PhD - MAEG

Lévy Processes and Applications

Exam

Duration: 2 hours

January 17, 2014

1.

(a) Discuss the main drawbacks of the classical Black-Scholes model (in finance) and explain how the Lévy processes can be used to overcome these drawbacks. (marks: 2)

(b) State the Lévy-Itô decomposition for a general Lévy process and present a financial interpretation for the jump terms of this decomposition. (marks: 2)

2. Consider a "jump-diffusion" model without compensation term for the jumps.

(a) Define the Lévy process associated to the Merton "jump-diffusion" model, give an interpretation of each term in the definition, present the characteristic function of the Lévy process at time $t = 1$ and present the characteristic triplet of the process. (marks: 2)

(b) Consider the "jump-diffusion" process where the distribution for the jump sizes has a probability density function given by:

$$f_J(x) = p\theta_1 e^{\theta_1 x}\mathbf{1}_{\{x<0\}} + (1-p)\,\theta_2 e^{-\theta_2 x}\mathbf{1}_{\{x>0\}}.$$

Calculate the characteristic function for the "jump-diffusion" process at time $t = 1$ and show that the measure ν associated to the process satisfies the conditions of a Lévy measure. (marks: 2)

3. Consider a distribution with characteristic function

$$\phi(u) = \exp\left(imu - \sigma|u|\left[1 + i\beta\frac{2}{\pi}\mathrm{sgn(u)}\log|u|\right]\right),$$

where $\sigma > 0$, $-1 \leq \beta \leq 1$ and $m \in \mathbb{R}$.

(a) Show that this distribution is infinitely divisible . (marks: 2)

(b) Let X be a random variable with the distribution associated to $\phi(u)$ in the case $\beta = 0$. State the usual name given to the distribution of X, present the probability density function of X, present the value of $\mathbb{E}[|X|]$ and discuss how is the decay of the distribution tail of X when $x \to +\infty$, i.e., how is the decay of $\mathbb{P}[X > x]$ as a function of x when $x \to +\infty$. (marks:2)

4. Let X be a Lévy process with Lévy measure $\nu(dx) = \frac{\exp(-x)}{x^2}\mathbf{1}_{\{x>0\}}$.

(a) Calculate the expected value of the Poisson integral

$$\mathbb{E}\left[\int_{\varepsilon}^{+\infty} e^x N(t, dx)\right].$$

where $\varepsilon > 0$ and deduce how is the function $h(t, \varepsilon)$ such that

$$\int_{\varepsilon}^{+\infty} e^{x} N(t, dx) - h(t, \varepsilon)$$

is a martingale.. (marks: 2)

(b) Consider that the process X is the solution of the stochastic differential equation

$$dX_t = 2X_{t-}dt + X_{t-}dB_t + X_{t-} \int_{|x| \geq 1} \left(e^{\frac{x}{2}} - 1 \right) N(dt, dx).$$

Solve this equation (hint: apply the Itô formula to $f(X_t) = \ln(X_t)$). (marks: 2,5)

5. Consider a financial market with one risky asset with price process S_t given as the solution of the stochastic differential equation

$$dS(t) = S(t-) \, dZ(t),$$

where $Z(t) = \sigma X(t) + \mu t$ and $X(t)$ is a Lévy process with decomposition

$$X(t) = mt + kB(t) + \int_{c}^{+\infty} x \widetilde{N}(t, dx),$$

where $k \geq 0$, $m \geq 0$ and $c < -1$. Assume that the riskless interest rate is $r > 0$.

(a) State the condition that σ, μ, k, m and r must satisfy in order for the discounted price process $\widetilde{S}$ to be a martingale with respect to an equivalent martingale measure Q and discuss the completeness of the market in the following cases:

(i) $X(t) = mt + \widetilde{N}(t)$, where $\widetilde{N}(t)$ is a compensated Poisson process

(ii) $X = mt + \widetilde{N}_1(t) + \widetilde{N}_2(t)$, where $\widetilde{N}_1$ and $\widetilde{N}_2$ are compensated Poisson processes with intensities λ_1 and λ_2 and with constant jump sizes c_1 e c_2, respectively ($\widetilde{N}_1$ and $\widetilde{N}_2$ are also independent). (marks: 2)

(b) Show that the general condition that must be satisfied in order for the discounted price process $\widetilde{S}$ to be a martingale with respect to an equivalent martingale measure Q is an equation that has, in general, an infinite number of solutions (F, H).

Hint: Consider that $\frac{dP}{dQ} = e^{Y(T)}$ where $dY(t) = G(t)dt + F(t)dB(t) + \int_{\mathbb{R}\setminus\{0\}} H(t,x)\widetilde{N}(dt, dx)$ and show that if the pair (F, H) is a solution then the pair $(F + \int_{\mathbb{R}\setminus\{0\}} f(x)\nu(dx), \log\left(e^{H} - \frac{kf(x)}{x}\right)$ is also a solution for any $f \in L^1(\mathbb{R}\setminus\{0\}, \nu)$. (marks: 2)