(-a(u-t)\right)du + b\int_t^T \left(1-\exp\left(-a(u-t)\right)\right)du + \sigma\int_t^T \exp(-au)\int_t^u \exp(as)\,dW_s du.$$

After some computation:

$$\int_t^T r(u)du = b(T-t) + \left(r(t)-b\right)\frac{1-\exp\left(-a(T-t)\right)}{a} + \frac{\sigma}{a}\int_t^T \left(1-\exp\left(-a(T-s)\right)\right)dW_s.$$

(iii) Hence, $\int_t^T r(u)du$ is also a Gaussian random variable.

(iv) Since the bond market is complete, the price of a zero-coupon bond can be written as

$$B(t,T) = E\left[\exp\left(-\int_t^T r(s)ds\right)\Big| F_t\right].$$

Since $\int_t^T r(u)du$ is a Gaussian random variable, we can compute explicitly the price of the zero-coupon bond in terms of the expected value and variance (conditional) of $\int_t^T r(u)du$:

$$B(t,T) = \exp\left[-E\left[\int_t^T r(s)ds \,\Big|\, F_t\right] + \frac{1}{2}V\left[\int_t^T r(s)ds \,\Big|\, F_t\right]\right].$$

11 (i) Let f denote the price of a call option, then

$$f(s,T) = s\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$d_1 = (\ln(S_0/K) + (r + ½\sigma^2)T)/\ \sigma\sqrt{T} \text{ and } d_2 = d_1 - \sigma\sqrt{T}.$$

It follows (since $\Phi'(x) = \exp(-x^2/2)/\sqrt{2\pi}$) that

$$\Delta = \partial f/\partial s = \Phi(d_1) + s\exp(-d_1^2/2)/\sqrt{2\pi}\,\partial d_1/\partial s - Ke^{-rT}\exp(-d_2^2/2)/\sqrt{2\pi}\,\partial d_2/\partial s.$$

If we now notice that

$$\partial d_1/\partial s = \partial d_2/\partial s$$

and

$$d_2^2 = d_1^2 - (2r + \sigma^2)T - 2\ln(s/K) + \sigma^2 T = d_1^2 - 2rT - 2\ln(s/K)$$

we see that the last two terms in the expression for Δ cancel and we are just left with $\Delta = \Phi\ (d_1)$.

In this case, we must have $100{,}000\Delta = 75{,}000$ and so $\Delta = 0.75$.

(ii) $\Delta = .75$ and so $d_1 = 0.6745$. It follows (rearranging the expression for d_1) that $(.02469 + .07 + 0.5\sigma^2) = 0.6745\sigma$. Solving the quadratic we obtain (choosing the root less than 1) $\sigma = 0.6745 \pm \sqrt{0.26557} = 0.159165 = 15.9\%$.

(iii) We need to calculate:

$$Ke^{-rT}\Phi(d_2) = 8e^{-r}\Phi(d_1 - \sigma\sqrt{T}\,) = 8e^{-r}\ 0.696825 = £5.19772$$

Clearly, the value of the loan is = £519,772 and the option price is

$$100{,}000 * 8.2 * 0.75 - 519{,}772 = £95{,}228.$$

(iv) Use put-call parity. This merely assumes that borrowing is allowed and the market is arbitrage free.

$$p_0 = c_0 + Ke^{-rT} - S_0 = .95228 + 8\ e^{-rT} - 8.2 = .21143$$

END OF EXAMINERS' REPORT

EXAMINATION

24 September 2008 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Two assets are available for investment. Asset 1 returns a percentage $4B$%, where B is a Binomial random variable with parameters $n = 3$ and $p = 0.5$. Asset 2 returns a percentage $2P$%, where P is a Poisson random variable with parameter $\mu = 3$. Assume a benchmark return of 3%.

(i) Calculate the following three measures of investment risk for each asset:

(a) variance
(b) semi-variance and
(c) shortfall probability [6]

An investor has £1,000 to invest in one of the assets.

(ii) Explain which asset the investor should choose assuming a utility function of the form:

$$u(x) = \begin{cases} -(1{,}060 - x)^2 & : x < 1{,}060; \\ 0 & : x \geq 1{,}060. \end{cases}$$

[2]
[Total 8]

2 (i) State the form of an equation that is appropriate to determine the relationship between the observed historical returns of a number of securities and a set of explanatory factors. Define all the terms you use. [2]

(ii) State the main features of:

(a) macroeconomic factor models
(b) fundamental factor models and
(c) statistical factor models

[6]
[Total 8]

3 (i) (a) Define Beta in the Capital Asset Pricing Model (CAPM)
(b) Explain why Beta is used in pricing securities.

[2]

In a market where the CAPM holds the following parameters are known:

Risk-free rate of interest = 6%
Expected market rate of return = 12%
Standard deviation of an efficient portfolio’s returns = 0.50
Standard deviation of the market returns = 0.7

(ii) Calculate the expected return on the portfolio. [2]

(iii) An investor is evaluating the risk and expected return of a portfolio of N securities.

Explain how many parameters need to be estimated if:

(a) the evaluation is made using mean-variance portfolio theory without any assumed cross-sectional structure in the variances of the securities.

(b) the CAPM is assumed to hold.

[3]
[Total 7]

4 One of your colleagues tells you that the work of Shiller conclusively proves that the stock-market overreacts.

(i) Outline the nature of Shiller's findings. [4]

(ii) Discuss whether you agree with your colleague. [4]
[Total 8]

5 State the defining properties of a standard Brownian motion. [4]

6 Consider an asset S paying a dividend at a constant instantaneous rate of δ, a forward contract with maturity T written on S and a constant, instantaneous (continuously compounded) risk-free rate of r.

Derive the price at time t of the forward contract, using the no-arbitrage principle. [8]

7 Consider a one-period Binomial model of a stock whose current price is $S_0 = 40$. Suppose that:

- over a single period, the stock price can either move up to 60 or down to 30
- the continuously compounded risk-free rate is r = 5% per period

(i) Show that there is no arbitrage in the market. [1]

(ii) Calculate the price of a European call option with maturity date in one period and strike price $K = 45$ using each of the following methods:

(a) by constructing a risk-neutral portfolio; and
(b) by constructing a replicating portfolio

[9]
[Total 10]

8 (i) Write down the formula for the value at time $t < T$ of a derivative payment due at time T in a market where the Black-Scholes formula applies, defining any notation that you use. [3]

A non-dividend paying stock has price S_t at time $t < T$. A derivative contract based on the stock pays \$1 at time T if and only if the stock price at time T, S_T, is at least K.

Assume the following:

risk-free interest rate: 7% p.a. continuously compounded
stock price volatility: 25% p.a.
dividend yield: nil
T = 18 months
S_0 = \$1.1

(ii) Derive a formula for V_0, the value of the derivative contract. [5]

(iii) Calculate the value at time 0 of a derivative which delivers one unit of the stock at time T if and only if the stock price at time T, S_T, is at least K. [3]

(iv) Calculate the value of a derivative which delivers 150,000 shares of the stock if and only if the stock price at time T, S_T, satisfies $\$1.2 \le S_T < \$1.5$. [6]

[Total 17]

9 (i) Explain why an investor might want to Vega-hedge a portfolio. [3]

(ii) Derive a formula for the Vega of a European call option on a non-dividend paying stock in a market where the assumptions underpinning Black-Scholes apply. [4]

(iii) (a) Determine the PDE satisfied by Vega by differentiating the Black-Scholes PDE.

(b) Show that if a portfolio is Gamma-hedged, then its Vega satisfies the Black-Scholes PDE. [4]

[Total 11]

10 State how the price at time t of a zero-coupon bond paying £1 at T (denoted by $B(t, T)$) is related to:

(a) spot rate curve
(b) instantaneous forward rate curve
(c) instantaneous risk free rate

Define all notation used. [4]

11 (i) Describe the Merton model for assessing credit risk. [5]

A company has just issued a zero-coupon bond of nominal value £8m with maturity of one year. The value of the assets of the company is £10.009m and this value is expected to grow at an average of 10% per annum compound with an annual volatility of 20%. The company is expected to be wound up after one year when the assets will be used to pay off the bond holders with the remainder being distributed to the equity holders. Shares in the company are currently traded at a market capitalisation of £2.9428m.

(ii) Estimate the risk-free rate of interest in the market to within 1% p.a., stating any additional assumptions that you make. [10]

[Total 15]

END OF PAPER

Subject CT8 — Financial Economics
Core Technical

EXAMINERS' REPORT

September 2008

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

November 2008

1 (i) Asset 1 (random return *X*):

Mean = 4‰*np* = 4‰3‰½ = 6%;
Variance = 4^2‰*npq* = 16‰3‰½‰½ = 12%%;
Lower semi-variance = 3/8‰ $(6 - 4)^2$+1/8‰ $(6 - 0)^2$ = 6%%
(can just use symmetry of distribution around the mean);
Shortfall probability = $P(4B < 3) = P(B = 0)$ = 1/8.

Asset 2 (random return *Y*):

Mean = 2‰μ = 2‰3 = 6%;
Variance = 2^2‰ μ =4‰3 = 12%%;
Lower semi-variance = $e^{-3}(3^2/2$‰ $(6 - 4)^2$ + 3/1‰ $(6 - 2)^2$ + 1.$(6 - 0)^2)$
= 5.078%%;
Shortfall probability = $P(2P < 3) = P(P = 0 \text{ or } 1) = e^{-3}$ (1 + 3) = .1991

(ii) Maximising the expected utility corresponds to minimising the lower semi-variance, hence choose asset 2.

2 (i) A multifactor model of security returns attempts to explain the observed historical return by an equation of the form

$$R_i = a_i + b_{i,1} I_1 + b_{i,2} I_2 + ... + b_{i,L} I_L + c_i \ ,$$

where R_i is the return on security *i*,
a_i and c_i are the constant and random parts respectively of the component of return unique to security *i*,
I_1 ... I_L are the changes in a set of *L* factors which explain the variation of R_i about the expected return a_i,
$b_{i,k}$ is the sensitivity of security *i* to factor *k*.

(ii) **Macroeconomic factor models**

These use observable economic time series as the factors. They could include factors such as the annual rates of inflation and economic growth, short term interest rates, the yields on long term government bonds, and the yield margin on corporate bonds over government bonds.

Fundamental factor models

Fundamental factor models are closely related to macroeconomic models but instead of (or in addition to) macroeconomic variables the factors used are company specific variables. These may include such fundamental factors as:

- the level of gearing
- the price earnings ratio
- the level of R&D spending

- the industry group to which the company belongs

Statistical factor models

Statistical factor models do not rely on specifying the factors independently of the historical returns data. Instead a technique called principal components analysis can be used to determine a set of indices which explain as much as possible of the observed variance. However, these indices are unlikely to have any meaningful economic interpretation and may vary considerably between different data sets.

3 (i) (a) Beta of security i = Covar[R_i, R_M]/V_M

(b) Beta is useful because it allows the expected return of any security to be expressed as a linear function of that security's covariance with the market as a whole.

(ii) Using the formula in (i)(a), Expected Return = 6 + 0.50/0.70(12 - 6) = 10.29%

(iii) (a) Need expected return for each security N, variance of each security N, covariance between each pair of securities $N(N - 1)/2$.

(b) Just need Beta for each security, expected market return, and market variance. Total of $N + 2$.

4 (i) The claim of "excessive volatility" was first formulated into a testable proposition by Shiller in 1981. He considered a discounted cashflow model of equities going back to 1870. By using the actual dividends that were paid and some terminal value for the stock he was able to calculate the perfect foresight price, the "correct equity" price if market participants had been able to predict future dividends correctly. The difference between the perfect foresight price and the actual price arise from the forecast errors of future dividends. If market participants are rational we would expect no systematic forecast errors. Also if markets are efficient broad movements in the perfect foresight price should be correlated with moves in the actual price as both react to the same news.

Shiller found strong evidence that the observed level of volatility contradicted the EMH.

(ii) However, subsequent studies using different formulations of the problem found that the violation of the EMH only had borderline statistical significance. Numerous criticisms were subsequently made of Shiller's methodology, these criticisms covered

- the choice of terminal value for the stock price

- the use of a constant discount rate

- bias in estimates of the variances because of autocorrelation

- possible non-stationarity of the series, i.e. the series may have stochastic trends which invalidate the measurements obtained for the variance of the stock price

Although subsequent studies by many authors have attempted to overcome the shortcomings in Shiller's original work there still remains the problem that a model for dividends and distributional assumptions are required. Some equilibrium models now exist which calibrate both to observed price volatility and also observed dividend behaviour. However, the vast literature on volatility tests can at best be described as inconclusive.

5 Standard Brownian motion (also called the Wiener process) is a stochastic process $\{B_t, t \geq 0\}$ with state space $S = \mathbf{R}$ and the following defining properties:

- B_t has independent increments, i.e. $B_t - B_s$ is independent of $\{B_r, r \leq s\}$ whenever $s < t$.

- B_t has stationary increments, i.e. the distribution of $B_t - B_s$ depends only on $t - s$.

- B_t has Gaussian increments, i.e. the distribution of $B_t - B_s$ is $N(0, t - s)$.

- B_t has continuous sample paths $t \to B_t$.

- $B_0 = 0$.

- (Note that the stationarity property is not needed separately if the Gaussian property is set out in detail.)

6 The proof of this result is an adaptation of that of the standard spot-forward parity. Two (self-financing) portfolios are considered:

- Portfolio A: take a long position in the forward contract at time t. Its value at time t is 0 and at time T, it is $S_T - F_t^T$.

- Portfolio B: buying a fraction $\exp\left(-\delta\left(T-t\right)\right)$ of the underlying asset and borrowing $F_t^T \exp\left(-r\left(T-t\right)\right)$ at time t. Its value at time t is then – ($F_t^T \exp\left(-r\left(T-t\right)\right) - \exp\left(-\delta\left(T-t\right)\right) S_t$). Its value at maturity is $S_T - F_t^T$ (assuming reinvestment of dividends).

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time t. Hence:

$$F_t^T = \exp\left(\left(r-\delta\right)\left(T-t\right)\right)S_t .$$

7 (i) There is no arbitrage in the market since $d = \dfrac{3}{4} < \exp\left(0.05\right) < u = \dfrac{3}{2}$.

(ii)

- First method: we construct a risk-neutral portfolio with 1 underlying asset and m call options. We choose the value of m such that this portfolio is risk neutral (its value in the upper state and in the lower state at time 1 should coincide). In this case, $m = -2$. Then, we use a no arbitrage argument: since the portfolio is risk-neutral, it should have the same rate of return as the risk-free asset. Hence, the initial value of the call:

  C_0 satisfies $S_0 - 2C_0 = 30e^{-r}$,

  so $C_0 = 5.732$.

- Second method: We use a replicating portfolio. This is a self-financing portfolio with φ_0 invested in the risk-free asset and φ_1 underlying asset at time 0. Its initial value is therefore $V_0 = \varphi_0 + \varphi_1 S_0$. At time 1, the portfolio should replicate the payoff of the call option. Therefore: $V_u = \varphi_0 \exp\left(r\right) + \varphi_1 S_u = C_u$ and $V_d = \varphi_0 \exp\left(r\right) + \varphi_1 S_d = C_d$. We can deduce the value of φ_0 and $\varphi_1 : \varphi_0 = -14.27$ and $\varphi_1 = 0.5$. By no arbitrage, the initial value of the portfolio and that of the call option should coincide.

  Hence $C_0 = 5.732$.

8 (i) $E_Q[e^{-r(T-t)}X|F_t]$, where X is the amount payable, Q is the risk-neutral measure and F_t is the sigma-algebra generated by the stock-price history up to time t.

(ii) From (a), the price is $E_Q[e^{-rT}1(K \leq S_T)|F_0] = e^{-rT}Q(K \leq S_T)|$.
Now, under Q, $S_T = S_0\exp(\sigma W_T + (r - \frac{1}{2}\sigma^2)T)$, where W is a standard Brownian motion. Thus, $V_0 = e^{-rT}Q(W_T > (\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)T)/\sigma)$
$= e^{-rT}(1 - \Phi((\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)T)/\sigma\sqrt{T})) = e^{-rT}\Phi(d_2)$, where Φ is the standard normal distribution function and d_2 is as in the Black-Scholes formula in the tables.

(iii) If we are long one unit of this derivative and short K units of the derivative in part (ii) then we effectively hold a call option. Thus the value of this derivative must be the sum of the value of the call and KV_0 i.e $S_0\Phi(d_1)$.

(iv) If we go long 150,000 contracts of the type in part (iii) with a strike of 120p and short 150,000 such contracts with a strike of 150p then we have duplicated the contract. Thus the fair price is

$$1.1\times 150{,}000\times(\Phi(\ln(S_0/120) + (r + ½\sigma^2)T)/\sigma\sqrt{T}) - \Phi(\ln(S_0/150) + (r + ½\sigma^2)T)/\sigma\sqrt{T})))$$

$=1.1\times 150{,}000\times(\Phi(.21184) - \Phi(-.51694)) = 165{,}000\times(.58389 - .30260)$
$= £46{,}413$.

9 (i) The value of a portfolio with a low value of vega will be relatively insensitive to changes in volatility. Put another way: it is less important to have an accurate estimate of σ if vega is low. Since σ is not directly observable, a low value of vega is important as a risk-management tool. Furthermore, it is recognised that σ can vary over time. Since many derivative pricing models assume that σ is constant through time the resulting approximation will be better if ν is small.

(ii) Let f denote the price of a call option, then $f(s,T) = s\Phi(d_1) - K e^{-rT}\Phi(d_2)$, where $d_1 = (\ln(S_0/K) + (r + ½\sigma^2)T)/\sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$. It follows (since $\Phi'(x) = \exp(-x^2/2)/\sqrt{2\pi}$) that $V= s(\exp(-d_1^2/2)/\sqrt{2\pi})\partial d_1/\partial\sigma - K e^{-rT}$ $(\exp(-d_2^2/2)/\sqrt{2\pi})\partial d_2/\partial\sigma = s(\exp(-d_1^2/2)/\sqrt{2\pi})\partial(d_1 - d_2)/\partial\sigma$
$= s(\exp(-d_1^2/2)/\sqrt{2\pi})\sqrt{T}$.

(iii) Differentiating the Black-Scholes PDE in σ gives us
$\partial V/\partial t + rs\partial V/\partial s+½\sigma^2 s^2\partial^2 V/\partial s^2+\sigma s^2\partial^2 f/\partial s^2=rV$.

Then, since $\Gamma=\partial^2 f/\partial s^2$, we see that in the case where $\Gamma=0$ we have
$\partial V/\partial t + rs\partial V/\partial s+½\sigma^2 s^2\partial^2 V/\partial s^2=rV$.

And so V satisfies the Black-Scholes PDE.

10 We will make use of the following notation:

$B(t,T)$	= Zero-coupon bond price
	= price at t for £1 payable at T
$r(t)$	= instantaneous risk-free rate of interest at t
$C(t)$	= unit price for investment at the risk-free rate
$F(t,T,S)$	= forward rate at t for delivery between T and S
$f(t,T)$	= instantaneous forward-rate curve
$R(t,T)$	= spot-rate (zero-coupon yield) curve

Zero-coupon bond prices are related to the spot-rate and forward-rate curves in the following way:

$$R(t,T) = \frac{-1}{T-t} \log B(t,T) \quad \text{for } t < T$$

or $$B(t,T) = \exp[-R(t,T)(T-t)]$$

$$F(t,T,S) = \frac{1}{S-T} \log \frac{B(t,T)}{B(t,S)} \quad \text{for } t < T < S$$

$$f(t,T) = \lim_{S \to T} F(t,T,S) = -\frac{\partial}{\partial T} \log B(t,T)$$

or $$B(t,T) = \exp\left[-\int_t^T f(t,u)du\right].$$

$$B(t,T) = E_Q\left[\exp\left(-\int_t^T r(u)du\right) \middle| r(t)\right]$$

for specific models.

It is important to remember that Q is an artificial computational tool. It is determined by combining (a) the model for $r(t)$ under the real world measure P and (b) the market price of risk established from knowledge of the dynamics of one bond.

11 (i) Merton's model assumes that a corporate entity has issued both equity and debt such that its total value at time t is of $F(t)$. $F(t)$ varies over time as a result of actions by the corporate entity which does not pay dividends on its equity or coupons on its bonds. Part of the corporate entity's value is zero-coupon debt with a promised repayment amount of L at a future time T. At time T the remainder of the value of the corporate entity will be distributed amongst the equity holders and the corporate entity will be wound up.

The corporate entity will default if the total value of its assets, $F(T)$ is less than the promised debt repayment at time T i.e. $F(T) < L$. In this situation, the bond holders will receive $F(T)$ instead of L and the equity holders will receive nothing. This can be regarded as treating the equity holders of the corporate entity as having a European call option on the assets of the company with maturity T and a strike price equal to the value of the debt.

The Merton model can be used to estimate either the risk-neutral probability that the company will default or the credit spread on the debt.

(ii) We assume the Merton model, so the value of the company is the value of a call on the assets. The underlying is the gross value and the strike is the debt.

Thus $S_0 = 10.009$, $\sigma = 0.2$, $T = 1$, $K = 8$, and 2.9428 is the value of the call (at time 0).

So, $2.9428 = 10.009\Phi((\ln(10.009/8) + .02 + r)/0.2) - 8e^{-r}\Phi((\ln(10.009/8) - .02 + r)/0.2) = 10.009\Phi(1.2202 + 5r) - 8e^{-r}\Phi(1.0202 + 5r)$. This is a differentiable and increasing function of r so interpolation should get a solution.

Setting $r = 10\%$, we get $10.009\Phi(1.2202 + 5r) - 8e^{-r}\Phi(1.0202 + 5r) = 10.009\Phi(1.7202) - 8e^{-.1}\Phi(1.5202) = 10.009 \times 0.95730 - 8e^{-.1} \times 0.93577 = 2.80786$, so we need to increase r.

Setting $r = 15\%$, we get $10.009\Phi(1.2202 + 5r) - 8e^{-r}\Phi(1.0202 + 5r) = 10.009\Phi(1.9702) - 8e^{-.1}\Phi(1.7702) = 10.009 \times 0.97559 - 8e^{-.15} \times 0.96166 = 3.14301$, so we need to decrease r.

Interpolating gives $r = 10 + 5X(2.9428 - 2.80786)/(3.14301 - 2.80786)\% = 12\%$.

If we try $r = 12\%$, we get $10.009\Phi(1.2202 + 5r) - 8e^{-r}\Phi(1.0202 + 5r) = 10.009\Phi(1.8202) - 8e^{-.12}\Phi(1.6202) = 10.009 \times 0.96564 - 8e^{-.12} \times 0.94740 = 2.9429$, so $r = 12\%$.

END OF EXAMINERS' REPORT

EXAMINATION

22 April 2009 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Describe what is meant by an arbitrage opportunity. [3]

2 One of your colleagues says that the stock market is not efficient because some accounting ratios have been shown to have predictive powers.

(i) Explain which of the main forms of efficiency is most relevant to this situation. [2]

(ii) Comment on whether you agree with your colleague. [2]

(iii) Explain the difference between active and passive fund management in terms of the concept of market efficiency. [2]
[Total 6]

3 (i) Outline the relevant empirical evidence and theoretical arguments regarding the behaviour of stock prices for each of the properties below.

(a) The volatility of returns over time.
(b) The expected value of returns over time.
(c) Whether stock prices are mean reverting.
(d) The statistical distribution of returns.

[8]

(ii) (a) Discuss the extent to which a continuous time lognormal model of security prices can capture the statistical properties empirically observed or expected in the stock market.

(b) Outline other possible processes which may be used.

[6]
[Total 14]

4 (i) State how investors are assumed to make decisions in modern portfolio theory (MPT). [1]

(ii) Define an efficient portfolio in the context of MPT. [1]

An investor can invest in only three assets which are uncorrelated with one another. The assets have the following characteristics:

Asset A	*Expected Rate of Return*	*Standard Deviation*
A	9%	18%
B	5%	8%
C	4%	0%

(iii) Calculate the efficient frontier for the investor taking into account the numbers provided in the table above. [8]

(iv) Explain how an investor with a quadratic utility function would select a portfolio from those making up the efficient frontier. [1]
[Total 11]

5 (i) Define the market price of risk in the CAPM. [1]

The table below gives the annual returns conditional on the state of the economy for all the assets in an investment market.

Economic State	*Asset*			*Probability*
	Stock	*Property*	*Bonds*	
Recession	0%	1%	2%	0.1
Normal	5%	3%	3%	0.7
Boom	10%	7%	3%	0.2
Value of asset (bn)	100	50	100	

(ii) Calculate the market price of risk given that the risk free annual rate of return is 2.5%. [4]

(iii) Discuss the particular issue a young investor might face in using the CAPM. [2]

[Total 7]

6 Describe three different approaches to modelling credit risk. [6]

7 (i) Set out a formula for the stock-price, S_t, in the Black-Scholes model under the equivalent martingale measure. [2]

A European call option on a stock has an exercise date one year away and a strike price of £2. The underlying stock has a current price of £1.80. The continuously compounded risk free rate of interest is 5% p.a. The option is priced at 20p.

(ii) Estimate the volatility of the stock price process to within 1% p.a., assuming the Black-Scholes model applies. [5]

A new derivative security has just been written on the underlying stock. This will pay a random amount D in one year's time, where D is £1 if $S_{0.5}$ > £2 and $S_1 > 2S_{0.5}$, is 50p if $S_{0.5}$ < £2 and $S_1 > 2\,S_{0.5}$ and is zero otherwise.

(iii) Derive an expression in terms of the distribution function and/or density function of the standard normal distribution for the fair price for this derivative security. [6]

[Total 13]

8 (i) Explain what is meant by self-financing in the context of continuous-time derivative pricing, defining all notation used. [4]

(ii) Define the delta of a derivative, defining all notation and terms used other than those already defined in your answer to (i). [2]

(iii) Explain how delta and self-financing are used in the martingale approach to valuing derivatives. [4]

[Total 10]

9 The zero-coupon bond market is assumed to be arbitrage-free and complete. Consider the following model for the instantaneous forward rate process:

$$df(t,T) = a(t,T)dt + \sigma(t,T)dW_t$$

where $(W_t; t \geq 0)$ is a standard Brownian motion with respect to the risk-neutral probability measure **Q**.

(i) State how the price of a zero-coupon bond is related to the instantaneous forward rate. [2]

Using Itô calculus, it is possible to prove that the dynamics for the zero-coupon bond price are given under **Q** as follows:

$$\frac{dB(t,T)}{B(t,T)} = m(t,T)dt + S(t,T)dW_t,$$

where

$$m(t,T) = r(t) - \int_t^T a(t,s)ds + \left(\int_t^T \sigma(t,s)ds\right)^2$$

$$S(t,T) = -\int_t^T \sigma(t,s)ds.$$

(ii) Explain the relationship between a and σ under the condition that the bond market is complete. Give reasons for your answer. [6]

[Total 8]

10 Consider a three-period binomial model for a stock with the following parameters: $u = 1.2$, $d = 0.9$ and $S_0 = 60$. Assume that the discretely compounded risk-free rate of interest is $r = 11\%$ per period.

(i) (a) Verify that there is no arbitrage in the market.
(b) Construct the binomial tree.
[3]

(ii) Calculate the price of a standard European call option with maturity date in three periods and strike price $K = 60$. [7]

A new "knock-in" option is introduced which has the following characteristics:

If the value of the stock crosses the level 80 during the whole life of the option, the contract holder has the right to obtain the difference between the value of the stock at maturity (in three periods) and 60.

(iii) Calculate the price of this new option. [4]
[Total 14]

11 Consider a forward contract on gold. Suppose that there is a fixed storage cost of £c per ounce, paid at the end of the period and c is the same for any time period less than one year. Let S_t be the spot price of one ounce of gold at time t and r be the continuously compounded risk-free rate of interest which is assumed to be constant.

Derive the price at time t of a forward contract written on one ounce of gold at the start of the year, with maturity T years ($T < 1$).
[8]

END OF PAPER

Subject CT8 — Financial Economics
Core Technical

EXAMINERS' REPORT

April 2009

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

June 2009

1 Put in simple terms, an *arbitrage opportunity* is a situation where we can make a sure profit with no risk. This is sometimes described as a free lunch. Put more precisely an arbitrage opportunity means that:

(a) We can start at time 0 with a portfolio which has a net value of zero (implying that we are long in some assets and short in others).

(b) At some future time T:

- the probability of a loss is 0
- the probability that we make a strictly positive profit is greater than 0

2 (i) Semi-strong form because linked to publically available information.

(ii) It is true that some ratios have predictive power.

This may not violate market-efficiency as the ratios may be acting as a proxy for risk.

(iii) Active managers believe market is not fully efficient hence they attempt to detect mispricings.

Passive managers believe in efficiency and just diversify across the whole market.

3 (i) (a) Direct statistical evidence shows volatility varies over time. Volatility implied from option prices also shows volatility/volatility expectations vary over time.

(b) Good theoretical reasons to expect this to vary over time. Equities should give a risk premium over bonds and bond yields vary over time. Empirically difficult to test.

(c) Empirically unsettled. Some evidence for mean reversion but rests heavily on the aftermath of a few dramatic crashes also conversely some evidence of momentum effects.

(d) Strong empirical evidence that prices are non-normal. Crashes happen more than would be expected. In addition more days with small/no changes than one would expect.

(ii) Some problems with random walk. Can consider the points in (i) again:

(a) Random walk assumes constant volatility — ARCH would be better in this respect. Also processes with non-normal returns can give a similar effect.

(b) Random walk assumes drift is constant.

(c) No allowance for mean reversion in random walk.

(d) Random walk does assume normality. Quite a few alternatives. Levy processes, jump processes like Poisson.

The above answer incorporates what is in the core reading. Quite a few processes which are not mentioned are valid e.g. GARCH, EGARCH, QGARCH.

4 (i) Select on the basis of expected return and variance of return over a single time horizon.

(ii) A portfolio is efficient if an investor cannot find a better one in the sense that it has both a higher expected return and a lower variance.

(iii) The basic idea is that the efficient frontier is a straight line which is the tangent to the efficient frontier (of risky assets) which passes through the point in (s.d., return) space corresponding to the risk-free asset.

Initially need to find the portfolio using A and B that maximises

(expected return – 4%)/standard deviation

Put say proportion x of assets in A and $(1 - x)$ in B.

Expected return of risky portfolio is $0.09x + 0.05(1 - x)$

Standard deviation of risky portfolio is

$[(0.18x)^2 + (0.08(1 - x))^2]^{0.5}$

Thus need to find x to maximise

$[0.09x + 0.05(1 - x) - 0.04]/ [(0.18x)^2 + (0.08(1 - x))^2]^{0.5}$

Method is to take logs.

Need to maximise

$\ln[0.01 + 0.04x] - 0.5\ln[0.0324x^2 + 0.0064(1 - x)^2]$

$= \ln[0.01 + 0.04x] - 0.5\ln[0.0064 - 0.0128x + 0.0388x^2]$

Differentiate and set to zero.

$0.04/[0.01 + 0.04x] - 0.5[- 0.0128 + 0.0388.2x]/ [0.0064 - 0.0128x + 0.0388x^2] = 0$

x is found to be 0.4969

(Can also calculate x using Lagrangian multipliers)

When x is 0.4969
Expected return of risky portfolio is 0.069876
Standard deviation of risky portfolio is 0.09808

Thus the efficient frontier is the straight line through (0.04.0) and (0.069876, 0.09808).

(iv) Portfolio would be that corresponding to the point where the utility indifference curve of the investor touched the efficient frontier.

5 (i) The market price of risk is $(E_m - r)/\sigma_{m\cdot}$

(ii) $E_m =$ (100 × (0% × 0.1 + 5% × 0.7 + 10% × 0.2) +
50 × (1% × 0.1 + 3% × 0.7 + 7% × 0.2) +
100 × (2% × 0.1 + 3% × 0.7 + 3% × 0.2)) ÷ 250

= 4.08%

$\sigma^2_{m\cdot} =$ [(100 × 0% + 50 × 1% + 100 × 2%) ÷ 250 – 3.36%]^2 × 0.1 +
[(100 × 5% + 50 × 3% + 100 × 3%) ÷ 250 – 3.36%]^2 × 0.7 +
[(100 × 10% + 50 × 7% + 100 × 3%) ÷ 250 – 3.36%]^2 × 0.2

= 0.00022736

Thus the market price of risk is (4.08% – 2.5%)/1.5078%

= 1.0479

(iii) The investor should consider their own human capital which is likely to be large and unpredictable compared to their other assets.

6 **Structural models**

Structural models are explicit models of a corporate entity issuing both equity and debt. They aim to link default events explicitly to the fortunes of the issuing corporate entity. An example of a structural model is the Merton model.

Reduced form models

Reduced form models are statistical models which use observed market statistics rather than specific data relating to the issuing corporate entity. The market statistics most commonly used are the credit ratings issued by credit rating agencies such as Standard and Poor's and Moody's.

Reduced form models use market statistics along with data on the default-free market to model the movement of the credit rating of the bonds issued by a corporate entity over time. The output of such models is a distribution of the time to default.

Intensity-based models

Intensity-based models model the factors influencing the credit events which lead to default and typically (but not always) do not consider what actually triggers the credit event.

7 (i) $S_t = S_0 \exp(\sigma Z_t + (r - ½\sigma^2)t)$, where Z is a standard Brownian motion.
Alternative solution: dS_t/S_t = r dt + σ dZ_t

(ii) Apply B-S formula with $K = 2$, $S = 1.8$, $r = 0.05$, $t = 1$ and sigma = 0.2, gives $c = 0.10$. With sigma = 0.4, $c = 0.24$. Try sigma = 0.2 + 0.2 * 0.1/0.14 = 0.34 and get $c = 0.20$.

(iii) The unique fair price is $V = E_P[e^{-rT}D]$, where P is the EMM. Thus

$V = e^{-r}[P(S_{0.5}>£2 \text{ and } S_1 > 2S_{0.5}) + 0.5\, P(S_{0.5} < £2 \text{ and } S_1 > 2S_{0.5})]$

$= e^{-r}$ {integral from 2 to infinity Prob[S_1 > 2*x*] * *f*(*x*) *dx* +
Integral from 0 to 2 0.5 Prob[S_1 > 2*x*] *f*(*x*) *dx*}

$= e^{-r}$ {integral from 2 to infinity Phi((ln(2x) – ln(1.8) – 0.05)/0.34)
phi((ln(*x*) – ln(1.8) – 0.025)/0.24) *dx* +

Integral from – infinity to 2 0.5 Phi((ln(2*x*) – ln(1.8) – 0.05)/0.34) phi
((ln(*x*) – ln(1.8) – 0.025)/0.24)*dx*}

Where Phi is the cumulative standard normal distribution function and phi is the standard normal density function. Standard deviation of ln(*S*_0.5) is 0.34/sqrt(2) = 0.24.

There is another way to solve the problem and the price can be calculated directly from

$P(S_{0.5} > 2 \text{ and } S_1 > 2\, S_{0.5}) = P(S_{0.5} > 2 \text{ and } S_1/S_{0.5} > 2)$
$= P(S_{0.5} > 2) * P(S_1/S_{0.5} > 2)$ since $S_1/S_{0.5}$ is independent of $S_{0.5}$.

$P(S_{0.5} > 2) = P(\sigma Z_{0.5} + (r - 0.5\sigma^2) *0.5 > \ln(2) - \ln(1.8))$
$= P(Z > x)$

with

$x = [1/(\sigma * \sqrt{0.5})] * [\ln(2) - \ln(1.8)) - (r - 0.5\sigma^2) * 0.5]$

Similarly, defining

$y = [1/(\sigma * \sqrt{0.5})] * [\ln(2)) - (r - 0.5\,\sigma^2) * 0.5\,]$

we get

$V = e^{-r} * [(1 - \varphi_x) * (1 - \varphi_y) + 0.5 * \varphi_x * (1 - \varphi_y)]$
where φ_x is the cumulative normal distribution at x.

8 (i) Suppose that at time t we hold the portfolio (ϕ_t, ψ_t) where ϕ_t represents the number of units of S_t held at time t and ψ_t is the number of units of the cash bond held at time t. We assume that S_t is a tradeable asset as described above. The only significant requirement on (ϕ_t, ψ_t) is that they are previsible: that is, that they are F_{t-}-measurable (so ϕ_t and ψ_t are known based upon information up to but not including time t).

Let $V(t)$ be the value at time t of this portfolio: that is, $V(t) = \phi_t S_t + \psi_t B_t$.

Now consider the instantaneous pure investment gain in the value of this portfolio over the period t up to $t + dt$: that is, assuming that there is no inflow or outflow of cash during the period $[t, t + dt]$. This is equal to

$$\phi_t dS_t + \psi_t dB_t$$

The instantaneous change in the value of the portfolio, allowing for cash inflows and outflows, is given by

$$dV(t) \equiv V(t + dt) - V(t) = \phi_t dS_t + S_t d\phi_t + d\phi_t .dS_t + B_t d\psi_t + \psi_t dB_t .$$

The portfolio strategy is described as self-financing if $dV(t)$ is equal to $\phi_t dS_t + \psi_t dB_t$: that is, at $t + dt$ there is no inflow or outflow of money necessary to make the value of the portfolio back up to $V(t+dt)$.

(ii) Delta is just one of what are called the Greeks. The Greeks are a group of mathematical derivatives which can be used to help us to manage or understand the risks in our portfolio.

Let $f(t,s)$ be the value at time t of a derivative when the price of the underlying asset at t is $S_t = s$.

The delta for an individual derivative is

$$\Delta = \frac{\partial f}{\partial s} \equiv \frac{\partial f}{\partial s}(t, S_t).$$

(iii) In the martingale approach we showed that there exists a portfolio strategy (ϕ_t, ψ_t) which would replicate the derivative payoff. We did not say what ϕ_t actually is or how we work it out. In fact this is quite straightforward.

First we can evaluate directly the price of the derivative

$$V_t = e^{-r(T-t)} E_Q[X \mid F_t]$$

either analytically (as in the Black-Scholes formula) or using numerical techniques.

In general, if S_t, represents the price of a tradeable asset

$$\phi_t = \frac{\partial V}{\partial s}(t, S_t).$$

ϕ_t is usually called the Delta of the derivative.

The martingale approach tells us that provided:

- we start at time 0 with V_0 invested in cash and shares
- we follow a self-financing portfolio strategy
- we continually rebalance the portfolio to hold exactly ϕ_t units of S_t with the rest in cash

then we will precisely replicate the derivative payoff.

9 (i) $B(t,T) = \exp\left[-\int_t^T f(t,u)du\right].$

(ii) Since the bond market is complete, the discounted price of a zero-coupon bond is a martingale with respect to the risk-neutral probability measure. Using Itô, the dynamics for the discounted zero-coupon bond price $\overline{B}(t,T)$ are:

$$\frac{d\overline{B}(t,T)}{\overline{B}(t,T)} = \big(m(t,T) - r(t)\big)dt + S(t,T)dW_t$$

As to be a martingale, the drift term should be equal to 0:

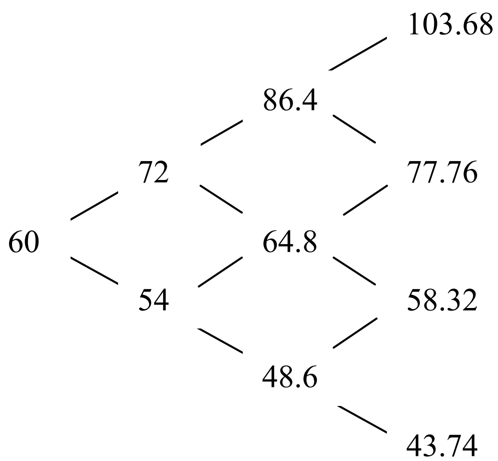
$$\big(m(t,T) - r(t)\big) = 0$$

In other words

$$\int_t^T a(t,s)ds = \left(\int_t^T \sigma(t,s)ds\right)^2.$$

10 (i) (a) There is no arbitrage in the market since $d = 0.9 < 1.11 < 1.2$.

(b)



(ii) To price the call option, we use the risk-neutral pricing formula. We use the following simplifying notation:

$$C_{uuu} = \left(u^3 S_0 - K\right)^+;$$
$$C_{uud} = \left(u^2 d S_0 - K\right)^+;$$
$$C_{udd} = \left(u d^2 S_0 - K\right)^+;$$
$$C_{ddd} = \left(d^3 S_0 - K\right)^+.$$

At time 2, we get in the upper state,

$$C_2(uu) = \frac{1}{1+r}\left[qC_{uuu} + (1-q)C_{uud}\right],$$

in the medium state,

$$C_2(ud) = \frac{1}{1+r}\left[qC_{uud} + (1-q)C_{udd}\right]$$

and in the lowest state,

$$C_2(dd) = \frac{1}{1+r}\left[qC_{udd} + (1-q)C_{ddd}\right],$$

where the risk-neutral probability of an upward move is

$$q = \frac{(1+r)-d}{u-d}.$$

At time 1, we get in the upper state,

$$C_1(u) = \frac{1}{1+r}\left[qC_2(uu) + (1-q)C_2(ud)\right],$$

and in the lower state,

$$C_1(d) = \frac{1}{1+r}\left[qC_2(ud) + (1-q)C_2(dd)\right].$$

At time 0,

$$C_0 = \frac{1}{1+r}\left[qC_1(u) + (1-q)C_1(d)\right].$$

Hence

$$C_0 = 16.68.$$

(iii) Two paths are relevant for this knock-in option: "up-up-up" and "up-up-down". The associated payoff are respectively C_{uuu} with probability q^3 and C_{uud} with probability $q^2(1-q)$. The price at time 0 of the option is therefore:

$$Knock-in_0 = \frac{1}{(1+r)^3}\left[q^3C_{uuu} + q^2(1-q)C_{uud}\right] = 12.82$$

11 The proof of this result is an adaptation of that of the standard spot-forward parity. Two (self-financing) portfolios are considered:

- Portfolio A: buying the forward contract at time t. Its value at time t is 0 and at time T, it is $S_T - F_t^T$.

- Portfolio B: buying the underlying asset and borrowing $\left(F_t^T - c\right)\exp\left(-r\left(T-t\right)\right)$ at time t. Its value at time t is then $\left(F_t^T - c\right)\exp\left(-r\left(T-t\right)\right) - S_t$. Its value at maturity is $S_T - F_t^T$ by taking into account the storage costs.

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time t. Hence:

$$F_t^T = S_t \exp\left(r\left(T-t\right)\right) + c .$$

END OF EXAMINERS' REPORT

EXAMINATION

1 October 2009 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 (i) State the general form of the equation used in multifactor models of security returns, defining any terms you use. [3]

(ii) Describe the different categories of factors that are used in these models and illustrate your answer with suitable examples. [6]
[Total 9]

2 (i) In the context of time series models of financial markets explain the difference between cross-sectional and longitudinal properties of statistical distributions. [2]

(ii) Discuss the difference between cross-sectional and longitudinal estimates of stock volatility assuming:

(a) stock prices follow a multiplicative random walk.
(b) the Wilkie model is being used to model financial variables.
[4]
[Total 6]

3 A small bank wishes to improve the performance of its investments by investing £1m in high returning assets. An investment bank has offered the bank two possible investments:

Investment A: A diversified portfolio of shares and derivatives which can be assumed to produce a return of £R_1 million where $R_1 = 0.1 + N$, where N is a normal $N(1,1)$ random variable.

Investment B: An over-the-counter derivative which will produce a return of £R_2 million where the investment bank estimates:

$R_2 =$ 1.5 with probability 0.99
−5.0 with probability 0.01.

The chief executive of the bank says that if one investment has a better expected return and a lower variance than the other then it is the best choice.

(i) (a) Calculate the expected return and variance of each investment A and B.

(b) Discuss the chief executive's comments in the light of your calculations.

[6]

(ii) Calculate the following risk measures for each of the two investments A and B:

(a) semi-variance of return
(b) shortfall probability of the returns falling below 0
(c) shortfall probability of the returns falling below –2

[3]

(iii) (a) Define other suitable risk measures that could be calculated.
(b) Discuss what the risk measures in (iii) (a) would show. [4]

(iv) Compare the merits of the two investments A and B. [2]
[Total 15]

4 An investor invests a proportion x_i of the assets in his portfolio in the ith of N securities.

(i) State the expected return and variance of his portfolio. Define any notation you use. [2]

Securities with the properties in the table below are available to an investor. The statistics in the table refer to the next year.

	A	*B*
Expected return	4%	3%
Variance of return	16%%	4%%
Correlation coefficient between assets	$\rho_{AB} = 1$	

The investor combines the securities to form a portfolio.

(ii) Calculate the relative amount which should be invested in each security to give a portfolio with the minimum possible variance. (Note: you may assume that short selling securities is allowable.) [4]

(iii) Show that if it is possible to borrow at the rate of 1% p.a. over the next year, it is possible for the investor to make a risk free profit over the year without using any of his own capital. [4]
[Total 10]

5 A derivative security entitles the holder to a payment, at time T, of $\max_{0\le t\le T} S_t$, where S_t is the price at time t of a security.

Assume that S satisfies $S_t = S_0\exp(\sigma B_t + (r - 1/2\,\sigma^2)t)$ under the risk neutral measure, where B is a standard Brownian motion and r is the risk-free rate of interest.

(i) Derive the probability density of $\max_{0\le s\le t} B_s + \mu s$. (Hint: use the formula in section 7.2 of the Formulae and Tables for Actuarial Examinations). [4]

(ii) Determine an expression for p_t, the fair price of the derivative security at time t. You need not evaluate the resulting integral. [4]
[Total 8]

6 (i) Describe the two-state model for credit-ratings. [4]

In a two state model a zero-coupon defaultable bond is due to redeem at par in two years' time. If default occurs the recovery rate is δ. The continuously compounded risk free rate of return is r, Under the probability measure P_λ, the default intensity is constant and equal to λ and the defaultable bond price is D_t, given by:

$$D_t = \delta e^{-r(2-t)}, \text{ if default has occurred prior to time } t$$

$$= e^{-r(2-t)}(\delta(1 - e^{-\lambda(2-t)}) + e^{-\lambda(2-t)}) \text{ otherwise.}$$

(ii) Show that P_λ is an equivalent martingale measure for this model. [3]

A derivative contract pays \$1,000 after two years if and only if the bond has defaulted.

(iii) (a) Determine a constant portfolio in the defaultable bond and cash which replicates the derivative.

(b) Calculate the fair price for the derivative. [5]

(iv) Explain how your answer to (iii) relates to the fact stated in part (ii). [3]
[Total 15]

7 (i) State the main assumptions underlying the Black-Scholes model for a security price. [4]

(ii) Comment on how realistic these assumptions are in practice. [4]
[Total 8]

8 (i) Define a state-price deflator in the context of continuous time models for security prices. [3]

(ii) Give a formula for the state-price deflator in the Black-Scholes model when the risk-free rate of interest is r and the stock price satisfies:

$$dS_t = S_t(\mu dt + \sigma dZ_t)$$

under the real-world measure P, where Z is a standard Brownian motion. [3]

A derivative contract pays $\exp(\gamma Z_1)$ if $Z_1 > 1$, and zero otherwise, where $Z_0 = 0$ and $\gamma = (\mu - r) / \sigma$.

(iii) Calculate the price p_t, at each time t, of this derivative contract, using your answer to part (ii), or otherwise. [5]

[Total 11]

9 Comment on the difference between real-world and risk-neutral measures in the context of the valuation of derivative securities using a binomial tree. [4]

10 Prove that it is never optimal to exercise an American call written on a non-dividend paying stock before maturity. [7]

11 (i) Define the market price of risk in the context of pricing zero coupon bonds using diffusion models for the short-rate of interest. Define any notation you use. [2]

(ii) Prove that the market price of risk at a given time t is constant for all zero-coupon bonds with maturities $T > t$ in the case where the diffusion model for the short-rate of interest has only one factor. Define any notation you use. [5]

Hint: construct a self-financing strategy involving zero coupon bonds of maturities T_1 and T_2 and a cash account.

[Total 7]

END OF PAPER

Subject CT8 — Financial Economics.
Core Technical

September 2009 Examinations

EXAMINERS REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

December 2009

Comments for individual questions are given with the solutions that follow.

1

(i) $R_i = a_i + b_{i,1} I_1 + b_{i,2} I_2 + \ldots\ldots. + b_{i,L} I_L + c_i$

Where

R_i is the return on security i.

a_i and c_i are the constant and random parts respectively of the component of return unique to security i.

$I_1 \ldots\ldots I_L$ are the changes in a set of L factors which explain the variation in R_i about the expected return.

$b_{i,k}$ is the sensitivity of security i to factor k.

(ii) **Macroeconomic factor models**

These use observable economic time series as the factors.

Examples: rate of inflation, economic growth, short term interest rates, yields on long-term government bonds, yield margin on corporate bonds over government bonds.

Fundamental factor models

These use company specific variables as the factors.

Examples: level of gearing, price earnings ratio, the level of R & D spending, the industry group to which the company belongs

Statistical factor models

Principal components analysis is used to determine a set of indices which explain as much as possible of the observed variance.

These indices are unlikely to have any meaningful economic interpretation and may vary considerably between different data sets.

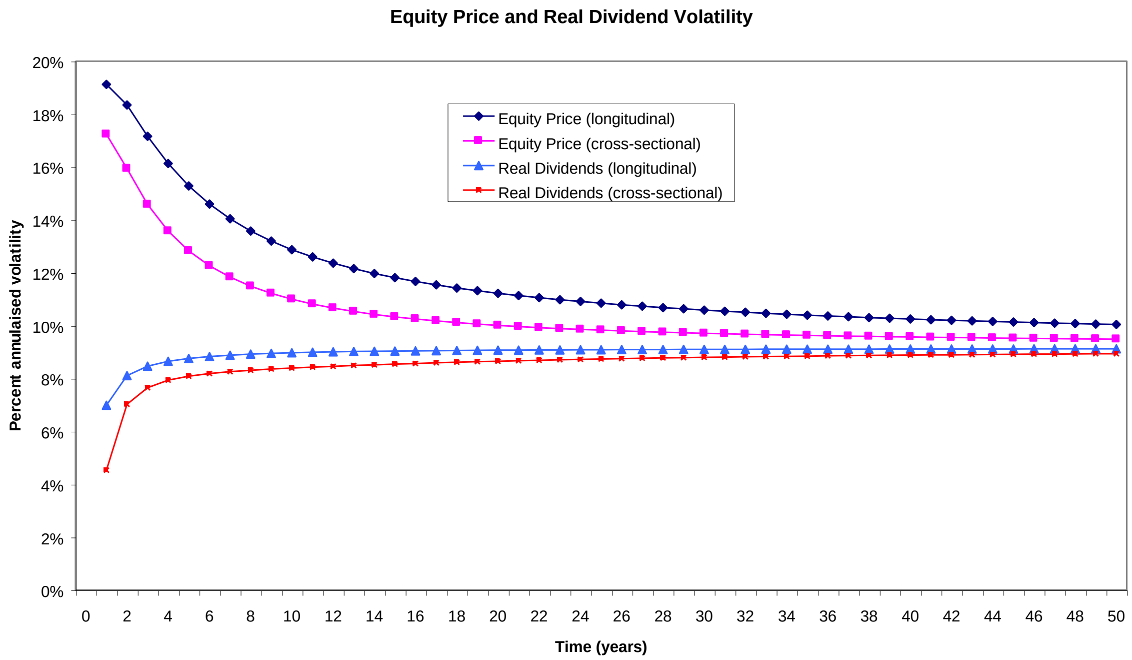
2

(i) Cross-sectional property fixes a time horizon and looks at the distribution over all the simulations. E.g. what will inflation be next year? The estimates are implicitly conditional on past information. They can be deduced from prices of options and other derivatives.

Longitudinal property looks at the distribution over a long period of time. E.g. what will the distribution of inflation be over the next 1000 years? Unlike cross sectional properties does not reflect market conditions at a particular time.

(ii)

a. Estimates are the same — random walk returns are independent across years.
b. The main point is longitudinal volatilities are higher. Longitudinal volatilities represent unconditional values whilst cross-sectional volatilities depend on the information set. The difference between the two shows the value of extra information. Over long horizons the two values converge to the same point. Students might draw a graph similar to that on Unit 6 page 13.



3

(i) **Investment A**

Expected return = $E[0.1 + N] = 0.1 + 1 = 1.1$
Variance = 1

Investment B

Expected return = $1.5 \times 0.99 - 5.0 \times 0.01 = 1.435$
Variance = $(1.435 - 1.5)^2 \times 0.99 + (1.435 - (-5))^2 \times 0.01 = 0.418275$

Investment B has both higher expected return and lower variance so would be preferred on this basis. However there is an issue with the possibility of very bad returns. Also there might be an issue with the estimated probabilities of investment B being somewhat unreliable as they are probably derived from the fat tail part of a distribution. Thus it might be wise to have a margin of error regarding this calculation in particular.

(ii)

a. **Investment A**

Semivariance = 0.5

Investment B

Semivariance = $(1.435 - (-5))^2 \times 0.01 = 0.41409$

b. **Investment A**

Shortfall probability of return below 0.
This is probability of the return from $N(1,1)$ being below $-0.1 = 0.13567$.

Investment B

Shortfall probability of return below 0 is 0.01.

c. **Investment A**

Shortfall probability of return below –2.
This is probability of the return from $N(1,1)$ being below $-2.1 = 0.00097$.

Investment B

Shortfall probability of return below –2 is 0.01.

(iii) Definitions of VAR, Tail VaR and Expected Shortfall. (Unit 1, page 3)
Clearly will show high but unlikely risk in the tail of Investment B.

(iv) Give points for sensible discussion:

Might not always be best to optimise on basis of expected return and variance.

No one "correct" measure of risk — different definitions give different orderings in this case.

Which investment is preferable depends somewhat on solvency of company — can they afford large, albeit unlikely, losses (analogies with "credit crunch").

Consider the rest of portfolio.

4

(i) Expected Return = $\Sigma_i x_i E_i$

where E_i is expected return on security i.

Variance is $\Sigma_i \Sigma_j x_i x_j C_{ij}$

where C_{ij} is the covariance of the returns on securities i and j and $C_{ii} = V_i$

where V_i is variance of security i.

(Unit 2 page 2)

(ii) Proportion in A = $(V_B - C_{AB}) / (V_A + V_B - 2C_{AB})$

From (Unit 2 page 3) or can fairly easily be calculated from first principles

$C_{AB} = 1 \times sd_A \times sd_B = 1 \times 4\% \times 2\% = 8\%\%$

Thus Proportion in A = (4%% – 8%%) / (16%% + 4%% – 2 × 8%%) = –1
Proportion in B = 2
i.e. Short sell a unit of A and buy 2 of B.

(iii) Expected Return of portfolio in (ii) is $-1 \times 4 + 2 \times 3 = 2\%$

Variance = $1^2 \times 16\%\% + 2^2 \times 4\%\% + 2 \times 2 \times -1 \times 2\% \times 4\% = 0$
(i.e. risk free)

Now if we borrow at 1% p.a. can invest in the portfolio in (ii) make a return of 2% pay back the loan and will have make 1% over the year.

5

(i) The formula states that

$P(\max_{0 \le s \le t} B_s + \mu s > y) = \Phi((-y + \mu t) / \sqrt{t}) + e^{2\mu y}\Phi(-(y + \mu t) / \sqrt{t})$.

Thus the density of $\max_{0 \le s \le t} B_s + \mu s$ is

$-d/dy(\Phi((-y + \mu t) / \sqrt{t}) + e^{2\mu y}\Phi(-(y + \mu t) / \sqrt{t}))$

$= 2\mu e^{2\mu y}\Phi(-(y + \mu t) / \sqrt{t})) + e^{2\mu y} \exp(-(y + \mu t)^2 / t) / \sqrt{(2\pi t)})$
$\qquad + \exp(-(-y + \mu t)^2/t) / \sqrt{(2\pi t)})$

$= 2\mu e^{2\mu y}\Phi(-(y + \mu t) / \sqrt{t})) + 2 \exp(-(-y + \mu t)^2 / t) / \sqrt{(2\pi t)})$. [4]

(ii) We need to price the derivative under the risk neutral measure. Under this measure,

$\max_{0\le s\le T} S_s = S_0 \exp(\max_{0\le s\le T} \sigma B_s + s(r - 1/2\ \sigma^2))$
$= S_0\exp(\sigma(\max_{0\le s\le T} B_s + \mu\ s))$,

with $\mu = (r - 1/2\ \sigma^2) / \sigma$, and B a standard Brownian motion. Thus the price is

$E[e^{-rT}S_0\exp(\sigma(\max_{0\le s\le T} B_s + \mu\ s))]$
$= e^{-rT}S_0\int[2\ e^{-(-y + \mu T)^2/2T} / \sqrt{(2\pi T)} - 2\mu e^{2\mu y}\Phi(-(y + \mu T)/\sqrt{T})]\exp(\sigma y)dy$ [4]

[Total 8]

6

(i) The model is a continuous time Markov with two states: N (not previously defaulted) and D (previously defaulted). Under this simple model it is assumed that the default-free interest rate term structure is deterministic with $r(t) = r$ for all t. If the transition intensity, under the real-world measure P, from N to D at time t is denoted by $\lambda(t)$, this model can be represented as:

$$\boxed{\text{No default, } N} \xrightarrow{\lambda(t)} \boxed{\text{Default, } D}$$

and D is an absorbing state.

If $X(t)$ is the state at time t, the transition intensity, $\lambda(t)$, can be interpreted as:

$\Pr_P(X(t + dt) = N \mid X(t) = N) = 1 - \lambda(t)\ dt + o(dt)$ as $dt \to 0$,
$\Pr_P(X(t + dt) = D \mid X(t) = N) = \lambda(t)\ dt + o(dt)$ as $dt \to 0$.

(ii) We need to show that under P_λ, $E[e^{-rt} D_t |F_s] = e^{-rs} D_s$.
If the default time $\tau \le s$ then $e^{-rs} D_s = e^{-rs}\ \delta\ e^{-r(2-s)} = e^{-rt} D_t$.
If $s \le \tau$, then $e^{-rt} D_t = \delta e^{-2r}$, if default has occurred prior to time t

$= e^{-2r}(\delta(1 - e^{-\lambda(2-t)}) + e^{-\lambda(2-t)})$ otherwise.
Thus,
$E[e^{-rt} D_t |F_s] = \delta e^{-2r}(1- e^{-\lambda(t-s)}) + e^{-2r}(\delta(1 - e^{-\lambda(2-t)}) + e^{-\lambda(2-t)})\ e^{-\lambda(t-s)}$

$= e^{-2r}((\delta(1 - e^{-\lambda(2-s)}) + e^{-\lambda(2-s)})$ which is $e^{-rs} D_s$ in this case.

(iii) Seeking a portfolio of the form $aD + bB$, we need the value to be 1000 at time 2 if default has occurred, so we need $a\delta + b = 1000$. Similarly we need the value to be zero at time 2 if default has not occurred, so we need $a + b = 0$. Hence $b = -a$ and $b = 1000/(1 - \delta)$.
The fair price for the derivative must be the set-up cost for this portfolio which is $be^{-2r} + aD_0 = 1000/(1 - \delta)(e^{-2r} - e^{-2r}(\delta(1 - e^{-2\lambda}) + e^{-2\lambda})) = 1000\, e^{-2r}(1 - e^{-2\lambda})$

(iv) We need to check that the initial value of this portfolio is $E[e^{-2r}V]$ under P_λ : under P_λ , $E[e^{-2r}V] = 1000e^{-2r}P(default) = 1000e^{-2r}(1 - e^{-2\lambda})$ as required (V is the final value of the derivative).

This then accords with the fact that if we can hedge without arbitrage then the price is that given by the EMM

7

(i) The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion.
2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
4. Unlimited short selling (that is, negative holdings) is allowed.
5. There are no taxes or transaction costs.
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

(ii) It is clear that each of these assumptions is unrealistic to some degree, for example:

- Share prices can jump. This invalidates assumption 1. since geometric Brownian motion has continuous sample paths. It also invalidates assumption 2. However, hedging strategies can still be constructed which substantially reduce the level of risk.
- The risk-free rate of interest does vary and in a unpredictable way. However, over the short term of a typical derivative the assumption of a constant risk-free rate of interest is not far from reality. (More specifically the model can be adapted in a simple way to allow for a stochastic risk-free rate, provided this is a predictable process.)
- Unlimited short selling may not be allowed except perhaps at penal rates of interest. These problems can be mitigated by holding mixtures of derivatives which reduce the need for short selling. This is part of a suitable risk management strategy as discussed in Section 2 below.
- Shares can normally only be dealt in integer multiples of one unit, not continuously and dealings attract transaction costs: invalidating assumptions 2., 5., 6. and 7. Again we are still able to construct suitable hedging strategies which substantially reduce risk.

- Distributions of share returns tend to have fatter tails than suggested by the log-normal model, invalidating assumption 1.

8

(i) A corollary to the Cameron-Martin-Girsanov theorem states that there exists a process η_t such that, for any F_T -measurable derivative payoff X at time T,

$$E_Q[X \mid F_t] = E_P\left[\frac{\eta_T}{\eta_t} X \mid F_t\right].$$

Define $A_t = e^{-rt}\eta_t$

The process A_t is called a state-price deflator (also *deflator*; *state-price density*; *pricing kernel*; or *stochastic discount factor*). [3]

(ii) If we define $\eta_t = \exp(-\gamma Z_t - ½\gamma^2 t)$, where $\gamma = (\mu - r)/\sigma$, then the state price deflator is $A_t = \eta_t e^{-rt}$. [3]

(iii) If a contract has terminal value V then its price at time t is $E_P[A_T V/A_t|F_t]$. So in this case we obtain

$p_t = E_P[\exp(\gamma Z_1)\ 1_{(Z1>1)}\exp(-\gamma Z_1 - ½\gamma^2)\ e^{-r}/\ (\exp(-\gamma Z_t - ½\gamma^2 t)\ e^{-rt})|F_t]$
$= P[Z_1 > 1|F_t]\exp(\gamma Z_t - (½\gamma^2 + r)(1 - t))$
$= (1 - \Phi((1 - Z_t) / \sqrt{(1 - t)}))\ \exp(\gamma Z_t - (½\gamma^2 + r)(1 - t))$ [5]

9

The real-world probability measure P can be interpreted in the following way. Let A be some event contained in F (for example, suppose that A is the event that S_1 is greater than or equal to 100). Then $P(A)$ is the *actual* probability that the event A will occur. On a more intuitive level with m independent realisations of the future instead of one we would find that the event A occurs on approximately a proportion $P(A)$ occasions (with the approximation getting better as m gets larger and larger).

Two measures P and Q which apply to the same sigma-algebra F are said to be *equivalent* if for any event E in F : $P(E) > 0$ if and only if $Q(E) > 0$, where $P(E)$ and $Q(E)$ are the probabilities of E under P and Q respectively.

In the context of the binomial model and using the above definition of equivalence the only constraint on the real-world measure P is that at any point in the binomial tree the probability of an up move lies strictly between 0 and 1. The only constraint on Q is the same but this can be equated to the requirement that the risk-free return must lie strictly between the return on a down move and the return on an up move. This gives us considerable flexibility in the range of possible equivalent measures.

10

Consider an investor holding an American call. She needs some cash at time $t < T$. Two strategies are available for her:

(i) She sells the options on the market and gets the price of it in exchange C_t^A,

(ii) She exercises the option and obtains the intrinsic value $S_t - K$.

But, we know that $C_t^A \geq C_t^E \geq \max\left(0; S_t - K\exp\left(-r\left(T-t\right)\right)\right)$.

Hence $C_t^A \geq \max\left(0; S_t - K\exp\left(-r\left(T-t\right)\right)\right) \geq \left(S_t - Ke^{-r(T-t)}\right) \geq S_t - K$.

As a consequence, the first strategy is better and it is never optimal for the agent to exercise her option early.

11

(i) $B(t,T)$ = Zero-coupon bond price
= price at t for £1 payable at T
$r(t)$ = instantaneous risk-free rate of interest at t

Take a specific bond with maturity at T_1. Suppose its SDE under the real-world measure P is

$$dB(t,T_1) = B(t,T_1)\left[m(t,T_1)dt + S(t,T_1)dW(t)\right]$$

where, besides $S(t, T_1)$, $m(t, T_1)$ might be stochastic. The *market price of risk* is defined as

$$\gamma(t,T_1) = \frac{m(t,T_1) - r(t)}{S(t,T_1)}.$$

(ii) Define C_t = cash account at time t

Portfolio A: a_t units of $B(t, T_2)$ and b_t units of C_t

Portfolio *B*: 1 unit of $B(t, T_1)$

Self financing implies:

$$a_tB(t, T_2) + b_tC_t = B(t, T_1)$$

and

$$a_t\, dB(t, T_2) + b_t\, dC_t = dB(t, T_1)$$

So

$$a_t\, B(t, T_2)\, [m(t, T_2)\, dt + S(t, T_2)\, dW_t]$$
$$+\, b_t r_t C_t\, dt = B(t, T_1)\, [m(t, T_1)\, dt + S(t, T_1)\, dW_t]$$

equating the coefficients of dt and dW_t we obtain

$$a_t = \frac{S(t,T_1)B(t,T_1)}{S(t,T_2)B(t,T_2)}$$

and $b_t = \dfrac{1}{r_t C_t}\left[m(t,T_1)B(t,T_1) - \dfrac{m(t,T_2)B(t,T_1)S(t_1 T_1)}{S(t,T_2)}\right]$

Substituting these back:

$$\frac{S(t_1 T_1)B(t,T_1)}{S(t,T_2)B(t,T_2)}B(t,T_2) +$$

$$B_t \frac{1}{r_t B_t}\left[m(t,T_1)B(t,T_1) - \frac{m(t,T_2)B(t,T_1)S(t,T_1)}{S(t,T_2)}\right]$$

$$= B(t,T_1)$$

Simplifying:

$$\frac{S(t,T_1)}{S(t,T_2)} + \frac{1}{r_t}\left[m(t,T_1) - \frac{m(t,T_2)S(t_1 T_1)}{S(t,T_2)}\right] = 1$$

$$\Rightarrow \quad \frac{m(t,T_1) - r_t}{S(t,T_1)} = \frac{m(t,T_2) - r_t}{S(t,T_2)}$$

END OF EXAMINERS' REPORT

EXAMINATION

28 April 2010 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all nine questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Let $\left(X_t; t \geq 0\right)$ be a stochastic process satisfying:

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

where W is a standard Brownian motion.

Let $f : \Re \times \Re \to \Re$ be a function, twice partially differentiable with respect to *x*, once with respect to *t*.

(i) State the stochastic differential equation for $f\left(t, X_t\right)$. [2]

Let $dX_t = -\gamma X_t dt + \sigma dW_t$.

(ii) Prove that the solution of this stochastic differential equation is given by:

$$X_t = X_0 \exp\left(-\gamma t\right) + \sigma \int_0^t \exp\left(\gamma\left(s - t\right)\right) dW_s$$ [6]

[Total 8]

2 Consider a stock paying a dividend at a rate δ and denote its price at any time *t* by S_t. The dividend earned between *t* and *T*, $T \geq t$, is $S_t(e^{\delta(T-t)} - 1)$.

Let C_t and P_t be the price at time *t* of a European call option and European put option respectively, written on the stock *S*, with strike price *K* and maturity $T \geq t$. The instantaneous risk-free rate is denoted by *r*.

Prove put-call parity in this context by adapting the proof of standard put-call parity that applies to put and call options on a non-dividend paying stock. [8]

3 Consider a two-period binomial model for a non-dividend paying stock whose current price is $S_0 = 100$. Assume that:

- over each six-month period, the stock price can either move up by a factor $u = 1.2$ or down by a factor $d = 0.8$
- the continuously compounded risk-free rate is $r = 5\%$ per six-month period

(i) (a) Prove that there is no arbitrage in the market.
(b) Construct the binomial tree.
[2]

(ii) Calculate the price of a standard European call option written on the stock S with strike price $K = 100$ and maturity one year. [5]

Consider a special type of call option with strike price $K = 100$ and maturity one year. The underlying asset for this special option is the average price of the stock over one year, calculated as the average of the prices at times 0, 0.5 and 1 measured in years.

(iii) Calculate the initial price of this call option assuming it can be exercised only at time 1. [5]
[Total 12]

4 Consider the following stochastic differential equation for the instantaneous risk free rate (also referred to as the short-rate):

$$dr(t) = a\left(b - r(t)\right)dt + \sigma dW_t$$

Its solution is given by:

$$r(t) = r_0 \exp(-at) + b\left(1 - \exp(-at)\right) + \sigma \exp(-at)\int_0^t \exp(as)\,dW_s$$

You may also use the fact that for $T > t$:

$$\int_t^T r(u)du = b(T-t) + \left(r(t) - b\right)\frac{1 - \exp\left(-a(T-t)\right)}{a} + \frac{\sigma}{a}\int_t^T \left(1 - \exp\left(-a(T-s)\right)\right)dW_s$$

(i) Derive the price at time t of a zero-coupon bond with maturity T. [10]

(ii) (a) State the main drawback of such a model for the short-rate.

(b) State the name and stochastic differential equation of an alternative model for the short-rate that is not subject to the drawback.
[2]
[Total 12]

5 A European call option on a stock has an exercise date one year away and a strike price of 320p. The underlying stock has a current price of 350p. The option is priced at 52.73p. The continuously compounded risk-free interest rate is 4% p.a.

(i) Estimate the stock price volatility to within 0.5% p.a. assuming the Black-Scholes model applies. [5]

A new derivative security has just been written on the underlying stock. This will pay a random amount D in one year's time, where D is 100 times the terminal value of the call option capped at 1p (i.e. 100 times the lesser of the terminal value and 1p).

(ii) (a) State the payoff for this derivative security in terms of two European call options.

(b) Calculate the fair price for this derivative security.

[5]

(iii) Calculate the risk neutral probability that the stock price is greater than 320p.

[4]

[Total 14]

6 (i) Describe the-two state model for credit ratings under the real world measure.

[9]

(ii) Explain how the two state model is generalised in the Jarrow-Lando-Turnbull model. [3]

[Total 12]

7 (i) State the Cameron-Martin-Girsanov Theorem. [3]

(ii) Derive the value of a which makes $\exp(\sigma B_t - at)$ a martingale when B is a standard Brownian Motion. [3]

In a Black-Scholes market, the stock price is given by:

$S_t = S_0 \exp(0.2B_t + 0.2t)$, where B is a standard Brownian Motion under the real-world measure.

A derivative security written on this stock in the same market has price:

$D_t = 2\exp(0.6B_t + 0.39t)$ at time t.

(iii) (a) Calculate the value of c such that $B_t + ct$ is a standard Brownian Motion under the Equivalent Martingale Measure.

(b) Calculate the risk-free rate of interest.

[8]

[Total 14]

8 Outline the main points you would make in a discussion of the statement:

The efficient markets hypothesis states that the market price is always correct and therefore it is not possible for investors to make money from investing in shares.

[10]

9 An asset is worth 100 at the start of the year and is funded by a senior loan and a junior loan of 50 each. The loans are due to be repaid at the end of the year; the senior one with interest at 6% p.a. and the junior one with interest of at 8% p.a. Interest is paid on the loans only if the asset sustains no losses.

Any losses of up to 50 sustained by the asset reduce the amount returned to the investor in the junior loan by the amount of the loss. Any losses of more than 50 mean that the investor in the junior loan gets 0 and the amount returned to the investor in the senior loan is reduced by the excess of the loss over 50.

The probability that the asset sustains a loss is 0.25. The size of a loss, L, if there is one, follows a uniform distribution between 0 and 100.

(i) Calculate the variances of return for the investors in the junior and senior loans. [8]

(ii) Calculate the shortfall probabilities for the investors in the junior and senior loans, using the full return of the amounts of the loans as the respective benchmarks. [2]

[Total 10]

END OF PAPER

EXAMINERS' REPORT

April 2010 examinations

Subject CT8 — Financial Economics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

July 2010

1 (i) Write down Ito's formula for $f(t,X_t)$ when $dX_t = \mu_t dt + \sigma_t dW_t$

$$\begin{aligned} df(t,X_t) &= \frac{\partial f(t,X_t)}{\partial t}dt + \frac{\partial f(t,X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\left(dX_t\right)^2 \\ &= \frac{\partial f(t,X_t)}{\partial t}dt + \frac{\partial f(t,X_t)}{\partial x}\left(\mu_t dt + \sigma_t dW_t\right) + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\left(\sigma_t^2 dt\right) \\ &= \left(\frac{\partial f(t,X_t)}{\partial t} + \frac{\partial f(t,X_t)}{\partial x}\mu_t + \frac{1}{2}\frac{\partial^2 f(t,X_t)}{\partial x^2}\sigma_t^2\right)dt + \frac{\partial f(t,X_t)}{\partial x}\sigma_t dW_t \end{aligned}$$

(ii) Consider $X_t = U_t\, e^{-\gamma t}$.

Then

$$dU_t = d(e^{\gamma t} X_t) = \gamma e^{\gamma t} X_t\, dt + e^{\gamma t}\, dX_t$$

$$= \gamma e^{\gamma t} X_t\, dt + e^{\gamma t}(-\gamma\, X_t\, dt + \sigma dW_t) = \sigma e^{\gamma t}\, dW_t\,.$$

Thus

$$U_t = U_0 + \sigma \int_0^t e^{\gamma s} dW_s$$

and consequently

$$X_t = e^{-\gamma t}\, U_t = X_0 e^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)} dW_s$$

2 The proof of this result is an adaptation of that of the standard call-put parity. Two (self-financing) portfolios are considered:

- Portfolio A: buying the call and selling the put at time t. Its value at time t is $P_t - C_t$ and at time T, it is $S_T - K$ in all states of the universe.

- Portfolio B: buying a fraction $\exp\left(-\delta\left(T-t\right)\right)$ of the underlying asset for $S_t \exp\left(-\delta\left(T-t\right)\right)$ and borrowing $K\exp\left(-r\left(T-t\right)\right)$ at time t. Its value at time t is then $K\exp\left(-r\left(T-t\right)\right) - S_t \exp\left(-\delta\left(T-t\right)\right)$. Its value at maturity is then $S_T - K$ by taking into account the dividends which are paid continuously at rate δ.

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time t. Hence:

$$C_t - P_t = S_t \exp\left(-\delta\left(T-t\right)\right) - K\exp\left(-r\left(T-t\right)\right).$$

Another proof can include the following portfolios:

Portfolio A: At time t, buying a call option and lending $K\exp\left(-r\left(T-t\right)\right)$
Portfolio B: At time t, buying the put option and buying one share.

3 (i) There is no arbitrage in the market since $d = 0.8 < \exp(0.05) < u = 1.2$.

(ii) To price the call option, we use the risk-neutral pricing formula. We use the following simplifying notation:

$$C_{uu} = \left(u^2 S_0 - K\right)^+ = 44;$$
$$C_{ud} = \left(udS_0 - K\right)^+ = 0;$$
$$C_{dd} = \left(d^2 S_0 - K\right)^+ = 0.$$

At time 1, we get in the upper state,

$$C_1\left(u\right) = \exp\left(-r\right)\left[qC_{uu} + \left(1-q\right)C_{ud}\right] = 26.29,$$

and in the lower state

$$C_1\left(d\right) = \exp(-r)\left[qC_{ud} + \left(1-q\right)C_{dd}\right] = 0$$

where the risk-neutral probability of an upward move is

$$q = \frac{\exp\left(r\right) - d}{u-d} = 0.628.$$

At time 0,

$$C_0 = \exp\left(-r\right)\left[qC_1(u) + \left(1-q\right)C_1(d)\right].$$

Hence

$$C_0 = 15.71.$$

(iii) For the special option, we need to compute the average for the different possible trajectories, the probability of each path and the associated payoff:

trajectory	*average*	*probability*	*payoff of the option*
$up-up$	$X_{uu}=121.33$	$q^2=0.394$	$(X_{uu}-K)^+=21.33$
$up-down$	$X_{ud}=105.33$	$q(1-q)=0.234$	$(X_{ud}-K)^+=5.33$
$down-up$	$X_{du}=92$	$(1-q)q=0.234$	$(X_{du}-K)^+=0$
$down-down$	$X_{dd}=81.33$	$(1-q)^2=0.138$	$(X_{dd}-K)^+=0$

The price of the option is obtained as

$$X_0=\exp(-2r)\left(q^2(X_{uu}-K)^+ + q(1-q)(X_{ud}-K)^+\right)=8.744.$$

4 (i) The price of a zero-coupon bond can be written as

$$B(t,T)=E\left[\exp\left(-\int_t^T r(s)ds\right)|F_t\right].$$

Since $\int_t^T r(u)du$ is a Gaussian random variable, we can compute explicitly the price of the zero-coupon bond in terms of the expected value and variance (conditional) of $\int_t^T r(u)du$:

$$B(t,T)=\exp\left[-E\left[\int_t^T r(s)ds|F_t\right]+\frac{1}{2}V\left[\int_t^T r(s)ds|F_t\right]\right]$$

where $E\left[\int_t^T r(s)ds|F_t\right]=b(T-t)+(r(t)-b)\left(\frac{1-\exp(-a(T-t))}{a}\right)$ and

$$V\left[\int_t^T r(s)ds|F_t\right]=\frac{\sigma^2}{a^2}(T-t)-\frac{\sigma^2}{2a^3}\left(\exp(-2a(T-t))-1\right)+\frac{2\sigma^2}{a^3}\left(\exp(-a(T-t))-1\right).$$

(ii) Main issue: possibility to have negative interest rates when using the Vasicek model. An alternative is the CIR model:

$$dr(t)=a(b-r(t))dt+\sigma\sqrt{r(t)}dW_t.$$

5 (i) Try $\sigma = 10\%$. Black Sholes formula gives a price of $p_{10} = 44.05$.
Try $\sigma = 40\%$. Black Sholes formula gives a price of $p_{40} = 76.05$.

Interpolating gives a trial value of
$(76.05 - 52.73) / (76.05 - 44.05) * 10 + (52.73 - 44.05) / (76.05 - 44.05) * 40 = 20.2\%$.

Evaluating gives $p_{20.2} = 52.96$.

Interpolation with p_{40} give
$\sigma = ((76.05 - 52.73) * 20.2 + (52.73 - 52.96) * 40) / (76.05 - 52.96) = 21.9\%$ (to the nearest .5%)

$p_{21.9} = 54.75$.

Actual answer is 20%.

(ii) The payoff is
$100\min(1, \max(S_T - 320,0)) = 100(\max(S_T - 320,0) - \max(S_T - 321,0))$
so is 100 times the difference between two call options with the corresponding strikes.

Using the Black-Scholes formula, the price of the second call option is 52.06p

and hence the value of the derivative is $p = 100 * (52.73 - 52.06) = 67$p.

(iii) The option essentially pays £1 if the final security price is greater than 320p.

Thus its price is approximately $e^{-r}P(S_1 > 320)$ (where P is the EMM).

So

$$P(S_1 > 320) = e^{.04} * p = 0.70.$$

Other answers are also possible. In particular, using the distribution of $\ln\left(\frac{S_1}{S_0}\right)$ and use it to calculate the probability directly.

6 (i) A model can be set up, in continuous time, with two states N (not previously defaulted) and D (previously defaulted). Under this simple model it is assumed that the default-free interest rate term structure is deterministic with $r(t) = r$ for all t. If the transition intensity, under the real-world measure P, from N to D at time t is denoted by $\lambda(t)$, this model can be represented as:

$$\boxed{\text{No default, } N} \xrightarrow{\lambda(t)} \boxed{\text{Default, } D}$$

and D is an absorbing state.

If $X(t)$ is the state at time t. The transition intensity, $\lambda(t)$, can be interpreted as:

$$\Pr_P(X(t+dt) = N \mid X(t) = N) = 1 - \lambda(t)\,dt + o(dt) \qquad \text{as } dt \to 0,$$
$$\Pr_P(X(t+dt) = D \mid X(t) = N) = \lambda(t)\,dt + o(dt) \qquad \text{as } dt \to 0.$$

Another correct answer will be:

$$\Pr_P(X(T) = N \mid X(t) = N) = \exp\left(-\int_t^T \lambda_s ds\right)$$

$$\Pr_P(X(T) = \mathrm{D} \mid X(t) = N) = 1 - \exp\left(-\int_t^T \lambda_s ds\right)$$

If τ is a stopping time defined as:

$$\tau = \inf\{t : X(t) = D\} \text{ (with } \inf \emptyset = \infty)$$

and if $N(t)$ is a counting process defined as:

$$N(t) = \begin{cases} 0 & \text{if } \tau > t, \\ 1 & \text{if } \tau \le t. \end{cases}$$

Then τ can be interpreted as the time of default and $N(t)$ can be interpreted is the number of defaults up to and including time t.

It is assumed that if the corporate entity defaults all bond payments will be reduced by a known, deterministic factor $(1 - \delta)$ where δ is the recovery rate, i.e. for a zero-coupon bond which is due to pay 1 at time T, the actual payment at time T will be 1 if $\tau > T$ and δ if $\tau \le T$.

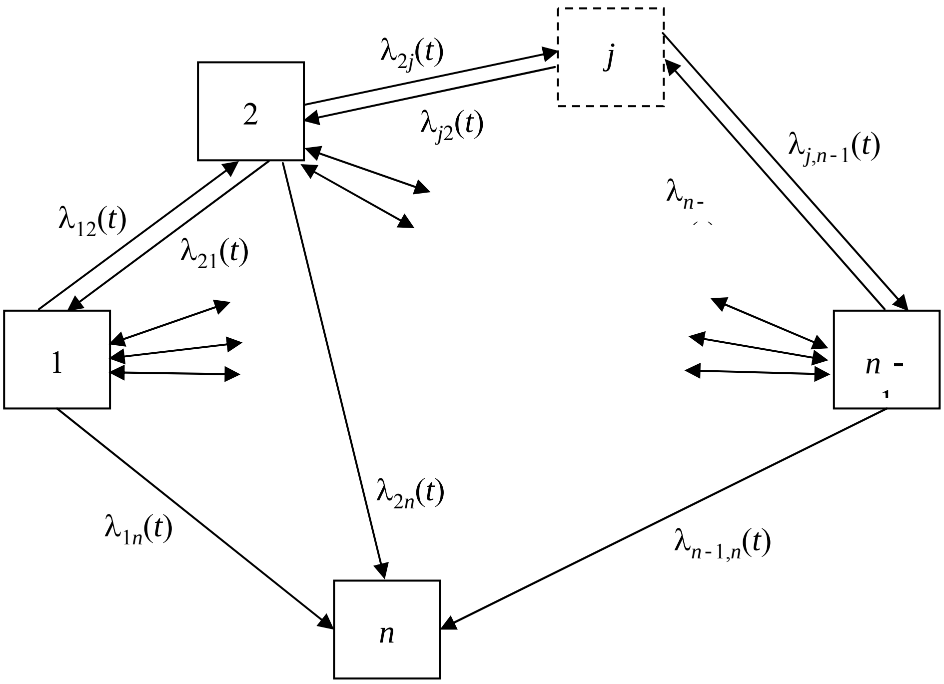
The formula for the zero-coupon bond price is

$$P(t,T) = B(t,T)\left(1-(1-\delta)\left(1-\exp\left(-\int_t^T \lambda_s ds\right)\right)\right)$$

Where $P(t,T)$ is the price at time t of a risky zero-coupon bond and $B(t,T)$ is the price at time t of a risk-free zero-coupon bond.

(ii) A more general and more realistic model with multiple credit ratings rather than the simple default/no default model, used above was developed by Jarrow, Lando and Turnbull. In this model there are n – 1 credit ratings plus default.

If the transition intensities, under the real-world measure P, from state i to state j at time t are denoted by $\lambda_{ij}(t)$ where the $\lambda_{ij}(t)$ are assumed to be deterministic then this model for default risk can be represented by the following diagram:



In this n-state model transfer is possible between all states except for the default state n, which is absorbing.

7 (i) (The Cameron-Martin-Girsanov theorem)

Suppose that Z_t is a standard Brownian motion under P. Furthermore suppose that γ_t is a previsible process. Then there exists a measure Q equivalent to P and where $\overline{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q.

Conversely, if Z_t is a standard Brownian motion under P and if Q is equivalent to P then there exists a previsible process γ_t such that $\overline{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a Brownian motion under Q.

(ii) The answer is $a = \frac{1}{2}\sigma^2$. This can be proved using two different approaches: for showing all working correctly using one of the approaches below.

(1) Write the martingale condition and consider the expected value of the process at time t, conditional on the filtration up to an earlier time s.

(2) Write Ito's formula for the function $f(t,B_t)$ = exp(sigma $B_t - a_t$), and set the drift term equal to 0.

(iii) We know that $e^{-rt} S_t$ is a martingale under the EMM and so is $e^{-rt}D_t$. So, setting $W_t = B_t + ct$ we can write $e^{-rt} S_t = S_0\exp(0.2W_t - (r + 0.2c - 0.2)t)$ and we require $r + 0.2c - 0.2 = 1/2(0.2)^2 = 0.02$.

Similarly, we can write $e^{-rt}D_t = 2\exp(0.6W_t - (r + 0.6c - 0.39)t)$ and we then require $r + 0.6c - 0.39 = 1/2(0.6)^2 = 0.18$.

Eliminating r from these two equations gives
$0.4c - 0.19 = 0.16$, or $0.4c = 0.35$ so $c = 0.875$.

Substituting in the first equation gives $r + 0.175 - 0.2 = 0.02$ so $r = 4.5\%$.

8

- EMH states that market fully reflects all available information and the implication is therefore that investors are not able to make "excess" returns (rather than any returns at all!).

- 3 forms of EMH defining what type of information is available: weak for historical price information, semi-strong for all public information and strong for all information.

- Although illegal, insider information appears to enable investors to make money. Reasonable to conclude the other way round as studies of directors' share dealings suggest that, even with inside information, it is difficult to out-perform.

- Difficult to define publicly available information – might be that some very difficult-to-obtain information enables profits but at a high cost of obtaining the information.

- Investors taking higher risks may earn higher returns – this does not contradict the EMH.

- EMH does not specify how information is priced, so very difficult to test.

- Conflicting empirical evidence from supporters and detractors.

- Difficult to determine when, precisely, information arrives.

9 (i) Let J be the return to the investor in the junior loan.

$J = 54$, with probability 0.75

$= 0$ with probability $0.25 * 0.5$

$= 50 * U$ with probability $0.25 * 0.5$, where U is uniform over (0,1)

$E[J] = 0.75 * 54 + 0.25 * 0.5 * 0 + 0.25 * 0.5 * 0.5 * 50 = 43.625$

$E[J^2] = 0.75 * 54^2 + 0 + 0.25 * 0.5 * 50^2 * E[U^2]$
$= 0.75 * 54^2 + 312.5 * (0.25 + 0.083) = 2291$

$\text{Var}[J] = 2291 - 43.625^2 = 388$

S = return to investor in senior loan

$S = 53$ with prob 0.75

$= 50$ with prob $0.25 * 0.5$

$= 50 * U$ with prob $0.25 * 0.5$

$E[S] = 0.75 * 53 + 0.125 * 50 + 0.125 * 50 * 0.5 = 49.125$

$E[S^2] = 0.75 * 53^2 + 0.125 * 50^2 + 0.125 * 50^2/3 = 2523$
$\text{Var}[S] = 2523 - 49.125^2 = 110$

Alternative answers:

The word "return" can be interpreted in different ways, leading to several possible answers.

In the detailed solution above, it is total return.

If using percentage return, as a percentage, then

$J = 0.08$ with probability 0.75, -1 with probability 0.25 * 0.5 and $U - 1$ with probability 0.25 * 0.5 with U uniformly distributed over [0,1]

The expected value is then $E(J) = -0.1275$ and the variance is $V(J) = 0.1552$

$S = 0.06$ with probability 0.75, 0 with probability 0.25 * 0.5 and $U - 1$ with probability 0.25 * 0.5 with U uniformly distributed over [0,1].

The expected value is then $E(S) = -0.0025$ and the variance is $V(S) = 0.0441$.

(ii) $\Pr(J < 50) = 0.25$
$\Pr(S < 50) = 0.125$

END OF EXAMINERS' REPORT

EXAMINATION

8 October 2010 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all nine questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 An investor holds an asset that produces a random rate of return, R, over the course of a year. The distribution of this rate of return is a mixture of normal distributions, i.e. R has a normal distribution with a mean of 0% and standard deviation of 10% with probability 0.8 and a normal distribution with a mean of 30% and a standard deviation of 10% with a probability of 0.2.

S is the normally distributed random rate of return on another asset that has the same mean and variance as R.

(i) Calculate the mean and variance of R. [3]

(ii) Calculate the shortfall probabilities for R and for S using:

(a) a benchmark rate of return of 0%
(b) a benchmark rate of return of –10% [4]

(iii) Comment on what the variance and shortfall probabilities at both benchmark levels illustrate about the asset returns, by referring to the calculations in (i) and (ii). [3]
[Total 10]

2 (i) Explain what is meant by "excessive volatility" of share prices. [2]

(ii) State two examples of empirical evidence of the "under-reaction" of share prices to events. [4]
[Total 6]

3 Discuss whether one-factor models are good models for the short-rate of interest (instantaneous risk free rate). Include discussion of extensions that may be considered to improve the model. Illustrate your discussion by defining and referring to particular models. [10]

4 (i) In the context of credit risk for defaultable bonds:

(a) give three examples of a credit event
(b) give three examples of an outcome of a default
(c) define the recovery rate [7]

(ii) Describe the two-state model for credit ratings. [4]

Two companies have zero coupon defaultable bonds in issue. Bond A has £2m nominal in issue. Bond B has £3m nominal in issue. Both bonds redeem in exactly 2 years time.

Under a risk neutral measure, each bond defaults (not necessarily independently) at a constant rate. Both bonds have a 60% recovery rate.

Assume:

- a continuously compounded risk free rate of interest of 3% p.a.
- the issue of bond A is priced at £1.6m
- the issue of bond B is priced at £2.2m

(iii) Evaluate the two default rates (under a risk-neutral measure). [4]

There is also a traded derivative security, D, priced at £52 which pays £100 after 2 years if (and only if) at least one of the bonds defaults.

(iv) (a) Determine a hedging portfolio for the security which pays £100 after 2 years if and only if both bonds default by considering fixed portfolios consisting of bond A, bond B and security D and a risk-free zero-coupon bond paying £100 at redemption in exactly 2 years.

(b) Calculate the fair price for the security that pays £100 if and only if both bonds default.

[8]
[Total 23]

5 (i) State an expression for the price of a derivative security in a Black-Scholes market in terms of the risk-neutral measure. [2]

A European call option on a stock has an exercise date one year away and a strike of £6. The underlying stock has a current price of £5.50. The option is priced at 60p. The stock price volatility has been estimated from other option prices as 20%.

(ii) Estimate the risk free rate of interest to within 0.5% p.a. assuming the Black-Scholes model applies. [5]

A new derivative security has just been written on the underlying stock. This will pay a random amount D in one year's time, where $D = S_1^2$.

(iii) Calculate the fair price for this new derivative security, quoting any further results that you use. [5]

(iv) Determine the initial hedging portfolio (in units of the underlying stock and cash) for this new derivative security. [4]

[Total 16]

6 Under the real-world measure P, W is a standard Brownian motion and the price of a stock, S, is given by $S_t = S_0\exp(\sigma\, W_t + (\mu - 1/2\,\sigma^2)t)$. The continuously compounded risk-free rate of interest is r and a zero coupon bond with maturity T has price $B_t = e^{-r(T-t)}$. Suppose that in the market any contract which pays $f(S_T)$ at time T is valued at:

$$p_t = E[e^{-r(T-t)} f(S_T)\, \Lambda_T | F_t],$$

where:

$$\Lambda_t = \exp(mW_t - 1/2m^2t) \text{ for } t \le T \text{ for some real number } m.$$

(i) (a) Prove, using Ito's formula, that Λ_t is a martingale.

(b) Show that $E[\exp(mW_t)] = \exp(1/2m^2t)$.

[5]

(ii) (a) Derive an expression for p_0 when $f(x) = x$.

(b) Show that there is an arbitrage in the market unless $m = (r - \mu)/\sigma$.

[5]
[Total 10]

7 (i) Define delta, gamma and vega for an individual derivative. [3]

(ii) Explain how gamma and vega can be used in the risk management of a portfolio that is delta-hedged. [4]
[Total 7]

8 Consider a particular stock and denote its price at any time t by S_t. This stock pays a dividend D at time T'.

Let C_t and P_t be the price at time t of a European call option and European put option respectively, written on S, with strike price K and maturity $T \ge T' \ge t$. The instantaneous risk-free rate is denoted by r.

Prove the put-call parity in this context by adapting the proof of standard put-call parity.

[Hint: assume that when the dividend is paid it is used to pay off any borrowed positions required as part of the proof.] [7]

9 Consider a two-period binomial model for a non-dividend paying stock whose current price is $S_0 = 100$. Assume that:

- over each of the next six-month periods, the stock price can either move up by a factor $u = 1.2$ or down by a factor $d = 0.8$
- the continuously compounded risk-free rate is $r = 6\%$ per period

(i) (a) Prove that there is no arbitrage in the market.

(b) Construct the binomial tree for the model. [2]

(ii) Calculate the price of a standard European call option written on the stock S with strike price $K = 100$ and maturity one year. [5]

Consider a special European call option with strike price $K = 100$ and maturity one year. The owner of such an option has the right to exercise her option at the end of the year only if the stock price goes above the level $L = 130$ during or at the end of the year.

(iii) (a) Calculate the initial price of this call option.

(b) Comment on the relationship between the price of the special call option and the option in (ii). [4]

[Total 11]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2010 examinations

Subject CT8 — Financial Economics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

December 2010

1 (i) $E[R] = 0.8 * 0\% + 0.2 * 30\% = 6\%$
$E[R^2] = 0.8 * 10\%^2 + 0.2*(30\%^2 + 10\%^2) = 0.028$

(ii) $\text{Var}(R) = 0.028 - 0.06^2 = 0.0244 = 0.1562^2$

$\text{Prob}(R < 0) = 0.8 * N(0; 0, 10\%) + 0.2 * N(0; 30\%, 10\%) = 0.8 * 0.5 + 0.2 * 0.00135 = 0.4 + 0.00027 = 0.40027$

$\text{Prob}(S < 0\%) = N(0; 6\%, 15.62\%) = 0.3504$

$\text{Prob}(R < -10\%) = 0.8 * N(-10\%; 0, 10\%) + 0.2 * N(-10\%; 30\%, 10\%) = 0.8 * 0.1587 + 0.2 * 0 = 0.1269$

$\text{Prob}(S < 10\%) = N(-10\%; 6\%, 15.62\%) = 0.1528$

(iii) Variance suggests risks are the same

Benchmark at 0% suggests R riskier than S – "weight" of probability around 0% with R makes R look riskier than S

Benchmark at −10% suggests S riskier than R – overall wider "spread" of S dominates at more extreme risk levels

(Note: candidate answers may differ slightly because approximations required in standard normal lookups from tables.)

2 (i) Excessive volatility is when the change in market value of stocks (observed volatility), cannot be justified by the news arriving. This is claimed to be evidence of market over-reaction which was not compatible with efficiency.

(ii) There are also well-documented examples of under-reaction to events (any two of these):

1. Stock prices continue to respond to earnings announcements up to a year after their announcement. An example of under-reaction to information which is slowly corrected.

2. Abnormal excess returns for both the parent and subsidiary firms following a de-merger. Another example of the market being slow to recognise the benefits of an event.

3. Abnormal negative returns following mergers (agreed takeovers leading to the poorest subsequent returns). The market appears to over-estimate the benefits from mergers, the stock price slowly reacts as its optimistic view is proved to be wrong.

3 One-factor models
All are arbitrage-free.
Vasicek: easy to implement but problem of possible negative interest rates

CIR: more tricky to implement but positive rates (for suitable choice of parameter values).

HW: more flexible as time-inhomogeneous, so better fit to market data (in particular option prices)., but negative rates are possible

Limitations:

1) historical data shows changes in the prices of bonds with different terms to maturity are not perfectly correlated

2) there have been sustained periods of both high and low interest rates with periods of both high and low volatility

3) we need more complex models to deal effectively with more complex derivative contracts e.g. any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated

Multiple-factor models: to capture more features of market data, better for pricing exotic derivatives.

There is no perfect model. A good model depends on the data available and the use of the model (basic assets, plain vanilla derivatives, more exotic derivatives, short or long maturities…).

Fit to historical data; realistic dynamics

4 (i) (a) A credit event is an event which will trigger the default of a bond and includes the following:

- failure to pay either capital or a coupon
- loss event
- bankruptcy
- rating downgrade of the bond by a rating agency such as Standard
- and Poor's or Moody's

(b) The outcome of a default may be that the contracted payment stream is:
- rescheduled
- cancelled by the payment of an amount which is less than the default-free value of the original contract
- continues but at a reduced rate
- totally wiped out

[any three of the above]

(c) In the event of a default, the fraction of the defaulted amount that can be recovered through bankruptcy proceedings or some other form of settlement is known as the recovery rate.

(ii) The model is in continuous time; it has two states N (not previously defaulted) and D (previously defaulted).

Under this simple model it is assumed that the default-free interest rate term structure is deterministic with $r(t) = r$ for all t.

If the transition intensity, under the real-world measure P, from N to D at time t is denoted by $\lambda(t)$, then if $X(t)$ is the state at time t:

$\Pr P(X(t + dt) = N \mid X(t) = N) = 1 - \lambda(t)\, dt + o(dt)$ as $dt \to 0$,

$\Pr P(X(t + dt) = D \mid X(t) = N) = \lambda(t)\, dt + o(dt)$ as $dt \to 0$.

(iii) The formula for the unit ZCB price is $e^{-rT}(1 - (1 - \delta)(1 - e^{-\lambda(i)T}))$, where δ is the recovery rate and $\lambda(i)$ is the (constant) default rate for bond i and T is the redemption time.

Thus

$1.6 = 2e^{-0.06}(1 - .4(1 - e^{-2\lambda(A)}))$ and
$2.2 = 3e^{-0.06}(1 - .4(1 - \mathrm{e}^{-2\lambda(B)}))$,

so $\lambda(A) = 0.2361$

and

$\lambda\,(B) = 0.4029$

(iv) (a) We seek a portfolio consisting of a units of £100 nominal of bond A, b units of £100 nominal of bond B, d units of the derivative, D, and c units of cash.

If this is to perfectly hedge the security then its value at time 2 should be zero unless both bonds default, in which case it should be 100.

At time 2 there are four possibilities: no defaults, bond A only has defaulted, bond B only has defaulted, both bonds have defaulted.

Equating the corresponding values of the portfolio and of the new security (at time 2) we obtain:

$100a + 100b + c = 0;$
$60a + 100b + 100d + c = 0;$
$100a + 60b + 100d + \mathrm{c} = 0$
$60a + 60b + 100d + c = 100$

Solving gives $a = b = -2.5$, $c = £500$ and $d = -1$.

(b) Since this is a perfect hedge, the initial value of the hedging portfolio is the fair price for the new security, so the fair price is
$500e^{-0.06} - 250(1.6/2) - 250(2.2/3) - 52 = £34.55$

5 (i) The unique fair price is $V = E_P[e^{-rT}D]$, where P is the EMM

(ii) Standard interpolation using the Black-Scholes formula gives $r = 14.55\%$ or 14.5%

(iii) The price of the security is given in (i) so equals
$E_P[e^{-r}\, S_1{}^2] = E_P[S_0{}^2 e^{-r} \exp(2\sigma B_1 + 2r - \sigma^2)] = S_0{}^2 \exp(r + \sigma^2)$=$5.5^2\exp(.185)$=£36.40.

(iv) The amount of stock to hold in the hedging portfolio is Delta = $\partial f/\partial S$, where f is the price as a function of current stock price S. Thus the initial hedging portfolio holds $2S_0\exp(r + \sigma^2)$=13.235 units of stock and is short £$S_0{}^2\exp(r + \sigma^2)$ =£36.40.

6 (i) Set $g(x,t) = \exp(kx - (1/2)k^2t)$, then $\Lambda_t = g(W_t\,, t)$.

It follows from Ito's formula that
$d\Lambda_t = (\partial g/\partial t\ (W_t\,, t) + (1/2)\ \partial^2 g/\partial^2 x(W_t\,, t))dt + \partial g/\partial x(W_t\,, t)\ dW_t$
$= (-(1/2)k^2 g + (1/2)k^2 g)dt + kg\ dW_t = kg\ dW_t$.

It follows that Λ is a (local) martingale.

Hence $1 = \Lambda_0 = E[\Lambda_T] = E[\exp(kW_T - 1/2k^2T)$
$= E[\exp(kW_T]\exp(-1/2k^2T)$ so $E[\exp(kW_T]=\exp(1/2k^2T)$

(ii) (a) When

$$
\begin{aligned}
f(x) &= x,\ p_0 = E[e^{-rt}\, S_T\, \Lambda_t | F_0] \\
&= E[e^{-rt}\, S_0\exp(\sigma\, W_t + (\mu - 1/2\ \sigma^2)t)\Lambda_t | F_0] \\
&= E[e^{-rt}\, S_0\exp((\sigma + m)\, W_t + (\mu - 1/2\ \sigma^2 - 1/2\ m^2)t) | F_0] \\
&= e^{-rt}\, S_0\exp(1/2\ (\sigma + m)^2 t + (\mu - 1/2\ \sigma^2 - 1/2\ m^2)t) \\
&= S_0\exp((\sigma m + \mu - r)t)).
\end{aligned}
$$

(b) Now the price at time 0 of a unit of stock is S_0, so unless $p_0 = S_0$, which holds if and only if $m = (r - \mu)/\sigma$, there is an arbitrage opportunity.

7 (i) Denote the individual derivative by f and assume this is written on an underlying security S

$$\Delta = \frac{\partial f}{\partial s} \equiv \frac{\partial f}{\partial s}(t, S_t).$$

$$\Gamma = \frac{\partial^2 f}{\partial s^2}$$

$$\nu = \frac{\partial f}{\partial \sigma}$$

(Marks should also be awarded if these are defined in words.)

(ii) If the portfolio is Delta-hedged and has a high value of Γ then it will require more frequent rebalancing or larger trades than one with a low value of gamma. The need for rebalancing can, therefore, be minimised by keeping gamma close to zero.

The value of a portfolio with a low value of vega will be relatively insensitive to changes in volatility. Since σ is not directly observable, a low value of vega is important as a risk-management tool. Furthermore, it is recognised that σ can vary over time. Since many derivative pricing models assume that σ is constant through time the resulting approximation will be better if ν is small.

8 The proof of this result is an adaptation of that of the standard call-put parity. Two (self-financing) portfolios are considered:

- Portfolio A: buying the call and selling the put at time t. Its value at time t is $C_t - P_t$ and at time T, it is $S_T - K$ in all states of the universe.

- Portfolio B: buying the underlying asset for S_t and borrowing $K\exp\left(-r\left(T-t\right)\right)+D\exp\left(-r\left(T'-t\right)\right)$ at time t. Its value at time t is then: $-\left(K\exp\left(-r\left(T-t\right)\right)+D\exp\left(-r\left(T'-t\right)\right)-S_t\right)$. At time T', the dividend D is paid and added to the portfolio. Therefore the value at maturity of the portfolio is then $S_T - K$, taking into account the dividend payment.

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time t. Hence:

$$C_t - P_t = S_t - K\exp\left(-r\left(T-t\right)\right) - D\exp\left(-r\left(T'-t\right)\right).$$

9 (i) (a) Key calculation in demonstrating no arbitrage is
$d = 0.8 < \exp(0.06) < u = 1.2$

(b) The binomial tree is:

		144
	120	
100		96
	80	
		64

(ii) To price the call option, we use the risk-neutral pricing formula. We use the following simplifying notation:

$$C_{uu} = \left(u^2 S_0 - K\right)^+ = 44 \ ; \ C_{ud} = \left(udS_0 - K\right)^+ = 0 \ ; \ C_{dd} = \left(d^2 S_0 - K\right)^+ = 0 .$$

At time 1, we get in the upper state,

$C_1(u) = \exp(-r)\left[qC_{uu} + (1-q)C_{ud}\right] = 27.12$, and in the lower state $C_1(d) = \exp\left[qC_{ud} + (1-q)C_{dd}\right] = 0$ where the risk-neutral probability of an upward move is $q = \dfrac{\exp(r) - d}{u - d} = 0.6545$.

At time 0, $C_0 = \exp(-r)\left[qC_1(u) + (1-q)C_1(d)\right]$.

Hence $C_0 = 16.72$. (this could be seen directly as C=e^{-2}rp^{2}*44)

(iii) (a) Only one path is relevant for this barrier option "up-up". Its probability of occurrence is q^2 and the associated payoff is X_{uu} = 44. Using the risk-neutral valuation formula, we get:

$$X_0 = \exp(-2r)\left(q^2 X_{uu}\right) = 16.72$$

(b) In practice this option "clearly" has less value than the option (ii) because it pays off in fewer cases. However it has the same price when calculated using the binomial tree approach – this reinforces the need for choosing binomial trees carefully when pricing derivatives.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

20 April 2011 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 (i) State the six assumptions underlying the Black-Scholes market.

(ii) Give the four defining characteristics of a Brownian Motion Z, such that $Z_0 = 0$.

[5]

2 (i) State the main assumptions of modern portfolio theory. [2]

Three assets have the following characteristics:

Asset i	*Expected return* E_i	*Volatility* σ_i
1	4%	6%
2	6%	12%
3	8%	18%

The correlation between assets 1 and 2 is 0.75; while the correlation between asset 3 and both of the other two assets is zero.

(ii) (a) State the Lagrangian function that can be minimised to find the minimum variance portfolio associated with a given expected return, defining any notation used.

(b) By taking five partial derivatives of this function, calculate the minimum variance portfolio which yields an expected return of 5%.

[7]
[Total 9]

3 A securities market has only three risky securities, A, B and C with the following annual return attributes:

	Asset A	*Asset B*	*Asset C*
Market capitalisation	£100bn	£150bn	£250bn
Annual expected return	4%	r_B	6%

Assume that:

- the assumptions of the Capital Asset Pricing Model hold
- the market price of risk is 10% per annum
- the risk free rate is 3.3% per annum
- the expected annual return on the market portfolio is 5.3% per annum.

(i) Calculate σ_M, the standard deviation of the annual return on the market portfolio. Quote any results that you use. [1]

(ii) Calculate r_B, the expected annual return on asset *B*. [2]

(iii) Calculate the covariance of the annual returns on each asset with the annual return on the market portfolio. State any further results that you use. [4]
[Total 7]

4 (i) Outline the three forms of the efficient market hypothesis. [6]

XYZ has just announced that its profits are up by 52% on last year. On the announcement XYZ shares fell in price by 20%. Analysts had been predicting a rise in profits of 65%. A friend says that this shows that the efficient markets hypothesis is false.

(ii) Comment on this statement. [3]

[Total 9]

5 Assume that a non-dividend-paying security with price S_t at time t can move to either $S_t u$ or $S_t d$ at time $t + 1$. The continuously compounded rate of interest is r, and $u > e^r > d$. A financial derivative pays α if $S_{t+1} = S_t u$ and β if $S_{t+1} = S_t d$.

A portfolio of cash (amount x) and the underlying security (value y) at time t exactly replicates the payoff of the derivative at time $t + 1$.

(i) Derive expressions for x and y in terms of r, u, d, α and β. [4]

(ii) Derive an expression for the risk-neutral probability of the security having value $S_t u$ at time $t + 1$ in terms of $(x + y)$, r, α and β. [2]

Assume $S_t = 100$, $u = 1.25$, $d = 0.8$ and $r = 0$.

(iii) (a) Calculate the prices of at-the-money call and put options.
(b) Check that the put-call parity holds for this model.

[4]

[Total 10]

 PLEASE TURN OVER

6 (i) Describe the lognormal model for security prices. [2]

A security price, S_t, is assumed to follow a lognormal model with drift $\mu = 4.28\%$ per annum and volatility 12% per annum. The price now is $S_0 = €1.83$. The continuously compounded risk-free rate of interest is 2% per annum.

(ii) Calculate, as at this date, the probability, p, that ($S_1 > €2.20$). [2]

Someone now offers you an option which will pay €1000 if and only if the stock price $S_1 > €2.20$. They propose to charge $€1000e^{-0.02}p$.

(iii) Explain whether or not you would buy this option. [4]

Assume now that the value of 4.28% for μ has been estimated from observations of the security price over 10 years using the estimator $\mu' = \{\log(S_0) - \log(S_{-10})\}/10$.

(iv) (a) Specify the distribution of $\mu' - \mu$.

(b) Deduce the probability that $\mu' - \mu > 3\%$. [4]

(v) (a) Explain how your answer to (iii) would change if you knew that $\mu <$ 1.28% rather than 4.28%.

(b) Comment on this in the light of your answer to part (iv)(b). [3]

[Total 15]

7 (a) List five factors that effect the price of a European put option on a non-dividend paying share.

(b) State how the premium for a European put option would change if each of these factors increased.

[5]

8 Assume the Black-Scholes model applies.

(i) State an expression for the price of a derivative security with payoff D at maturity date T in terms of the risk-neutral measure. [2]

An at the money European call option on a stock has an exercise date one year away and a strike price of £118.57. The option is priced at £10. The continuously compounded risk-free rate is 1% per annum.

(ii) (a) Estimate the implied volatility to within 1% per annum.

(b) Calculate the corresponding hedging portfolio in shares and cash for 1000 options on the share, quoting any results that you use.

(c) Calculate the option's Vega.

[10]

(iii) Price a put on the same stock with the same expiry date and a strike price of £110. [2]

The hedging portfolio of the call option has the same value, the same Delta and the same Vega as the option.

The Delta of the put option is −0.29975 and its Vega is 39.435.

(iv) Determine the hedging portfolio of the call option in terms of shares, cash and the put option. [4]

[Total 18]

9 In an extension of the Merton model, a very highly geared company has two tiers of debt, a senior debt and a junior debt. Both consist of zero coupon bonds payable in three years time. The senior debt is paid before the junior debt.

Let F_t be the value of the company at time t, L_1 the nominal of the senior debt and L_2 the nominal of the junior debt.

(i) (a) State the value of the senior debt at maturity.
(b) Deduce the value of the junior debt at maturity.

[4]

The current gross value of the company is £3.2m. The nominal of the senior debt is £1.2m and that of the junior debt is £2m. The continuously compounded risk-free rate is 4% per annum, the volatility of the value of the company is 30% per annum and the price of £100 nominal of the senior bond is £88.26.

(ii) Calculate the theoretical price of £100 nominal of the junior debt. [6]

[Total 10]

10 Let $B(t,T)$ be the price at time t of a zero-coupon bond paying £1 at time T, r_t be the short-rate of interest, $\mathbb{P}$ be the real world probability measure and $\mathbb{Q}$ the risk neutral probability measure.

(i) Write down two equations for the price of a zero-coupon bond, one of which uses the risk-neutral approach to pricing and the other of which uses the state-price-deflator approach to pricing. [2]

(ii) State the Stochastic Differential Equation (SDE) of the short rate r_t under Q for the Vasicek model and the general type of process this SDE represents. [3]

(iii) Solve the SDE for the short rate r_t from (ii). [5]

(iv) Deduce the form of the distribution of the zero-coupon bond price under this model. [2]

[Total 12]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2011 examinations

Subject CT8 — Financial Economics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

July 2011

Overall the paper was answered well and candidates' performance was satisfactory. The comments below each question indicate where candidates had the most difficulty.

1 (i) [Unit 13 pp1-2, Unit 8 p2] I

The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion.

2. There are no risk-free arbitrage opportunities.

3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.

4. Unlimited short selling (that is, negative holdings) is allowed.

5. There are no taxes or transaction costs.

6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

(ii) [Unit 8 p3 para 1]

A Brownian Motion Z has the following properties:

(1) Zt has independent increments, i.e. $Zt - Z_s$ is independent of $\{Z_r, r \le s\}$ whenever $s < t$.

(2) Z has stationary increments, i.e. the distribution of $Zt - Z_s$ depends only on $t - s$.

(3) Z has Gaussian increments, i.e. the distribution of $Zt - Z_s$ is $N(0, t - s)$.

(4) Z has continuous sample paths $t \to Zt$

Candidates seemed to know this material well, and had no particular problems with this question.

2 (i)

- It is assumed that investors select their portfolios on the basis of the expected return and the variance of that return over a single time horizon.

- It is assumed that the expected returns, variance of returns and covariance of returns are known for all assets and pairs of assets.

- Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

- Investors dislike risk. For a given level of return, they will always prefer a portfolio with lower variance to one with higher variance.

(ii) (a) Let the proportion invested in asset *i*, be *xi* , with expected return *Ei* , variance *Vi* and correlation ρ12. Let *E* be the return on the portfolio of the three assets and let λ and μ be Lagrange multipliers. Then, the Lagrangian function *W* satisfies:

$$W = \sum_{i=1}^{3} x_i^2 V_i + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2 - \lambda(E_1x_1 + E_2x_2 + E_3x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$$

$$= 36x_1^2 + 144x_2^2 + 324x_3^2 + 108x_1x_2 - \lambda(4x_1 + 6x_2 + 8x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$$

(b) $\dfrac{\partial W}{\partial x_1} = 0 \Rightarrow 72x1 + 108x2 - 4\lambda - \mu = 0$

$\dfrac{\partial W}{\partial x_2} = 0 \Rightarrow 108x1 + 288x2 - 6\lambda - \mu = 0$

$\dfrac{\partial W}{\partial x_3} = 0 \Rightarrow 648x3 - 8\lambda - \mu = 0$

$\dfrac{\partial W}{\partial \lambda} = 0 \Rightarrow 4x1 + 6x2 + 8x3 = 5$

$\dfrac{\partial W}{\partial \mu} = 0 \Rightarrow x1 + x2 + x3 = 1$

Solving this set of simultaneous equations gives $x1 = 0.7125$, $x2 = 0.075$ and $x3 = 0.2125$.

$72x1 + 108x2 - 4\lambda - \mu = 0$ (1)
$108x1 + 288x2 - 6\lambda - \mu = 0$ (2)
$648x3 - 8\lambda - \mu = 0$ (3)
$4x1 + 6x2 + 8x3 = 5$ (4)
$x1 + x2 + x3 = 1$ (5)
(1) $\Rightarrow \mu = 72x1 + 108x2 - 4\lambda$ (6)
into (2) $36x1 + 180x2 - 2\lambda = 0 \Rightarrow \lambda = 18x1 + 90x2$ (7)
(4) and (5) into (3) $648x3 - 144x1 - 468x2 = 0$ (8)
(5) $\Rightarrow x3 = 1 - x1 - x2$ (9)
(7) into (4) $\Rightarrow 4x1 + 6x2 + 8 - 8x1 - 8x2 = 5$
$\Rightarrow 4x1 + 2x2 = 3$
$\Rightarrow x2 = 1.5 - 2x1$ (10)

(10) and (9) into (8) $648 - 792x1 - 1116x2 = 0$

$648 - 792x1 - 1674 + 2232x1 = 0$

$1440\ x1 - 1026 = 0$

$\Rightarrow x1 = 0.7125$

$\Rightarrow x2 = 0.075$

$\Rightarrow x3 = 0.2125$

Although most candidates could write down the Lagrangian, several missed the factor of 2 in front of the covariance term. The handling of the Lagrangian showed that many candidates could write down the partial differential equations for optimisation, but were unable to solve them simultaneously.

3 (i) The market price of risk is $(EM - r)/\sigma M$ so $\sigma M = (EM - r)/0.1 = .02/.1 = 20\%$

(ii) The market portfolio is in proportion to the market capitalisation since every investor holds risky assets in proportion to that portfolio. Thus the market portfolio is $.2A + .3B + .5C$ and so $EM = .2EA + .3EB + .5EC$ so $E_B = (.053 - .2\times.04 - .5\times.06)/.3 = 5\%$.

(iii) Assets all lie on the securities market line, so $Ei - r = \beta i(EM - r)$, where $\beta i = \text{Cov}(Ri, RM)/\text{Var}(RM)$.

It follows that $\beta A = .007/.02 = .35$, $\beta B = .017/.02 = .85$ and $\beta C = .027/.02 = 1.35$.

Then $\text{Var}(RM) = .04$ (from part (i)) so $\text{Cov}(RA, RM) = 0.014$, $\text{Cov}(RB, RM) = 0.034$ and $\text{Cov}(RC, RM) = 0.054$.

Generally well-answered by most candidates.

4 (i) Bookwork Unit

Strong form EMH: market prices incorporate all information, both publicly available and also that available only to insiders.

Semi-strong form EMH: market prices incorporate all publicly available information.

Weak form EMH: the market price of an investment incorporates all information contained in the price history of that investment.

(ii) Any reasonable comments:-the market was expecting more and reacted efficiently on the release of insider information. This does suggest that Strong form EMH doesn't hold. It doesn't seem to contradict weak or semi-strong EMH. However, the price fall could be an over-reaction which would contradict the semi-strong form.

Part (i) was well-answered by most candidates. In part (ii) the comments on the statement were disappointingly unclear.

5 (i) Consider an investment of x in cash and y in the stock at time t. Equating the value of this portfolio to the value of the derivative at time $t = 1$ we find the two simultaneous equations:

$xer + yu = \alpha$,
$xer + yd = \beta$.

Rearranging we find:

$$y = \frac{\alpha - \beta}{u - d}, \text{ and}$$

$$x = e^{-r}\frac{\beta u - \alpha d}{u - d}.$$

(ii) $x + y = e{-}r[q\alpha + (1 - q)\beta]$

where q is the risk-neutral probability we are seeking.

$$\text{So } q = \frac{(x+y)e^r - \beta}{\alpha - \beta}.$$

(iii) (a) For the call option we have:

$$y = 55\tfrac{5}{9},\ x = -44\tfrac{4}{9}, \text{ and so } x + y = 11\tfrac{1}{9}.$$

For the put option we have:

$$x = 55\tfrac{5}{9},\ y = -44\tfrac{4}{9}, \text{ and so } x + y = 11\tfrac{1}{9}.$$

(b) The strike price (for the at-the-money option) is just $St = 100$. Therefore, the put-call parity relation holds.

Several candidates misread the question and took y to denote the number of shares rather than their initial value. There were also a significant number of careless errors in the calculation.

6 (i) The lognormal model has independent, stationary normal increments for the log of the asset price. Thus, if S_u denotes the stock price at time u, then $\log(S_t / S_s) \sim N(\mu(t-s), \sigma^2(t-s))$ where μ is the drift and σ is the volatility parameter.

(ii) $p = P(S_1 > €2.20) = P(\log(S_1/S_0) > \log(2.2/1.83)) = P(N(0,1) > (\log(2.2/1.83) - .0428)/.12) = 1 - \Phi(1.17784) = 0.1194$

(iii) No, I would not buy the contract. Assuming the log normal model, we are in a Black-Scholes market and the fair price for the option is $f = E_Q[e^{-.02} C]$ where C is the contract value at expiry date, and Q is the EMM. Under the EMM, the discounted stock price will be a martingale i.e S will be lognormal with drift $.02 - ½\sigma^2 = .0128$ and volatility σ. Now $f = €1000e^{-.02} p'$, where $p' = Q(S_1 > €2.20)$, and since S has a smaller drift under Q than under the real-world measure, this will be a smaller price than I am being offered.

(iv) (a) μ' is $N(\mu, \sigma^2/10)$ so $\mu' - \mu \sim N(0, 0.00144)$.

(b) A priori, therefore, $P(\mu' - \mu > 0.03) = 1 - \Phi(.03/\sqrt{.00144})$
$= 1 - \Phi(.79057) = .21459$.

(v) (a) If the true value of μ is <0.0128 then p is smaller than p' and so the option is a bargain!

(b) The probability of this level of error in the estimate of μ is relatively large even though we have 10 years of data.

In fact, this shows the difficulty in estimating drifts in market models generally.

The poorer candidates answered this question in a way that is inconsistent with the Core Reading, taking the drift parameter to refer to the parameter in the Black Scholes model. This resulted in incorrect numerical answers.

7 According to the Core Reading the factors and the effect they would have are:

(1) The premium would decrease as the underlying share price increased.
(2) The premium would increase as the strike price increased.
(3) The premium would increase as the time to expiry increased.
(4) The premium would increase as the volatility of the underlying share increased.
(5) The premium would decrease as interest rates increased.

A very well answered question.

8 (i) The unique fair price is $V = E_Q[e{-}rTD]$, where Q is the EMM

(ii) (a) Standard interpolation using the Black-Scholes formula gives σ = 20% as follows:

using Black-Scholes, $C = S0\Phi(d1) - ke{-}rT\Phi\,(d2)$, with
$d1 = (rT + ½\sigma 2T)/\sigma\sqrt{T} = (.01 + ½\sigma 2)/\sigma$
and $d2 = (.01 + ½\sigma 2)/\sigma$, $S0 = k = 118.57$ and $C = 10$.

Trying σ = 15% gives a value of $d1 = .14167$ and $d2 = -.00833$ which gives $\Phi(d1) = .55633$, $\Phi(d2) = .49668$, and thus a trial value for C of $118.57 \times (.55633 - e{-}.01 \times .49668) = 7.65868$.

Trying σ = 25% gives a value of $d1 = .165$ and $d2 = -.085$ which gives $\Phi(d1)$=.56553, $\Phi(d2)$=. 46613, and thus a trial value for C of $118.57 \times (.56553 - e{-}.01 \times .46613) = 12.33579$.

Interpolation gives a new trial value of σ of
$15 + (10 - 7.65868)/(12.33579 - 7.65868) \times 10 = 20\%$.

With this value for σ we get a value of
$d1 = 0.15$ and $d2 = -.05$ which gives
$\Phi(d1) = 0.5596$, $\Phi(d2) = 0.4801$, and thus a trial value for C of
$118.57 \times (0.5596 - e{-}.01 \times 0.4801) = 9.993$.

Thus σ = 20%.

(b) The call's Delta = $\Delta C = \partial f/\partial S = \Phi(d1)$, where
$d1 = (\log(S/K) + rT + ½\sigma 2T) / \sigma\sqrt{T} = 0.15$ and
$\Phi(.15) = 0.55962$, so the hedge is $1000\Delta = 559.62$ units of stock and
£10,000 − 118.57 × 559.62 = − £56,354 in cash.

(c) Vega = $VC = \partial f/\partial\sigma = \partial/\partial\sigma(S\Phi(d1) - Ke^{-rT}\,\Phi(d_2))$
$= (S\varphi(d_1)\,\partial d1/\partial\sigma - Ke{-}rT\,\varphi(d2)\,\partial\, d2/\partial\sigma)$
$= (S\varphi(0.15)(½ - r/\sigma 2) + Ke^{-rT}\,\varphi(\,-0.05)(½ + r/\sigma 2))$
$= 118.57\times\,(0.25Xe{-}.01125 + 0.75\times e{-}.00125\times e{-}.01)/\sqrt{(2\pi)}$
$= 46.773$
[since $d2 = (\log(S/K) + r - ½\sigma 2T)/\sigma\sqrt{T} = -0.05$ and
$\partial d2/\partial\sigma = -(½ + (r + \log(S/K))/\sigma 2)$]

(iii) The put price is $p = Ke^{-rT}\,\Phi(-d2) - S\Phi(-d1)$, where
$d1 = (S/K) + r + ½\sigma 2T)/\sigma\,\sqrt{T} = 0.52512$ and
$d2 = (\log(S/K) + r - ½\sigma 2T)/\sigma\,\sqrt{T} = 0.32512$.
So the price is $110Xe{-}.01\times.37254 - 118.57\times.29975 = £5.0303$

(iv) If we have a portfolio of a shares, b puts and m cash we require to match the value, Delta and Vega of the option. This gives three equations:

(1) $aS + bp + m = 10$
(2) $a + b\,\Delta P = \Delta C$
(3) $bVP = VC$

Equation (3) gives $b = 1.18506$;
equation (2) then gives $a = 0.91484$;
equation (1) gives $m = -£104.43$.

This question was generally not well answered, with errors being made in simple calculations of hedging portfolios.

9 (i) (a) Under the Merton model, the value, Ft, of the firm follows a Geometric BM under the EMM. It follows that the terminal value of the debt is $\min(FT, L1)$, where Li is the tier i nominal debt (since FT is available to pay the senior debt).

(b) Subtracting this value from the value of the firm we see that the assets available to redeem the junior debt are $\max(FT - L1, 0)$. It follows that the terminal value of the junior debt is $\min(L2, \max(FT - L1, 0))$.

(ii) Using a Black-Scholes approach, the current value of the senior debt is $V1 = E[e{-}rT \min(FT, L1)] = E[e{-}rT\,(FT - \max(FT - L1, 0)] = F0 - C1$, where $C1$ is the initial value of a call on the value of the firm with strike $L1$. The current value of the junior debt is $V2 = E[e{-}rT\min(L2, \max(FT - L1, 0))]$.

We obtain immediately $V1 = 88.26*12{,}000 = 1059120$

Now the value of the junior debt is $C1 - C2 = F0 - V1 - C2$ – where $C2 = E[e{-}rT\max(F_T - (L1 + L2), 0))]$.

Using Black-Scholes, $C2 = F_0\Phi(d1) - (L1 + L2)\, e{-}rT\, \Phi(d2)$, with
$d1 = (\ln(F0/L1 + L2) + rT + ½σ2T)/σ\sqrt{T}$
$= (\ln(1) + .12 + ½*.09*3)/.3*\sqrt{3} = 0.49075$
and $d2 = (\ln(F0/L1 + L2) + rT - ½σ2T)/σ\sqrt{T} = -.02887$.

$\Phi(d1) = 0.68819$ and $\Phi\,(d_2) = 0.48848$ and so
$C_2 = 3.2*.68819 - e{-}0.12*3.2*.48848 = 0.81583m = £815{,}830$. Thus the junior debt is worth $= C1 - C2 = 32000000 - 1059120 - 815830 = 1325050$. This is the value of £2m nominal so the value of £100 nominal is £66.25.

With some notable exceptions, this question was generally very poorly answered. Candidates were unable to perform calculations related to the Merton model, and were unable to identify the payoffs from simple contingent contracts.

10 (i) Risk-neutral approach:

$$B(t,T) = \mathbb{E}_Q\left[\exp\left(-\int_t^T r_u du\right)|F_t\right]$$

State-price deflator approach:

$$B(t,T) = \frac{\mathbb{E}_{\mathbb{P}}\{A(T)|F_t\}}{A(t)}$$

Where $A(t)$ is the deflator.

(ii) The dynamics of the short rate rt under $\mathbb{Q}$ for the Vasicek model are:

$drt = \alpha(\mu - rt\,)dt + \sigma dZt,$

where Z is a $\mathbb{Q}$-Brownian motion.

This is an Ornstein-Uhlenbeck process.

(iii) Consider $st = e\alpha trt$. Then

$dst = \alpha e\alpha trtdt + e\alpha t\ drt$
$= \alpha\mu e\alpha tdt + \sigma e\alpha t\ dZt$

Thus $st = s0 + \mu(e\alpha t - 1) + \sigma\int_0^t e^{\alpha u}dZ_u$

and consequently

$rt = e{-}\alpha tr0 + \mu(1 - e{-}\alpha t) + \sigma\int_0^t e^{\alpha(u-t)}dZ_u$

(iv) So rt has a Normal distribution and hence from (i), $B(t, T)$ has a lognormal distribution.

This question was largely from a section of the core reading with which some candidates seemed unfamiliar. Candidates need to study the sections relating to interest rate models more carefully. Candidates who knew the bookwork performed well.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

28 September 2011 (am)

Subject CT8 — Financial Economics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 (i) Define an efficient portfolio in the context of modern portfolio theory. [1]

A market consists of two assets A and B. Annual returns on the two assets (R_A and R_B) have the following characteristics:

Asset	*Expected return %*	*Standard deviation %*
A	6	20
B	10	20

The correlation between the returns on the two assets is 0.25.

(ii) (a) Calculate the proportion that would be invested in each of the two assets in a minimum variance portfolio.

(b) Calculate the expected return of that portfolio.

[3]
[Total 4]

2 Consider the following three-factor model of security returns:

$$R_i = \alpha_i + \beta_{i1}I_1 + \beta_{i2}I_2 + \beta_{i3}I_3 + \varepsilon_i$$

Where:

- R_i is the return on security i
- α_i, β_{i1}, β_{i2} and β_{i3} are security-specific parameters
- I_1, I_2 and I_3 are the changes in the three factors on which the model is based; and
- ε_i are independent random normal variables, each with variance σ^2

(i) Describe three categories of model that could be used to help choose the factors I_1, I_2 and I_3. [6]

(ii) List examples of the variables that could be used for the factors I_1, I_2 and I_3, for two of these three categories of model. [2]
[Total 8]

3 An investor wishes to save for a retirement fund of £100,000 in 10 years' time. The instantaneous, constant continuously compounded risk-free rate of interest is 4% per annum. The investor can purchase shares on a non-dividend paying security with price S_t governed by the Stochastic Differential Equation (SDE):

$$dS_t = S_t(\mu dt + \sigma dZ_t)$$

where:

- Z_t is a standard Brownian motion
- $\mu = 12\%$
- $\sigma = 25\%$
- t is the time from now measured in years; and
- $S_0 = 1$

(i) (a) Derive the distribution of S_t.

(b) Calculate the amount, A, that the investor would need to invest in shares to give a 50:50 probability of building up a retirement fund of £100,000 in 10 year's time.

[4]

(ii) Calculate the following risk measures applied to the difference between the value of the fund and £100,000, if the investor invests A.

(a) Variance
(b) Shortfall probability relative to £90,000
(c) 99% Value at Risk

[6]

The investor decides that they do not need more than £100,000 so they write a call option giving up any upside return above £100,000. They also buy a put option to remove the downside risk of receiving less than £100,000.

(iii) Calculate the net cost at time zero of purchasing enough shares to give themselves a 50:50 chance of building up a retirement fund of £100,000, writing the call option on those shares and buying the put option on the shares.

[2]
[Total 12]

4 Assume that there is no arbitrage in the market. A forward contract is available on a physical asset. The continuously compounded costs of managing the asset are x% of its value, and it provides an income stream of £y per ton payable at six monthly intervals, a payment has just been made.

Let S_t be the spot price of one ton of the asset at time t and let r be the continuously compounded risk-free rate of interest per annum which is assumed to be constant.

Derive the current price of a forward contract written on one ton of the asset with maturity T years where (6 months < T < 1 year). [8]

5 (i) List the desirable characteristics of a model for the term structure of interest rates. [4]

(ii) Write down the stochastic differential equation for the short rate r_t under $\mathbb{Q}$ in the Hull-White model. [1]

(iii) Indicate whether or not the Hull-White model shows the characteristics listed in (i). [4]

[Total 9]

6 Under the real-world probability measure, $\mathbb{P}$, the price of a zero-coupon bond with maturity T is given by:

$$B(t, T) = \exp\left\{-(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3\right\}$$

where r_t is the short rate of interest at time t and satisfies the following stochastic differential equation under the real-world measure $\mathbb{P}$:

$$dr_t = \mu r_t dt + \sigma dZ_t,$$

where $\mu > 0$ and Z_t is a standard Brownian motion under $\mathbb{P}$.

(i) Derive a formula for the instantaneous forward rate $f(t, T)$, based on this model. [2]

(ii) Derive an expression for the market price of risk. [4]

(iii) Deduce the stochastic differential equation for r_t under the risk-neutral measure $\mathbb{Q}$ defining all terms used. [2]

[Total 8]

7 A non-dividend-paying stock, S_t, has a current price of 200p. After 6 months the price of the stock could increase to 230p or decrease to 170p. After a further 6 months, the price could increase from 230p to 250p, or decrease from 230p to 200p. From 170p the price could increase to 200p or decrease to 150p. The semi-annually compounded risk-free rate of interest is 6% per annum and the real-world probability that the share price increases at any time step is 0.75. Adopt a binomial tree approach with semi-annual time-steps.

(i) Calculate the state-price deflator after one year. [5]

(ii) Calculate, using the state-price deflator from (i), the price of a non-standard option which pays out $\max\{0, \log(S_1 - 180)\}$ one year from now. [4]

(iii) State how the answer to (ii) would change if the real-world probability of a share price increase at each time step was 0.6. [1]

[Total 10]

8 A non-standard derivative is written on a stock with current price S_0 = \$2 and is exercisable at two dates, after exactly one year and at expiry, after exactly two years. If it is exercised at expiry it returns \$1000 if and only if the stock price is below \$2. If it is exercised after one year it returns \$500 if and only if the stock price is above \$2.

Assume the market is a Black-Scholes one with a continuously compounded risk-free rate of 2% per annum and a stock volatility of 30% per annum.

(i) (a) Explain how the option should be priced after $t = 1$ (assuming that it is not exercised at $t = 1$).

(b) Give an expression for the corresponding price, p_t.

[4]

(ii) Denoting the price just after 1 year by p_{1+}, explain why the fair price, p_1, at $t = 1$, is given by $p_1 = \max(p_{1+}, 500)$ if $S_1 <$ \$2 and by $p_1 = p_{1+}$ if $S_1 >$ \$2. [2]

(iii) (a) Show that a holder should exercise the option at $t = 1$ if $S_1 > k$ for a suitable value of k.

(b) Calculate the value of k.

[4]

[Total 10]

9 A European call option and a European put option on the same stock with the same strike price have an exercise date one year away and both are priced at 12p. The current stock price is 300p.

The continuously compounded risk free rate of interest is 2% per annum.

(i) Calculate the common strike price, quoting any results that you use. [3]

Assume the Black-Scholes model applies.

(ii) Calculate the implied volatility of the stock. [4]

(iii) Construct the corresponding hedging portfolio in shares and cash for 5000 of the call options. [2]
[Total 9]

10 (i) In the Wilkie model, the force of inflation, $I(t)$, is a mean-reverting AR(1) process.

(a) Explain what the statement above means.

(b) Show that the mean of $I(t)$ converges to m, using the formula:

$$I(t) = m + a(I(t-1) - m) + Z(t),$$

where the $Z(t)$'s are iid $N(0, \sigma^2)$ random variables and $0 < a < 1$.
[4]

(ii) Discuss the differences between and suitability of mean-reverting and random walk models for share prices, interest rates and inflation. [5]
[Total 9]

11 (i) Draw a diagram to illustrate the Jarrow-Lando-Turnbull model for credit default, defining any notation used. [4]

A model was proposed for a country's sovereign debt as follows:

The economy is in one of three states: 1 (good), 2 (bad) and 3 (default).
Transition intensities $\lambda_{i,j}$ are constant and as follows:

$$\lambda_{1,2} = 1; \lambda_{1,3} = 0; \lambda_{2,1} = 0.25, \lambda_{2,3} = 0.75; \lambda_{3j} = 0 \text{ for all } j \text{ and } \lambda_{1,1} = \lambda_{2,2} = -1.$$

It follows that if $p_i(t)$ is the probability that the economy is in state i at time t then:

$$\frac{dp_1(t)}{dt} = -p_1(t) + 0.25 p_2(t)$$

and

$$\frac{dp_2(t)}{dt} = p_1(t) - p_2(t) .$$

Set $h(t) = 2p_1(t) - p_2(t)$.

(ii) (a) Show that $\dfrac{dh(t)}{dt} = -1.5h(t)$.

(b) Derive a similar equation for k defined by $k(t) = 2p_1(t) + p_2(t)$.

[2]

Suppose that this country's economy is in state 2 at time 0.

(iii) Find the probability that it is in default at time 2. [4]

Assume a continuously compounded risk-free interest rate of 2% per annum and a recovery rate of 60%.

(iv) (a) Deduce the price under this model for a zero-coupon bond in this country with a redemption value of 100 and a redemption date in two years' time.

(b) Calculate the credit spread.

[3]
[Total 13]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2011 examinations

Subject CT8 — Financial Economics
Core Technical

Purpose of Examiners' Reports

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

December 2011

General comments on Subject CT8

Subject CT8 introduces the main concepts and principles of financial economics. These are developed in later subjects in the ST series of exams. This subject combines various types of skills. In particular, along with CT7, it is one of the first where candidates are expected to write lengthy passages of reasoned thought, rather than just complete calculations. This is a skill that will be new to many, and candidates are advised to pay particular attention to the answers to this type of question by studying many past papers.

Comments on the September 2011 paper

The general performance was slightly worse than in April 2011 and candidates found this paper more challenging, but well-prepared candidates scored well across the whole paper and the best candidates scored close to full marks. As in previous diets, questions that required an element of application of the core reading to situations that were not immediately familiar proved very challenging to most candidates. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Candidates approaching the subject for the first time are advised to concentrate their revision in these areas and the ability to apply the core reading to similar situations.

1 (i) A portfolio is efficient if the investor cannot find a better one in the sense that it has the same expected return and a lower variance, or the same variance and a higher expected return.

(ii) We have:

$$V = x_A^2 V_A + x_B^2 V_B + 2x_A x_B C_{AB}$$

Which is a minimum at

$$x_A = \frac{V_B - C_{AB}}{V_A - 2C_{AB} + V_B}$$

= 0.5

So $x_B = 0.5$

And the expected return on the portfolio is 8%.

Generally candidates scored well on this question. Some students struggled to calculate the weighting in each asset class or failed to distinguish between the correlation and the correlation coefficient.

2 (i) **Macroeconomic factor models**

These use observable economic time series as the factors. They could include factors such as the annual rates of inflation and economic growth, short term interest rates, the yield on long term government bonds, and the yield margin on corporate bonds over government bonds. A related call of model uses a market index plus a set of industry indices as the factors.

Fundamental factor models

These are closely related to macroeconomic models but instead of (or, in addition to) macroeconomic variables the factors used are company specific variables. These may include such fundamental factors as:

- the level of gearing;
- the price earnings ratio;
- the level of R&D spending; or
- the industry group to which the company belongs.

Commercial fundamental factor models are available which use many tens of factors. They are used for risk control by comparing the sensitivity of a portfolio to one of the factors with the sensitivity of a benchmark portfolio.

Statistical factor models

These do not rely on specifying the factors independently of the historical returns data. Instead a technique called principal components analysis can be used to determine a set of indices which explain as much as possible of the observed variance. However, these indices are unlikely to have any meaningful economic interpretation and may vary considerably between different data sets.

(ii) There are many acceptable answers, but for example:

Macroeconomic factor model

I_1 = annual inflation; I_2 = annual GDP; I_3 = equity dividend yield

Fundamental factor model

I_1 = quick ratio; I_2 = book value; I_2 = industry group to which the company belongs

The candidates who were familiar with the bookwork scored very well. Some candidates were able to score some marks using economic knowledge from subject CT7.

3 (i) Using Ito's Lemma:

$$d\log S_t = \frac{1}{S_t}dS_t + \frac{-1}{2S_t^2}(dS_t)^2$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ_t$$

Written in integral form, this reads

$$\log S_t = \log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t .$$

Or, finally, $S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t\right\}$.

So, S_t has a lognormal distribution with parameters $\left(\mu - \frac{\sigma^2}{2}\right)t = 0.08875t$ and $\sigma^2 t = 0.0625t$.

The initial investment (based on a 50:50 chance) can be calculated by choosing the 50th percentile point of $Z_t = 0$, i.e. the initial investment is:

$$\frac{£100,000}{\exp(0.08875 \times 10 + 0.25 \times 0)} = £41,168 = A$$

(ii) (a) We know that:

$$\text{Var}(S_t) = e^{2\mu t}(e^{\sigma^2 t} - 1) \text{ or equivalently,}$$
$$\text{Var}(AS_t) = 100,000^2 e^{\sigma^2 t}\left(e^{\sigma^2 t} - 1\right)$$

So the variance of the investment is:

$£41,168^2 \text{Var}(S_{10}) = £41,168^2 \, e^{2.4}(e^{0.625} - 1)$
$= £41,168^2 \times 9.571$
$= 16,220,971,227.90$

(b) As S_t has a lognormal distribution

$\mathbf{P}(£41,168 \; S_{10} < £90,000) = \mathbf{P}\;(S_{10} < 2.1862) = \mathbf{P}\;(\log S_{10} < 0.7821)$
$= \mathbf{P}((\log S_{10} - 0.8875)/\sqrt{0.625} < -0.1333) = 0.4470$

(c) The 99th percentile of the Normal distribution is given by $Z_t = 2.3263$. So the 99th percentile worst outcome for the investment is:

$S_{10} = £41,168 \; e^{0.8875 - \sqrt{0.625} \times 2.3263} = £15,896.$
So the VaR relative to A is £25,272 and relative to £100,000 is £84,104.

(iii) In this case the investor has removed all risk, so by the principle of no arbitrage the portfolio will earn the risk free rate. Therefore, the amount they need to invest at time 0 is:

$$\frac{£100,000}{e^{10 \times 4\%}} = £67,032.$$

Many candidates scored well on part (i) which was a fairly standard proof using Ito's lemma.
Many struggled with manipulating the log-normal distribution and calculating risk metrics relating to it.

4 The proof of this result is an adaptation of that of the standard no arbitrage approach to pricing forward contracts. For ease of exposition we use 100x% rather than x% in the calculations.

Two self-financing portfolios are considered at time zero:

Portfolio A: entering into the forward contract to receive one ton of the asset at time T. Its value at time zero is zero, and at time T it is $S_T - F_0^T$.

Portfolio B: buying e^{xT} units of the underlying asset and borrowing $F_0^T e^{-rT} + ye^{x(T-\frac{1}{2})-\frac{r}{2}}$ at time zero. Its value at maturity is $S_T - F_0^T$ by taking account of the storage costs and the income stream.

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time 0. Hence:

$$F_0^T = S_0 e^{(x+r)T} - ye^{(x+r)(T-\frac{1}{2})}.$$

Many candidates struggled with the concept of creating two portfolios using the principle of no arbitrage. They were unable to apply the core reading to a related situation. The question was challenging overall, with many candidates struggling to score well.

5 (i) Arbitrage free.
Positive interest rates.
Mean reversion of rates.
Ease of calculation of bonds and certain derivative contracts.
Realistic dynamics.
Goodness of fit to historical data.
Ease of calibration to current market data.
Flexible enough to cope with a range of derivative contracts.

(ii) $dr_t = \alpha(\mu_t - r_t)dt + \sigma dZ_t$ or alternatively $dr = \left[\theta(t) - ar\right]dt + \sigma dz$
Where in both cases Z is a Brownian motion under $\mathbb{Q}$.

(iii) Arbitrage free. Yes
Positive interest rates. No
Mean reversion of rates. Yes
Ease of calculation of bonds and certain derivative contracts. Yes
Realistic dynamics. No
Goodness of fit to historical data. Yes.
Ease of calibration to current market data. Yes
Flexible enough to cope with a range of derivative contracts. No.

This was standard material from the core reading and more successful candidates tended to score well, although some struggled to get all points required for full marks.

6 (i) $f(t, T) = -\frac{\partial}{\partial T}\log B(t,T)$

$= [r_t - \frac{\sigma^2}{2}(T-t)^2]$

(ii) The market price of risk, γ_t, is defined as:

$$\gamma_t = \frac{m(t,T) - r_t}{S(t,T)},$$

where

$dB(t, T) = B(t, T)[m(t, T)\, dt + S(t, T)\, dZ_t]$.

Now,

$$\frac{\partial B(t,T)}{\partial t} = B(t,T)\left[r_t - \frac{\sigma^2}{2}(T-t)^2\right]$$

$$\frac{\partial B(t,T)}{\partial r_t} = B(t,T)[-(T-t)]$$

$$\frac{\partial^2 B(t,T)}{\partial r_t^2} = B(t,T)(T-t)^2$$

So, using Itô's lemma, we have

$dB(t, T) = B(t, T)\{[-\mu(T-t)\, r_t + r_t]\, dt - \sigma(T-t)\, dZ_t\}$

and so

$$\gamma_t = \frac{\mu r_t}{\sigma}.$$

(iii) The stochastic differential equation for r_t under the risk-neutral measure $\mathbb{Q}$ is given by

$dr_t = \sigma d\tilde{Z}$

where $\tilde{Z}$ is a standard Brownian motion under $\mathbb{Q}$

$$dr_t = \mu r_t dt + \sigma(d\tilde{Z} - \gamma_t dt)$$

$$= \mu r_t dt + \sigma\left(d\tilde{Z} - \frac{\mu r_t dt}{\sigma}\right)$$

$$= \mu r_t\, dt - \mu r_t dt + \sigma d\tilde{Z}$$

$$= \sigma d\tilde{Z}\,.$$

Question 6 was generally challenging. While part (i) was generally straightforward for most candidates, part (ii) where application of first principles was necessary was only answered well by the best candidates.

7 (i) First we calculate the risk-neutral probability of an upwards movement in the share price from each state:

$$q(200) = \frac{1.03 \times 200 - 170}{230 - 170} = 0.6$$

$$q(230) = \frac{1.03 \times 230 - 200}{250 - 200} = 0.738$$

$$q(170) = \frac{1.03 \times 170 - 150}{200 - 150} = 0.502$$

We can use these to calculate the state-price deflators:

$$A_2(250) = \frac{q(200)q(230)}{(0.75 \times 1.03)^2} = 0.742$$

$$A_2(200) = \frac{q(200)[1 - q(230)] + [1 - q(200)]q(170)}{2 \times 0.75 \times 0.25 \times 1.03^2} = 0.900$$

$$A_2(150) = \frac{[1 - q(200)][1 - q(170)]}{(0.25 \times 1.03)^2} = 3.004$$

(ii) The option premium, V, can be calculated as

$$V = E_{\mathbb{P}}(A_2 V_2)$$

$$= p^2 A_2(250) \log(70) + 2p(1 - p) A_2(200) \log(20) + (1 - p)^2 A_2(150) \times 0$$

$$= 2.784$$

(iii) It would not change at all.

This question was overall well answered, showing that many candidates have understood the broad concept of state price deflators. Well- prepared candidates were able to score near full marks on all three parts of the question.

Some candidates lost marks through ignoring the semi-annual interest rate. Part (iii) was designed to test the understanding of the candidates on how option pricing theory works in practice, but disappointingly many candidates got this part wrong.

8 (i) If the first exercise date has passed then the owner now has a derivative contract which pays \$1000 at time 2 years if and only if the stock price $S_2 < 2$.

The derivative should then be priced using the formula

$$p_t = E_Q[e^{-r(2-t)}C|F_t],$$

where C is the claim value at $t = 2$ and Q is the risk-neutral probability measure.

This gives a value of p_t of

$$1000\, e^{-r(2-t)}\, Q(S_2 < 2|F_t)$$
$$= 1000\, e^{-r(2-t)}\, Q(S_2/S_t < 2/S_t)$$
$$= 1000\, e^{-r(2-t)}\, Q(\log(S_2/S_t) < \log(2/S_t))$$
$$= 1000\, e^{-r(2-t)}\, \Phi(\{\log(2/S_t) - (.02\text{- }0.045)(2-t)\}/\, (.3\surd\, (2-t))).$$

(ii) At t=1 the holder can choose between the value of the residual contract: p_{1+} and the current immediate exercise reward of \$500 if $S_1 > 2$ (and 0 otherwise). A rational holder will maximise value by choosing whichever has a greater current value.

There was a typo in the question where the inequalities were the wrong way around, full credit will be given to students who assumed this part was correct and had the inequalities the other way around.

In other words, an acceptable answer would be: At t=1 the holder can choose between the value of the residual contract: p_{1+} and the current immediate exercise reward of \$500 if $S_1 < 2$ (and 0 otherwise).

A rational holder will maximise value by choosing whichever has a greater current value.

(iii) (a) Since the current exercise value increases with S_1 and the value of p_{1+} decreases with S_1, the holder will choose to exercise the option at $t = 1$ if and only if the stock price is greater than some critical value k.

(b) At the critical value the holder should be indifferent i.e. we should have $p_{1+} =$ \$500. So we seek k such that

$$1000\, e^{-r}\, \Phi(\{\log(2/k) + .025\}/.3) = 500$$
$$\text{so } \Phi(\{\log(2/k) + .025\}/.3) = 0.51010$$

so $\{\log(2/k) + .025\}/.3 = .02531$
so $k = 2.0343$

Performance on this question was very variable.
Part (i) was generally well-answered. A number of candidates highlighted that the inequality was the wrong way around in part (ii). Part (iii) was generally poorly answered.

9 (i) Using put-call parity, $0 = S - Ke^{-rT}$, so $K = Se^{rT} = 306.06\text{p}$

(ii) $d_1 = (\log(S/K) + r + ½\sigma^2 T) / \sigma\sqrt{T} = ½\sigma$,
while $d_2 = (\log(S/K) + r - ½\sigma^2 T)/\sigma\sqrt{T} = -½\sigma$.

Thus $C = S\Phi(d_1) - Ke^{-rT}\,\Phi(d_2) = S(\Phi(½\sigma) - \Phi(-½\sigma)) = 300(2\Phi(½\sigma) - 1)$
so $\Phi(½\sigma) = 0.52$ so $\sigma = .1003 = 10.0\%$.

(iii) $\Phi(d_1) = 0.52$ so the hedge is $5000 \times 0.52 = 2600$ shares
and $600 - 2600 \times 3 = £7200$ short in cash.

Generally answered well by candidates. Most candidates were able to score full marks on part (i) and proceed to score well on parts (ii) and part (ii).

10 (i) (a) Mean reversion means that the force of inflation will tend to move towards its average value.

An AR(1) process is a linear auto-regressive model of order one (i.e. the impulse at time t depends on the process one step before) whose formula is of the form of the equation given in the question.

(b) Denote by $i(t)$ the mean value of $I(t)$, then taking expectations in the formula, we see that $i(t) = m + a(i(t-1) - m)$
or $i(t) - m = a(i(t-1) - m)$. It follows that $i(t) - m$ tends to zero at a geometric rate.

(ii) A random walk process can be expected to grow arbitrarily large with time.

If share prices follow a random walk, with positive drift, then those share prices would be expected to tend to infinity for large time horizons.

However, there are many quantities which should not behave like this. For example, we do not expect interest rates to jump off to infinity, or to collapse back to zero.

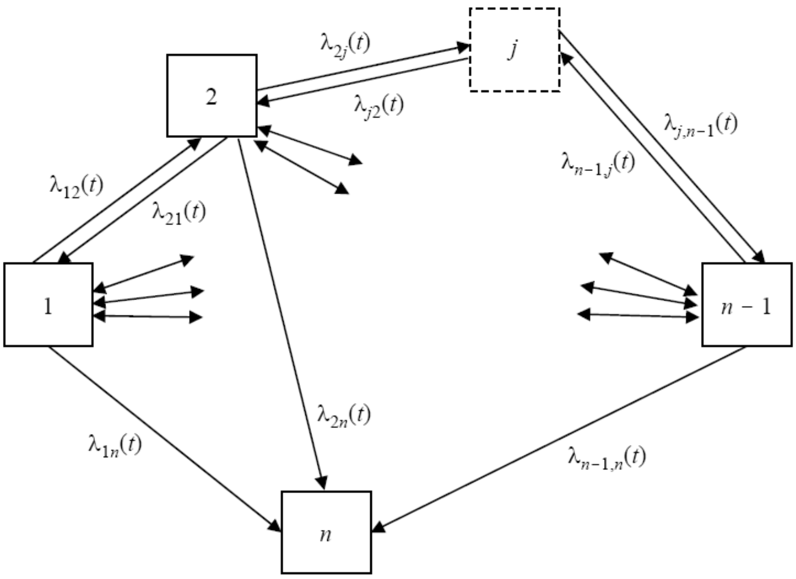
Instead, we would expect some mean reverting force to pull interest rates back to some normal range. In the same way, while inflation can change substantially over time, we would expect them, in the long run, to form some stationary distribution, and not run off to infinity. Similar considerations apply to the annual rate of growth in share prices.

In each case, these quantities are not independent from one year to the next; times of high interest rates or high inflation tend to bunch together i.e. the models are auto-regressive.

One method of modelling this is to consider a vector of mean reverting processes. These processes might include (log) yields, or the instantaneous growth rate of income streams. The reason for the log transformation is to prevent negative yields.

The question was straightforward bookwork. Candidates struggled to score full marks on part (ii) but were generally able to describe the basic concepts of the two models. Unfortunately many candidates chose to write extensive details about share price models and their characteristics rather than focus on the question about the random walk versus mean reverting models.

11 (i)



The n states represent $n-1$ credit ratings plus default.

$\lambda_{ij}(t)$ are the deterministic transition intensities from state *i* to state *j* at time *t* under the real world measure *P*.

(ii) (a) $h'(t) = 2p_1'(t) - p_2'(t) = -3p_1(t) + 3/2p_2(t) = -3/2h(t)$.

Similarly

$k'(t) = 2p_1'(t) + p_2'(t) = -p_1(t) - ½p_2(t) = -½k(t)$.

(b) Solving these linear differential equations with initial conditions $h(0) = -1$ and $k(0) = 1$ we get $h(t) = -e^{-3/2t}$ and $k(t) = e^{-½t}$.

It follows that $p_1(t) = ¼(h(t) + k(t)) = ¼(e^{-½t} - e^{-3/2t})$
while $p_2(t) = ½(k(t) - h(t)) = ½(e^{-½t} + e^{-3/2t})$.

Now, since $p_3(t) = 1 - p_1(t) - p_2(t)$,
we obtain $p_3(t) = 1 - 3/4e^{-½t} \; -1/4\, e^{-3/2t}$.

And so $p_3(2) = .71164$.

(iv) (a) The bond price is thus $e^{-.04}\,(1 - p_3(2))£100 + p_3(2)£60) = £68.729$.

(b) The equivalent no-default interest rate is ½log(100/68.729) = 18.75%. Thus the credit spread is 16.75%.

Few candidates failed to score well on parts (i) and (ii). In contrast, very few students were able to apply the results to part (iii) where scores were disappointing and often nil. Candidates did not understand the relevance of h(t) and k(t) and they may have gotten further if they had worked with them. Marks were picked up in question (iv) where candidates continued with the question.

END OF EXAMINERS' REPORT