PART IV

Models of Equilibrium in Capital Markets

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Capital Asset Pricing Model (CAPM)

1. Capital Asset Pricing Model (CAPM)

- Standard CAPM
- Testing CAPM

1.1 Standard CAPM

- Learning objectives
- Deriving the *equilibrium*
- Assumptions
- Implementing CAPM
- Questions

Learning objectives

- state the CAPM equation,
- state the assumptions necessary to derive CAPM,
- discuss proxies for the market portfolio in CAPM,
- relate the equilibrium return of a portfolio to its covariance with the tangent portfolio.

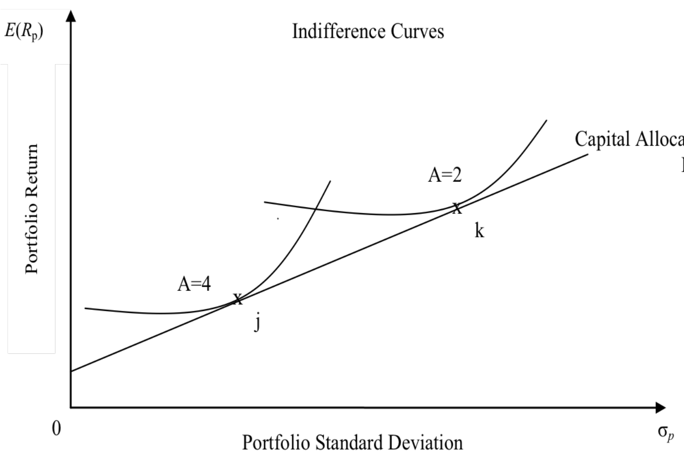
From MVT + EUT to CAPM

- Under certain assumptions, principally that an investor only cares about mean and variance, we derived an expression for a portfolio – the tangent portfolio – such that the investor would invest in a multiple of it and the risk-free asset.
- How much each investor invests in the tangent portfolio depends on various things that vary from investor to investor.
- If one makes assumptions that stops this variation, then one can develop a simple relationship between the expected return of an asset and its covariance with the tangent portfolio.

$\Downarrow$

This is called the Capital Asset Pricing Model (CAPM).

From MVT + EUT to CAPM



Tangent Portfolio

- We developed an algorithm for finding weights, X_T of the tangent portfolio for a given risk-free rate R_f.
- To find X_T we solve

$$VZ = \bar{R} - R_f \mathbb{1}, \text{ and put}$$
$$X_T = \frac{Z}{\langle Z, \mathbb{1} \rangle}.$$

Recall that

$$\bar{R} = (\bar{R}_1, \ldots, \bar{R}_n)'.$$

Argument Logic

We will argue in three parts.

1. In the first part all we assume is that a portfolio T has weights given by the algorithm above, and use that to deduce an equation relating a general portfolio's expected return to its covariance with T.
2. In the second part, we assume that, *in equilibrium*, all investors have the same tangent portfolio, and use that to argue that T is the market.
3. In the third part, we look at what assumptions would imply that all investors have the same tangent portfolio, *in equilibrium*.

First Part

1. Expected return and the tangent portfolio

We can write $Z = \lambda X_T$ for some $\lambda \in \mathbb{R}$. (Of course, $\lambda = \langle Z, \mathbb{1} \rangle$.)

We then have

$$\lambda V X_T = \bar{R} - R_f \mathbb{1},$$

as a vector equation.

Now suppose we take coordinate i, this then becomes

$$\lambda (V X_T)_i = \mathbb{E}(R_i) - R_f,$$

which is equivalent to

$$\mathbb{E}(R_i) = R_f + \lambda (V X_T)_i.$$

Recap

The weights X_T are the weights of the tangent portfolio, so $(VX_T)_i$ is the covariance between the tangent portfolio and asset i.

We therefore conclude

$$\mathbb{E}(R_i) = R_f + \lambda \text{Cov}(R_i, R_T) .$$

Since λ is independent of i, the covariance of an asset with the tangent portfolio determines its expected return.

For this equation to be useful, we need to compute λ.

OBS: We already know λ! Q: When did we derived it?

(Re) Computing λ

We can also find it using the fact that portfolio T has weights x_i^T

$$\begin{aligned}
\mathbb{E}(R_T) &= \sum x_i^T \mathbb{E}(R_i), \\
&= \left(\sum x_i^T R_f\right) + \gamma \sum x_i^T \text{Cov}(R_i, R_T), \\
&= R_f + \lambda \text{Cov}(\sum x_i^T R_i, R_T), \\
&= R_f + \lambda \text{Cov}(R_T, R_T), \\
&= R_f + \lambda \text{Var}(R_T) \\
\Longrightarrow & \qquad \lambda = \frac{\mathbb{E}(R_T) - R_f}{\text{Var}(R_T)}
\end{aligned}$$

Using λ we get,

$$\Downarrow$$

Expected return for a general portfolio

- For any asset i

$$\mathbb{E}(R_i) = R_f + \frac{\mathbb{E}(R_T) - R_f}{\mathrm{Var}(R_T)} \mathrm{Cov}(R_i, R_T).$$

- For any portfolio P with weights x_i on assets with returns R_i,

$$\mathbb{E}(R_P) = \sum x_i \mathbb{E}(R_i), \quad \text{and} \quad \mathrm{Cov}(R_P, R_T) = \sum x_i \, \mathrm{Cov}(R_i, R_T).$$

Since the other terms do not change, we also have

$$\mathbb{E}(R_P) = R_f + \frac{\mathbb{E}(R_T) - R_f}{\mathrm{Var}(R_T)} \mathrm{Cov}(R_P, R_T).$$

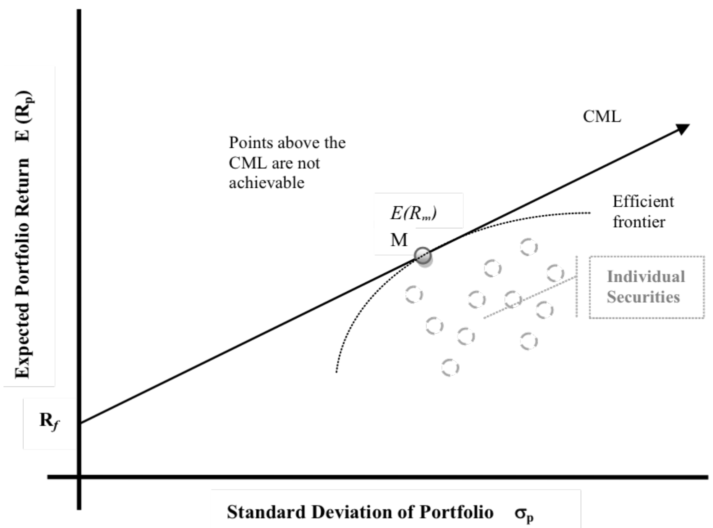
OBS: The expected return of individual assets or portfolios is determined by the covariance with the tangent portfolio.

From tangent to market

2. If, *in equilibrium*, the tangent portfolio is the same for every investor.

- Every investor will then hold a multiple of the risk-free asset and the same tangent portfolio.
- *In equilibrium*, every asset that exists must be owned by someone.
- The tangent portfolio is therefore the market portfolio – *everything in the investment universe in the same proportions as they exist.*
- Since the tangent is portfolio is efficient, this also implies that the market portfolio is efficient, when every investor has the same tangent portfolio.
- But, given an R_f, there is only one combination that is efficient so the tangent portfolio must be the market portfolio.

Capital Market Line



If everyone holds the same...

- If every investor holds the same market portfolio, we then have that the *equilibrium expected return* of any portfolio P, is given by

$$\bar{R}_P^e = R_f + \frac{\bar{R}_M - R_f}{\mathrm{Var}(R_M)} \mathrm{Cov}(R_M, R_P).$$

- We set

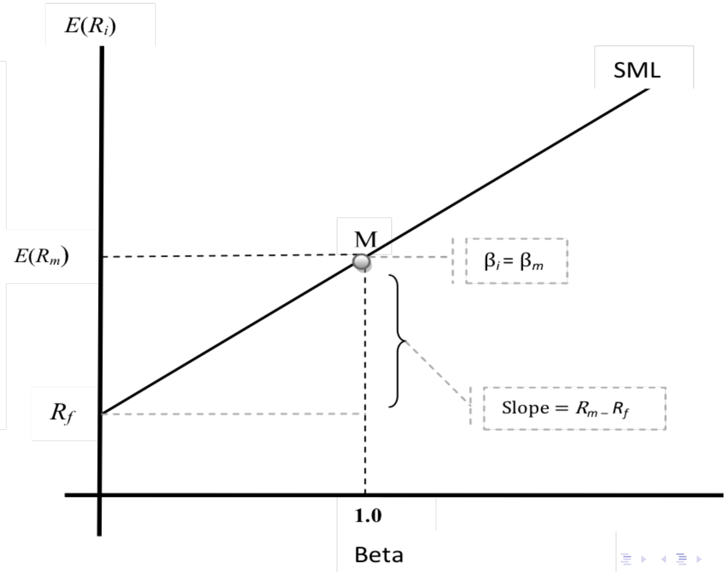
$$\beta_P = \frac{\mathrm{Cov}(R_P, R_M)}{\mathrm{Var}(R_M)} \; .$$

- This yields the CAPM equilibrium equation:

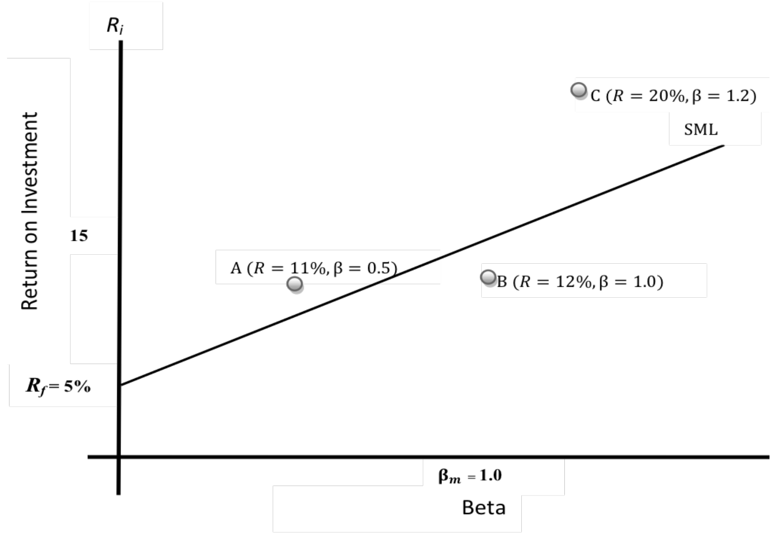
$$\bar{R}_P^e = R_f + \beta_P(\bar{R}_M - R_f).$$

OBS: The CAPM says that, in equilibrium, the return on any portfolio is determined by its covariance with the market, and nothing else! We get no compensation for other sorts of risks!

Securities Market Line



Securities Market Line



The assumptions of CAPM

3. We need enough assumptions to guarantee that, *in equilibrium*, every investor has the same tangent portfolio that is the market portfolio.
   - No transaction costs
     If there were transaction costs, then an investor's choice of investments would depend on existing holdings, since he would attempt to avoid incurring costs. These would vary from investor to investor.
   - Assets are divisible
     If every asset has to be held in the same proportion for each investor, then the amounts are unlikely to come out as integers for all investors. So we must assume that they can be held in fractions.
   - No tax effects
     Different investors are exposed to different tax rates. Often there are different rates for capital growth, and earned/saved income. These will affect different investors differently and so affect investment decisions. We therefore assume tax is the same for everyone and for all types and amounts of income.

The assumptions of CAPM

3. Assumptions (cont).
   - Not moving the market
     Whilst buying and selling shares moves market prices, we don't want any single investor's trading to move the market. This helps guarantee that all investors face the same decisions.
   - Mean-variance investors
     Investors are assumed to only use mean-variance analysis to make their decisions. This was a crucial part of our derivation of the tangent portfolio.
   - Short sales allowed
     Our derivation of the tangent portfolio does not require the weights to be positive. So we have to allow short sales.
     However, it turns out that the CAPM predicts that no one short sells so this axiom is not as necessary as it first appears: if everyone is holding the same proportion of each asset then that proportion must be positive for every asset.

The assumptions of CAPM

3. Assumptions (cont).
   - The risk-free asset
     Our derivation of the tangent portfolio and the argument that all investments would be a multiple of it assumed that there was infinite capacity to buy and sell at the risk-free rate.
   - Time horizon
     The mean returns and covariances of returns that determine the tangent portfolio depend upon the time horizon. We therefore must assume that the time horizon is the same for everybody.
   - Opinions
     The tangent portfolio depends on estimates of means and variances. For agreement on the tangent portfolio, these estimates must be the same across investors.
   - All assets are marketable
     An investor's optimum portfolio will be affected by any assets he holds which cannot be marketed. We therefore have to assume that even human capital is marketable and makes up part of the market portfolio.

Assessing a model

- The assumptions of CAPM are many and highly dubious.
- We also know that investors hold things other than the market portfolio.
- How do we assess a model? We should look on *how good its predictions are, not how good its assumptions are.*

So the question is: *"Does the CAPM equation hold in practice?"*

i.e,

- Does the beta of a stock determine its expected return in a linear fashion?
- Do zero beta portfolios have return equal to the risk-free rates?

Implementation issues

- We need to know the beta.
  As with the single-factor model, we only ever have an estimate of beta.
- We need to know the market portfolio, and we need to estimate its expected return.
  The theoretical market portfolio contains every asset you could possibly invest in, or even have exposure to! It is not feasible for us to invest in every possible investment. We need to rely on a proxy to the market portfolio.
  Examples of proxies: market indices such as S & P 500 or the FTSE 100!

Q: *Can we really test CAPM?*

Theory questions

1. What is the CAPM equation?
2. If we know the covariance of an asset with the tangent portfolio, how do we find its return? Give the derivation.
3. If everyone holds the same tangent portfolio, what can we say about its composition?
4. What are the assumptions of CAPM?
5. What do we typically use as the market portfolio when using CAPM?

1.2 Non-standard CAPM

- Learning objectives
- Two-factor CAPM
- CAPM with no borrowing
- CAPM with different lending and borrowing rates
- Other CAPM extensions
- Questions

Learning objectives

- derive the two-factor CAPM model,
- perform simple computations with the two-factor CAPM model,
- show that zero beta portfolios are not efficient in two-factor CAPM,
- state the problems with testing CAPM,
- give three different methods of testing CAPM,
- discuss known results from testing CAPM,
- explain Roll's objection to tests of two-factor CAPM,
- state the principle of falsification,
- contrast the Tobin separation theorem and CAPM.

CAPM without risk-free rate

- Suppose there is no risk-free asset.
- Suppose we take some efficient portfolio E with return R_E. Then we know by our previous work that E is the tangent portfolio for a *hypothetical* risk-free rate $R_{f,E}$.
- Our derivation of the equilibrium expected return equation used nothing else about the risk-free rate.
- So repeating it, we get for any portfolio P

$$\mathbb{E}(R_P) = R_{f,E} + \frac{\mathbb{E}(R_E) - R_{f,E}}{\mathrm{Var}(R_E)} \mathrm{Cov}(R_P, R_E).$$

Efficiency of the market portfolio

From CAPM assumptions

- Every investor is a mean-variance investor.
- Investor agrees on what portfolios are efficient.

We also know

- The efficient portfolios lie on a straight line in weight space.
- All investors' portfolios lie on a straight line.
- So the portfolio held by any group of investors taken together will be efficient.
- So the portfolio held by everyone taken together is efficient $\Longrightarrow$ the market portfolio, M, is efficient.

Two-factor CAPM

- We can use the fact that, *in equilibrium* the market portfolio is mean-variance efficient, to conclude there is some rate $R_{f,m}$ such that

$$\bar{R}^e_P = R_{f,m} + (\bar{R}_M - R_{f,m})\frac{\text{Cov}(R_P, R_M)}{\text{Var}(R_M)},$$

or

$$\bar{R}^e_P = R_{f,m} + (\bar{R}_M - R_{f,m})\beta_P.$$

- Also, if a portfolio Z has zero beta then

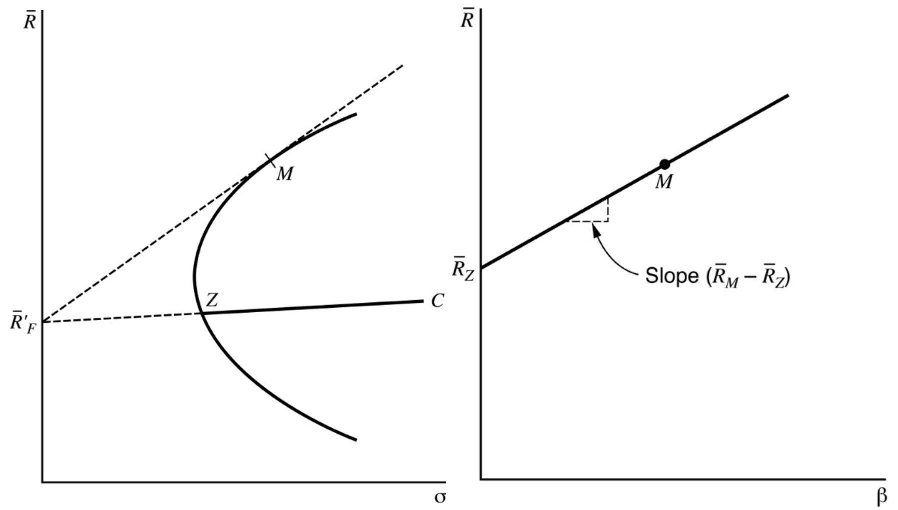
$$\bar{R}^e_Z = R_{f,m}.$$

- The equilibrium relation becomes :

$$\bar{R}^e_P = \bar{R}^e_Z + \beta_P(\bar{R}_M - \bar{R}^e_Z).$$

This is called the two-factor version of CAPM.

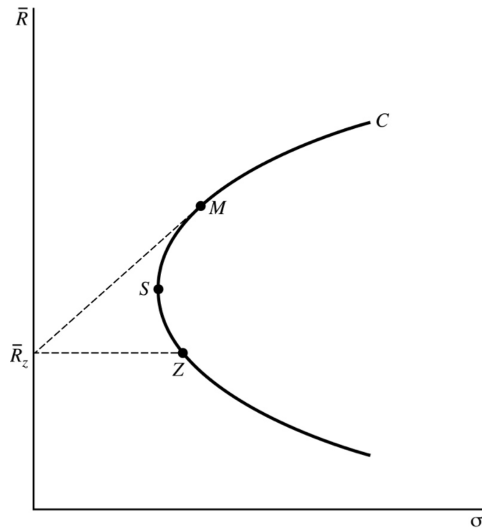
CAPM with no riskless asset



Zero beta portfolios

- Zero beta portfolios exist – just use a combination of long and short positions to cancel out betas.
- In fact, there will be many zero beta portfolios. For concreteness, we will choose one Z that minimizes variance. (This actually specifies it uniquely.)
- As in CAPM, the equilibrium expected return is a linear function of the β.
- The difference is that the straight line does not have to go through the risk-free rate, but through the zero beta expected return in equilibrium.

Z is not efficient

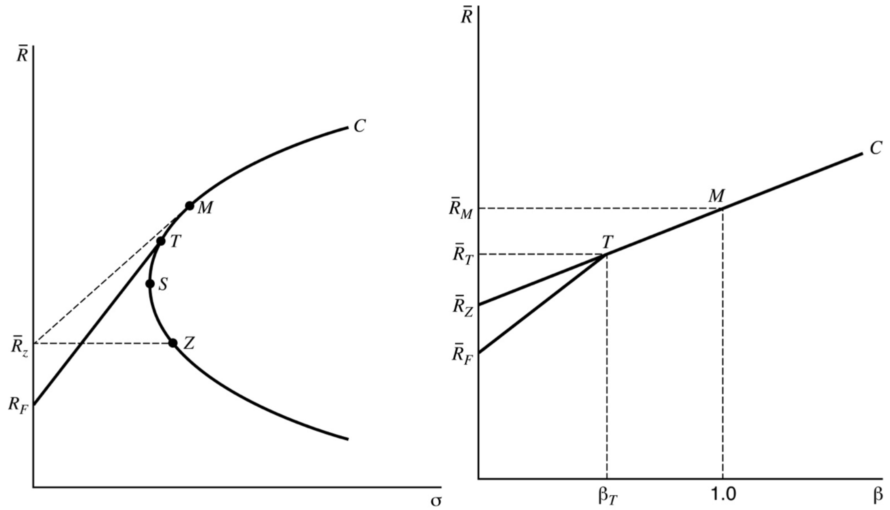


Consider S the minimum risk combination of Z and M.
Since Z and M have zero covariance (Z has zero beta), the holdings of Z and M are

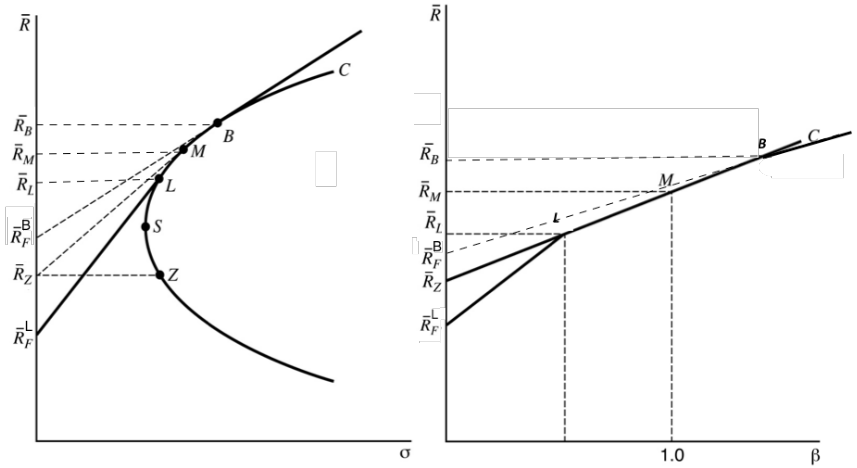
$$\frac{\sigma^2_M}{\sigma^2_M + \sigma^2_Z} \text{ and } \frac{\sigma^2_Z}{\sigma^2_M + \sigma^2_Z}.$$

Since it contains a positive amount of M, its beta is positive, and its return is greater than $\mathbb{E}(R_Z)$.
Also, its variance is lower since S is the minimal variance combination.

CAPM with no borrowing



CAPM with different lending and borrowing rates



Other CAPM extensions

We looked at:

- Short sales disallowed
- Modifications of riskless lending and borrowing

Other extensions:

- Personal taxes
- Nonmarketable assets
- Heterogeneous expectations
- Non-price-taking behaviour
- Multiperiod CAPM
- ...

Theory questions

1. Derive the two-factor CAPM equation.
2. Show that a zero beta portfolio is not efficient in the two-factor CAPM.
3. Derive the CAPM equations for other hypothesis concerning lending and borrowing.
4. Identify other extensions and discuss their relevance.

1.3 Testing CAPM

- Learning objectives
- Problems testing CAPM
- Well-known tests
- Roll's critique
- Questions

Learning Objectives

- To identify the main problems faced by empirical tests of CAPM.
- Explain the findings of well-known studies about CAPM.
- Understand the Roll's critique to empirical tests.

Testing CAPM

- CAPM makes the prediction that a stock's *equilibrium expected return* is determined by its covariance with the market portfolio and lies on a straight line through the risk-free rate.
- Two- factor CAPM makes the prediction that a stock's *equilibrium expected return* is determined by its covariance with the market and lies on a straight line which does not have to be through the riskless rate.

Q: Do these predictions hold?

Q: How do we test them?

Problems Testing CAPM

- Problem 1: expectations versus realizations
  Our first problem is that the CAPM is a statement about expectations. We cannot measure expectations. We can only measure returns. Returns are therefore typically used as a proxy for expectations.
- Problem 2: beta estimation
  Recall from single-factor models, the problems with measuring the beta. Betas are not known. We can only measure historical betas. We therefore have to use the historical beta as a proxy for the future beta. We are essentially assuming that betas are stable over time, and ignoring the noise in the beta estimation.
- Problem 3: finding the market portfolio
  CAPM is a statement about the *market portfolio*. What does that actually mean? The market portfolio is supposed to mean all possible assets that can be purchased. It is not practical to work with that set, so we can only test proxies.

Tests

This all means that any test is not a test of CAPM directly but instead atest of CAPM jointly with the assumptions needed implement the test such as

- Historical returns are a good proxy for expectation.
- Historical betas are a good proxy for future beta.
- The S&P is a good proxy for the market portfolio.

A simple test

- Suppose we are a fund manager and we decide to use the CAPM.
- We would decide what level of multiplier to market risk is appropriate and then form a portfolio of stocks with that level of beta.
- If CAPM was correct then the returns across time would be close to multiples of the market returns.
- We could look back historically to see if this was true.

Sharpe and Cooper (1972) results

Sharpe and Cooper carried out such a test.

- Divide stocks into ten portfolios once a year according to their betas.
- Roll-over once a year for each level of betas.
- Use beta estimated from previous 5 years.
- Measure average return and average beta for each portfolio over a long period of time.

Their numbers lie close to a straight line with the equation

$$R_i = 5.54\% + 12.75\%\beta_i,$$

this suggests a linear relationship between historical returns and betas.

However, the risk-free rate in that period was 2%. This suggests that two-factor CAPM may be good, but one-factor is bad.

Two-pass regressions

A standard methodology for testing CAPM and two-factor CAPM is the two-pass regression test.

- One takes stock returns over several years
- Regression 1: The returns are then regressed against market returns to estimate the betas, β_i, and the residual variances, $\mathrm{Var}(e_i)$.
- Regression 2: We then perform a second regression of the returns against a constant, the beta and the residual variances.

$$R_i = a_1 + a_2\beta_i + a_3\,\mathrm{Var}(e_i).$$

- CAPM says

$$a_1 = R_f,$$
$$a_2 = \mathbb{E}(R_M - R_f),$$
$$a_3 = 0.$$

Lintner test of two-factor CAPM

Lintner (1965) carried out this test and found
$$a_1 = 0.108,$$
$$a_2 = 0.063,$$
$$a_3 = 0.237.$$

These are very different from CAPM:

- a_1 is way too big when compared to the risk-free rate;
- a_3 is very far from zero predicted by CAPM and two-factor CAPM.

Other authors have done similar tests.

- There are suggestions that Lintner's results actually arise from sampling bias and that using portfolios rather than stocks one gets that a_3 is close to zero as predicted.
- But that a_1 is not close to R_f.
- So, CAPM looks bad, but two-factor looks better.

Roll's objection

- Suppose we decide to test the two-factor CAPM.
- Suppose we use a portfolio, L, that turned out to be efficient as our proxy for the market portfolio.
- Then if we pick Z to have zero historical beta against L, the two-factor model derivation holds since we required no other assumptions to make it work.
- Since the two-factor CAPM's derivation holds, its predictions are correct by construction.

Roll's critique: correctness by construction

So,

- If we use a portfolio that turned out to be efficient as a proxy for the market portfolio, we have tested nothing at all.
- If we do not, then all we are testing is the efficiency of our proxy NOT the two-factor CAPM.
- It is important to realize Roll is not arguing anything about the validity of CAPM.
- He is arguing that the tests are meaningless not the model.

He says

"Unfortunately, it has never been subjected to an unambiguous empirical test. There is considerable doubt, at least by me, that it ever will."

Principle of falsification

A statement is only meaningful if one can imagine circumstances under which it has been disproved.

This is the basis of scientific method:

- A model predicts the results of new experiments.
- We do the experiments.
- If the experiments disagree with the model, then the model is wrong.

CAPM versus Tobin

Tobin's separation theorem say:

- that if TWO mean-variance investors have the same situation except risk preferences, then they hold the same tangent portfolio but differing amounts in the risk-free asset and it.
- CAPM says that if ALL investors are mean-variance investors and ALL have the same situation then the tangent portfolio is the market portfolio.
- If we make people's situations vary even a little then CAPM does not hold.
- Tobin is only a statement about two investors.

Theory questions

1. What are the problems with testing CAPM?
2. What was Sharpe and Cooper's test of CAPM and what did they find?
3. What was Linter's test of CAPM and what did he find?
4. Explain Roll's objection to tests of the two-factor CAPM model.
5. What is the principle of falsification?