Master in Mathematical Finance - ISEG/Universidade de Lisboa

# Lévy Processes and Applications

## Exam (Época Normal)

Duration: 2 hours

January 20, 2017

**1.** Consider a Lévy process $X(t)$ and a subordinator process $T(t)$, which is independent of $X(t)$.

(a) Define what is a subordinator, present the general characteristic exponent of a subordinator and present four different examples of subordinators.

(b) Consider the process $Y(t) = X(T(t))$. Is $Y$ a Lévy process? If your answer is yes, deduce the formula for the characteristic exponent $\eta_Y$ of the process $Y$ in terms of the Laplace exponent $\psi_T$ and the characteristic exponent $\eta_X$ (if you cannot deduce the formula, just present it). Moreover, apply this formula in order to obtain the characteristic exponent of $Y$ if $T$ is an $\alpha$-stable process with $\alpha = \frac{1}{4}$ and $X = \sqrt{2}B(t)$, where $B$ is a Brownian motion.

**2.** Consider a Lévy process $L$ defined by

$$L_t = \sqrt{3}B_t + X_1 + X_2 + \cdots + X_{N(t)} - 8t,$$

where $B_t$ is a Brownian motion, the r.v. $X_1, ..., X_k, ...$ are i.i.d. with mean value $\mathbb{E}[X_k] = 4$ and $N(t)$ is a Poisson process with intensity $\lambda = 2$.

(a) (i) What kind of process is $L_t$? Is it a martingale?; (ii) Assuming that the random variables $X_k$ have a normal distribution with mean 4 and variance 2, deduce the characteristic function of the process $L_t$.

(b) Show that the distribution of $L_t$ is infinitely divisible.

**3.** Consider the process

$$X(t) = \exp(Y(t)),$$

where $Y(t)$ is a Lévy-type stochastic integral such that

$$dY(t) = b(t)dt + u(t)dB(t) + \int_{|x|\geq 1} w(t,x)N(dt,dx).$$

Deduce the conditions that the processes $b(t)$, $u(t)$ and $w(t,x)$ must satisfy if $Y(t)$ and $X(t)$ are both martingales.

**4.** Let $X$ be a Lévy process and consider the measure $\nu(dx) = x^{\alpha}\mathbf{1}_{\{0<x<1\}} + x^{\beta}\mathbf{1}_{\{x>1\}} + |x|^{\gamma}\mathbf{1}_{\{x<-1\}}$, where $\alpha, \beta$ and $\gamma$ are real parameters.

(a) For what values of $\alpha$, $\beta$ and $\gamma$ is $\nu(dx)$ a Lévy measure? Justify and then choosing the values of $\alpha = -1$, $\beta = -4$ and $\gamma = -5$, calculate the expected value of the Poisson integral

$$\mathbb{E}\left[\int_{\mathbb{R}\setminus[-1,1]} |x|^2 N(t,dx)\right].$$

and explain what is the interpretation of this value in terms of the sizes of the jumps.

(b) Consider that the process $Y$ is the solution of the stochastic differential equation

$$dY(t) = 3Y(t-)dt + 2Y(t-)dB_t + 5Y(t-)\int_{|x|<1} x^2 N\left(dt, dx\right), \text{ with } Y(0) = 4.$$

Solve this equation (hint: apply the Itô formula using the log function).

**5.** Consider a financial market with one risky asset with price process $S_t$ given as the solution of the stochastic differential equation

$$dS(t) = S\left(t-\right) dZ\left(t\right),$$

where $Z\left(t\right) = \sigma X\left(t\right) + \mu t$ and $X\left(t\right)$ is a Lévy process with decomposition

$$X\left(t\right) = mt + kB\left(t\right) + \int_{c}^{+\infty} x\widetilde{N}(t, dx),$$

where $k \geq 0$, $m \geq 0$ and $c := -\sigma^{-1}$. Assume that the riskless interest rate is $r > 0$.

(a) State the condition that $\sigma, \mu, k, m$ and $r$ must satisfy in order for the discounted price process $\widetilde{S}$ to be a martingale with respect to an equivalent martingale measure $Q$ and discuss the existence and uniqueness of this martingale measure, when

(i) $X(t) = B(t) + \widetilde{N}(t)$, where $B(t)$ is a Brownian motion and $\widetilde{N}(t)$ is an independent compensated Poisson process.

(ii) in the context of the Esscher transform measure $Q_u$, with $u(t) = -ku$, $h(t,x) = -ux$, discuss the equivalent martingale measure condition and the uniqueness of the Esscher transform measure.

(b) Considering that the price of the risky asset is modeled as an ordinary exponential of a Lévy process $Y(t)$, that is

$$S(t) = S_0 \exp(Y\left(t\right)),$$

in which of the following cases can we ensure that the market model is arbitrage free (explain why) ? :

(i) $Y(t)$ is a Brownian motion with drift: $Y(t) = mt + B(t)$.

(ii) $Y(t)$ is a compound Poisson process: $Y(t) = \sum_{k=1}^{N(t)} J_k$, where the random variables $J_k$ have uniform distribution on the interval $]0, 10[$.

(iii) $Y(t)$ is a pure jump process with Lévy measure $\nu(dx) = x^{-2}\mathbf{1}_{\{0<x<1\}}$.

(iv) $Y(t)$ is a compound Poisson process with drift: $Y(t) = -3t + \sum_{k=1}^{N(t)} J_k$, where the random variables $J_k$ have exponential distribution.

**Marks**: 1(a):2, 1(b):2.5, 2(a):2.5, 2(b):2, 3:2, 4(a):2, 4(b):2.5, 5(a):2.5, 5(b):2