Quantitative Finance

Economics, Finance and Management

Joaquim Montezuma de Carvalho, Alfredo D. Egídio dos Reis

LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

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Available income of people may be applied in two main forms:

- **Consumption**: Expenditure in goods or services that have a defined life time, which does not permit any return on that which has been spent.
- **Saving**: That may be changed into liquid currency without any type of income or by **investment**:

  Application in the form of real property or financial assets with the intention of attaining an income.

Capitalization: (or, **Accumulation of Capital**) Transformation, provoked by time, of a capital into capital and interest;

Principal: Amount of money invested or borrowed (present value).
It is a stock variable, and that is always related to a moment/instant, at the beginning or the end of a period of capitalization;

Interest: Cost or charge for the use of borrowed money.
The Interest is a flow variable, is always related to a period.

Interest Rate: Interest is usually expressed as a percentage for a given period. Percentage of the amount of interest charged for a loan for a given time period.

Example

$i_A = 5\%$: 5% per annum, per year, $i_M = 1.5\%$: 1.5% per month.

Time, Term: Length of the loan in time units (period). There's a correspondence to the rate.
Refers to the period during which the capital is applied periodically.

Period: Accumulation period, Payment period, Interest time period.

Example

The period can be: Annual, Semi-annual, Quarterly, Monthly, etc. Correspondingly, the Year, the Semester, the Quarter, the Month, etc.

Example

Similarly, the **Payment Period** could be: Annually: once a year; Semi-annually: Twice a year; Quarterly: 4 times a year, etc.

Golden Rules of financial mathematics

- **1st Golden rule**:
  The presence of principal, the presence of time and the absence of interest is an impossibility.
  The absence of principal or the absence of time and the presence of interest is another impossibility.
- **2nd Golden rule**:
  Any mathematical operation on two or more capitals requires homogenization in time.
  That is, capitals cannot be added unless they are valued at the same point in time.
- **3rd Golden rule** ($I = P \times i$):
  The interest in each period of capitalization is equal to the principal at the beginning of the period multiplied by the interest rate.

Some Basic Concepts

Approximate Time: Each month is assumed to be 30 days with exact time used for any portion of a month ;

Ordinary Interest: The length of a year is assumed to be 360 days (Bankers Rule);

Exact Time: Every day of the term except the first day;

Exact Interest: The length of a year is taken as 365 days (366 for a leap year, February = 29 days)

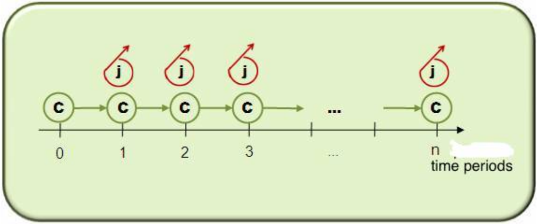
Accummulation in Simple Interest Regime

The Interest produced in each period is not added to the Principal.

Two sub-regimes may be distinguished:

- The regime of simple interest: The interest produced in each period is paid;
- The regime of interest: The interest produced in each period is retained.

Interest gains are constant throughout the period of time, unless interest of each period only varies if the interest rate varies.



Definition (Interest calculation)

The interest produced throughout the period of time is given as:
Let I := Interest ; P := Principal; t := Time (in time units); i := Interest Rate, referred to the time unit:

$$I = P \times i \times t = P\,i\,t$$

Definition (Maturity Value or Future Value)

The Maturity Value or Future Value (FV) is given as:

$$FV = P + I = P\,(1 + it)$$

Definition (Present Value)

Equivalently, Present Value, PV

$$PV = P = \frac{FV}{(1 + it)} = FV\,(1 + it)^{-1}$$

Definition (Equation of Value, Focal Date)

Equation of Value is a mathematical expression that equates several pieces of money, due at certain dates, at a same chosen date (**Focal Date**)

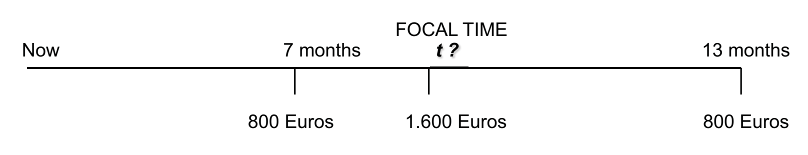
Definition (Equivalent Time)

Equivalent time: When a single payment equals the sum of the original debts.
The unknown date of that payment will be called the **Average Due Date**, that is usually measured in days from the present.

Example

Assume that two payments of €800 are due in 7 and 13 months, respectively. Using an interest rate of $i_A = 8.0\%$ find the date when a single payment of € 1600.

What is the equivalent time t?



$$1600(1+0.08\,t)^{-1} = 800\left(1+\frac{7}{12}0.08\right)^{-1} + 800\left(1+\frac{13}{12}0.08\right)^{-1}$$

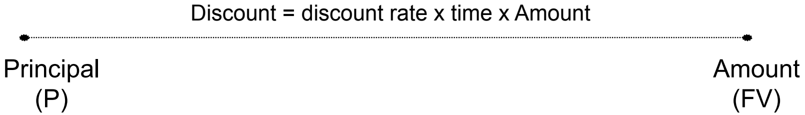
$$t \simeq 0.82\,864 \text{ years} \rightarrow 9 \text{ months } 28 \text{ days } 7\text{h}30\text{m}$$

Simple Discount (or Bank Discount)

Definition (Simple Discount Interest, Amount, Discount)

Simple discount interest is based on the **Future Value**, where interest is paid at the beginning of the term (upfront).

- The money that is borrowed is called the **Amount** (Future Value).
- The rate used to compute the interest charges is called the **Discount Rate**.
- The interest charged for the use of borrowed money is called the **Discount**.
- The money received is called **Proceeds** or **Principal** (Present Value)



Principal: Money that the borrower receives at the time zero.
Amount: what is due at the end of period

Formula (The Simple Discount Formula)

$$
\begin{aligned}
D &= FV \times d \times t \\
P &= FV - D
\end{aligned}
$$

D: Amount of Simple Discount; FV: Maturity Value;
d: Interest Discount Rate (for a unit period)
t: time

Formula (Simple Discount Formulae)

$$D = FV \times d \times t$$

$$P = FV - D$$

$$P = FV - FV\, d\, t$$

$$P = FV(1 - d\, t)$$

$$FV = \frac{P}{(1 - dt)}$$

Examples of Simple discount instruments:

- **Promissory note**: Document on which one party writes his or her promise to pay another party the principal and the interest for a loan due at some date in the future.
- **Treasury Bills, T-Bills**: Short-term loans to the U.S. federal government, carrying terms ranging from a few days to 6 months, though the most common terms are 4, 13, or 26 weeks
- **Commercial paper**: A type of promissory note used by corporations, and is short-term (term is lower than 12 months).

Example

A €10,000 face value discount note has a term of 4 months. The simple discount rate is 6%/year. Find the amount of the discount.

Solution:

$$
\begin{aligned}
D &= FV\, d\, t \\
D &= 10,000 \times 0.06 \times 4/12 \\
D &= 200
\end{aligned}
$$

- Discount interest is calculated over **Future Value** and paid upfront
- *Normal* Interest is paid over **Principal (Present Value)** and paid at the end.

Example

Mr. X borrows €1, 000 for 1 year, paying 10% interest upfront.

- He gets $P =$€**900** only; with a Discount of €**100**; Pays a $FV =$**1000** at the end.
- Effective **interest rate** is

$$i_A = \frac{100}{900} = \frac{1000 - 900}{900} = \frac{1}{9} = 0.111(1) \neq 10\%$$

- **Discount rate** $d = 10\%$; **Interest rate** $i_A = 11,1(1)\%$:

$$900(1 + 0.111(1)) = €1,000. \text{ Discount} = €100 .$$

Interest *versus* discount rate

- **Discount Rate**, d:

$$d = \frac{\textbf{Discount}}{\text{Future Value}} = \frac{FV - PV}{FV} \Leftrightarrow PV = FV\,(1-d)$$

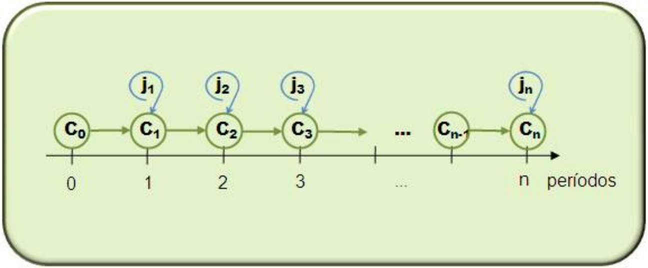
- Discount Rate d & Interest Rate i

$$\begin{aligned} PV &= FV\,(1+i)^{-1} \\ FV\,(1-d) &= FV\,(1+i)^{-1} \\ d &= \frac{1+i}{i} \Leftrightarrow i = \frac{d}{1-d}\;. \end{aligned}$$

- Other way,

$$d = \frac{FV - PV}{FV} \quad \text{and} \quad i = \frac{FV - PV}{PV}$$

Compound Interest



$$
\begin{aligned}
P_1 &= P_0 + I_1 = P_0 + P_0 i \cdot i = P_0\left(1+i\right) \\
P_2 &= P_1 + I_2 = P_1\left(1+i\right) = P_0\left(1+i\right)^2 \\
P_3 &= P_2 + I_3 = P_2\left(1+i\right) = P_0\left(1+i\right)^3 \\
&\ldots \\
P_n &= P_{n-1} + P_n = P_{n-1}\left(1+i\right) = P_0\left(1+i\right)^n,\ n = 0, 1, 2, \ldots
\end{aligned}
$$

Formula (Future Value or Acummulated Value Formula)

$$FV = S = P(1+i_A)^t\,, \quad t \geq 0$$

$(1+i)$ *is the Acummulation factor.*

Equivalently,

Formula (Present Value or Discounted Value)

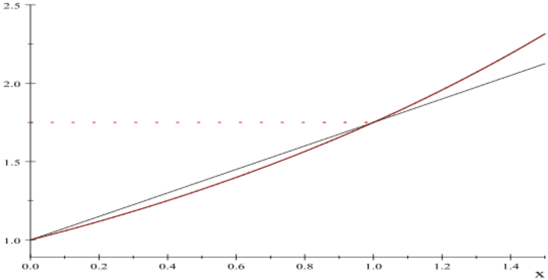
$$\begin{aligned} P &= S(1+i_A)^{-t} = \frac{S}{(1+i_A)}\,, \quad t \geq 0 \\ &= S\,v^t \end{aligned}$$

$$v = (1+i_A)^{-1}$$

$v = (1+i_A)^{-1}$ *is the Discount Factor.*

Simple Interest Formula: $S_t = P(1 + t.i_{\grave{A}})$, i_A: Annual rate.

Compound Interest Formula: $S_t = P(1 + i_A)^t$, $t \geq 0$



$0 < t < 1$: Simple interest acummulates more money;

$t = 1$: Both acummulate the same;

$t > 1$: Compound interest acummulates ("a lot") more money.

Example ($i_A = 30\%$)

t years	Simple	Compound
0	€100,000	€100,000
0.01	100,300	100,263
0.1	103,000	102,658
0.2	106,000	105,387
0.4	112,000	111,065
0.5	115,000	114,018
1	130,000	130,000
2	160,000	169,000
4	220,000	285,610

Effective Rates and Nominal Rates

Often, banks and others, Nominal Interest Rates are the rates presented to clients and investors. These, are rates proportional to the effective ones.

Definition (Annual Nominal Rate $i_A^{(m)}$)

The **Nominal Interest Rate**, denoted as $i^{(m)}$, is the annual percentage rate.
m is the number of conversion periods per year (usually);
Conversion Period (interest, accumulation or capitalization period) is the time between successive computations of interest;

Definition (Effective Rate i)

Is the interest rate per conversion period, proportional to $i^{(m)}$:

$$i = i^{(m)} / m \, .$$

Example (Conversion periods)

Commonly: Annual, bi-annual (semi-annual), quarterly, monthly, daily...

Example

$P = €1,000.00$, converted semi-annually, biannual rate of 5%, term is one year. Annual effective rate, equivalent rate:

- Annual nominal rate, with biannual conversion: $i^{(2)} = 10\%$;
- Effective rate: $i^{(2)}/2 = 5\%$;
- Annual Effective Rate: $i_A = 10.25\%$ (Equivalent rate to i_S of 5%):

$$
\begin{aligned}
P(1+0.05)^2 &= P(1+i_A) \Leftrightarrow (1.05)^2 = (1+i_A) \\
i_A &= (1.05)^2 - 1
\end{aligned}
$$

Definição (Equivalent rates)

Rates that produce the same return for a given investment.

Definição (Annual Effective Rate (Annual Percentage Yield-APY))

An Annual Effective Rate is an equivalent rate that is an annually converted rate $i(1)$ such that gives the same interest earnings as the nominal rate $i(m)$ converted m times per year, where $m \neq 1$.

Remark

Mathematically we can put $m = 1$, although it is not used.

Formula (APY calculation)

For each unit amount invested $P = €1$

$$(1+i) \quad = \quad \left(1+\frac{i^{(m)}}{m}\right)^m$$

Formula (APY calculation)

$$(1+i) = \left(1+\frac{i^{(m)}}{m}\right)^{m}$$

$$i = \left(1+\frac{i^{(m)}}{m}\right)^{m} - 1$$

Formula (Nominal Rate calculation)

$$(1+i) = \left(1+\frac{i^{(m)}}{m}\right)^{m}$$

$$(1+i)^{\frac{1}{m}} = \left(1+\frac{i^{(m)}}{m}\right)$$

$$i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$$

Definition (Compound Rate)

It is a growth rate that equates the present value and the future value for a number of compounding periods.

Formula (Compound Rate Formula, calculation)

$$
\begin{aligned}
S &= P(1+i)^n \\
\frac{S}{P} &= (1+i)^n \\
i &= \left(\frac{S}{P}\right)^{1/n} - 1
\end{aligned}
$$

How long does it take a (present) sum of money to increase certain (future) amount? Find time t.

Formula

$$\begin{aligned} S &= P(1+i)^t \Leftrightarrow \frac{S}{P} = (1+i)^t \\ \ln\frac{S}{P} &= t\, ln(1+i) \\ t &= \frac{ln\,(S/P)}{ln(1+i)} \end{aligned}$$

Remark: Time must be shown in Years, Months & Days. In most investment operations the "day " is taken as the smalest time unit

Definition (Equation of Value, Focal Date [Compound Interest])

Equation of Value is a mathematical expression that equates several pieces of money, due at certain dates, at a same chosen date (**Focal Date**)

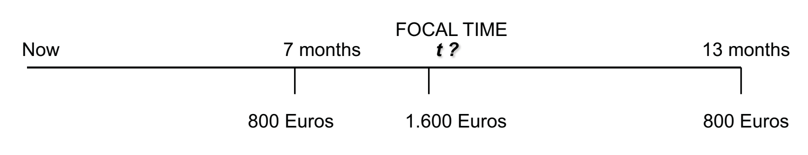
Definition (Equivalent Time)

Equivalent time: When a single payment equals the sum of the original debts.
The unknown date of that payment will be called the **Average Due Date**, that is usually measured in days from the present.

Example ([previous ex.] in Compound interest)

Assume that two payments of €800 are due in 7 and 13 months, respectively. Using an interest rate of $i_A = 8.0\%$ find the date when a single payment of € 1600.

What is the equivalent time t (from today)?



$$
\begin{aligned}
1600(1.08)^{-t} &= 800(1.08)^{-7/12} + 800(1.08)^{-13/12} \\
-t \, ln\,(1.08) &= ln\left(\left[(1.08)^{-7/12} + (1.08)^{-13/12}\right]/2\right) \\
t &\simeq 0.83\,093 \text{ years} \rightarrow 9 \text{ months} + 29 \text{ days}
\end{aligned}
$$

Annual Effective Rate, i_A, “vs" Annual Nominal Rate, $i_A^{(m)}$

$$1 + i_A = \left(1 + \frac{i_A^{(m)}}{m}\right)^m$$

$$i_A^{(m)} = m\left[(1 + i_A)^{1/m} - 1\right]$$

$$i_A^{(m)} = \frac{\left[(1 + i_A)^{1/m} - 1\right]}{\frac{1}{m}}$$

Compute the limit, $\lim_{m\to\infty} i_A^{(m)}$, directly (indertermination):

$$\lim_{m\to\infty} i_A^{(m)} = \lim_{m\to\infty} \frac{\left[(1 + i_A)^{1/m} - 1\right]}{\frac{1}{m}} = \frac{0}{0}$$

Using *l'Hôpital*'s rule, we have.

$$\begin{aligned}
\lim_{m\to\infty} i_A^{(m)} &= \lim_{m\to\infty} \frac{\left(\left[(1+i_A)^{1/m}-1\right]\right)'}{\left(\frac{1}{m}\right)'} = \lim_{m\to\infty} \frac{\left(e^{\frac{1}{m}\ln(1+i_A)}-1\right)'}{-\frac{1}{m^2}} \\
&= \lim_{m\to\infty} \frac{\left[e^{\frac{1}{m}\ln(1+i_A)}\ln(1+i_A)\right]\left(-\frac{1}{m^2}\right)}{\left(-\frac{1}{m^2}\right)} \\
&= \lim_{m\to\infty} (1+i_A)^{1/m}\ln(1+i_A) \; .
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{m\to\infty} i_A^{(m)} &= \lim_{m\to\infty} (1+i_A)^{1/m}\ln(1+i_A) = (1+i_A)^{1/\infty}\ln(1+i_A) \\
i_A^{(\infty)} &= \delta = \ln(1+i_A)
\end{aligned}$$

Nominal rate of interest compounded continuously or Interest Force (constant)

Where i_A is the Annual Effective Rate

$$\begin{aligned} \delta &= i_A^{(\infty)} = \ln(1+i) \\ e^{\delta} &= (1+i_A)\,. \end{aligned}$$

Future and Present Value Formulas for Continuous Compounding

$$\begin{aligned} S &= P(1+i_A)^t\,, \quad (t \text{ years}) \\ S &= P\,e^{\delta\,t} \\ \\ P &= S\,e^{-\delta\,t} \end{aligned}$$

Example

Future Value for an investment of €5 000, for $t = 5$ years and interest force of 5,5%?

$$S = 5000e^{0,055\times 5} = 6582,65€$$
$$i_A = e^{\delta} - 1 = e^{0.055} - 1 \Rightarrow 5,65\% \text{ approx.}$$

Definition (Ordinary Annuity)

An **Annuity** is a sequence of payments (sometimes equal) dispersed or received at equal intervals of time.
By default, or usually, at the end, sometimes at the beginning.

Ex.: A house rent, instalments of a loan.

Definition (Term)

The **Term** is the time of the operation, from start to end.

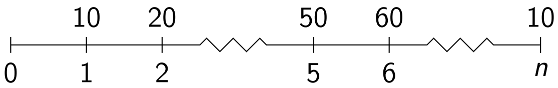


Figure: Time diagram for an annuity-immediate

Different classifications

- Certain or Contingent:
  Number of payments is known in advance. ⇒ *Annuity Certain* or *Guaranteed Annuity*
  Sometimes payments are done under some circumstances, like having a beginning or ending date that depends on some event. ⇒ *Contingent Annuity*. Ex.: Life Annuity.
- Temporary or Perpetual:
  Perpetuity: Annuity with a specific starting time but an infinite number of payments. Ex.: Perpetual bonds.
- Immediate or Due:
  Deals with the placement of the periodic payments.
  Ordinary Annuity or annuity-immediate: When the payments are done at the end of each time period.
  Annuity Due: If the payments are done at the beginning of each period.

Ordinary Annuity (or Annuity immediate), **Present Value** of a unit payment annuity

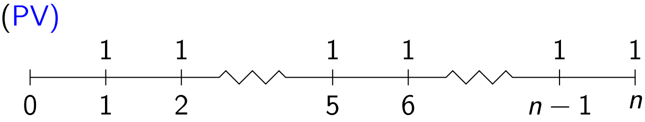


Figure: Time diagram for an Ordinary Annuity

$$
\begin{aligned}
PV &= 1 \times (1+i)^{-1} + 1 \times (1+i)^{-2} + ... + 1 \times (1+i)^{-n} \\
a_{\overline{n}|i} &= 1\left[(1+i)^{-1} + (1+i)^{-2} + ... + (1+i)^{-n}\right] \\
a_{\overline{n}|i} &= v + v^2 + ... + v^n,
\end{aligned}
$$

Geometric progression sum, with rate $r = v = (1+i)^{-1}$.

$$a_{\overline{n}|i} = \frac{u_1 - u_n \times r}{1-r} = \frac{v - v^n v}{1-v} = \frac{1-v^n}{1/v - 1} = \frac{1-(1+i)^{-n}}{i}$$

Annuity-immediate, **Future Value** of a unit payment annuity

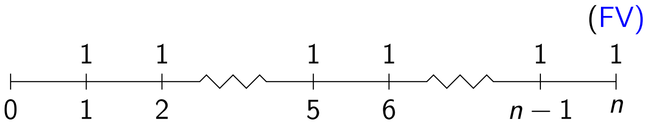


Figure: Time diagram for an Annuity-immediate

From $a_{\overline{n|}i}$ it's easy to find formula for $FV = s_{\overline{n|}i}$

Formula for $s_{\overline{n|}i}$

$$FV = s_{\overline{n|}i} = a_{\overline{n|}i}(1+i)^n$$

$$= \frac{1-(1+i)^{-n}}{i}(1+i)^n = \frac{(1+i)^n - 1}{i}$$

Relationship between $a_{\overline{n}\rvert}$ and $s_{\overline{n}\rvert}$

$$s_{\overline{n}\rvert i} = a_{\overline{n}\rvert i}\,(1+i)$$

Present Value of an Ordinary Annuity, with constant installment R

$$A_n = R\,a_{\overline{n}\rvert i} = R\left(\frac{1-(1+i)^{-n}}{i}\right)$$

Future Value of an Ordinary Annuity, with constant installment R

$$S_n = R\,s_{\overline{n}\rvert i} = R\left(\frac{1-(1+i)^{-n}}{i}\right)$$

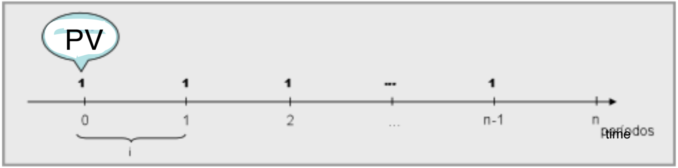
A_n: Present value; S_n: Present value; n: No. of payments; R: Periodical payment; $i = i^{(m)}/m$: Interest rate.

Knowing the rate of an annuity provides a realistic way to compare different investment or loan rates.
However, for instance, money lenders sometimes hide the true rate that they are charging for lending money. Apart from the interest rate they add other costs to the payments. Besides, different institutions act differently.
The best way to compare different lend proposals or investments is to calculate the Annual percentage Rate (**APR**):

Annual percentage Rate (APR)

It is the rate at which the cash value of the loan equals the present value of the payments

Annuity due, Present Value

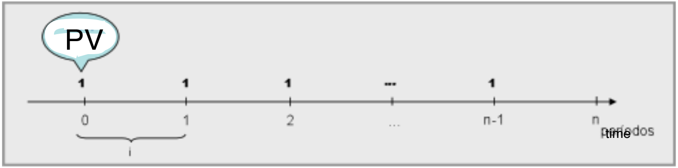


$$\begin{aligned}\ddot{a}_{\overline{n}|i} &= 1 + v + v^2 + \ldots + v^{n-2} + v^{n-1} \\ &= 1 + a_{\overline{n-1}|i} = a_{\overline{n}|i}(1+i)\end{aligned}$$

Geometric progression sum, rate $v = (1+i)^{-1}$,

$$PV = \ddot{a}_{\overline{n}|i} = \frac{1 - v^{n-1} \times v}{1 - v} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{i}(1+i) = a_{\overline{n}|i}(1+i)$$

Annuity due, Future Value



$$\begin{aligned} \textbf{Sum} &= (1+i)^n + (1+i)^{n-1} + \ldots + (1+i)^2 + (1+i) \\ \ddot{s}_{\overline{n|}} &= \frac{(1+i)^n - 1}{1-v} = \frac{(1+i)^n - 1}{i}(1+i) \end{aligned}$$

Geometric progression sum, rate $v = (1+i)^{-1}$,

$$FV = \ddot{s}_{\overline{n|}} = s_{\overline{n|}}(1+i)$$

Example (Deferred Annuity)

$$PV = 650\ {}_{2|}a_{\overline{4}|i} = 650\ {}_{3|}\ddot{a}_{\overline{4}|i}$$

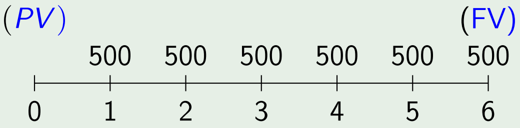
Definition (Deferred Annuity)

It's an annuity under which the first payment occurs at some specified future time.
The *PV* (Present Value) of an **annuity-due deferred** k **years**, with constant payment R, is given by

$$\begin{aligned} PV &= R\ {}_{k|}\ddot{a}_{\overline{n}|i} = R\, v^k\, \ddot{a}_{\overline{n}|i} \\ &= R\ {}_{k-1|}a_{\overline{n}|i} = R\, v^{k-1}\, a_{\overline{n}|i} \end{aligned}$$

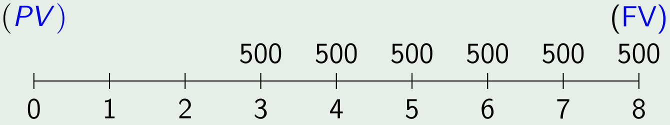
Example (Ordinary *vs* Deferred)

Ordinary annuity



$$PV_0 = 500\, a_{\overline{6}|1\%} = 2869.05; \quad FV_6 = 500\, s_{\overline{6}|1\%} = 3076.01$$

Deferred annuity



$$PV_0 = 500\, a_{\overline{6}|1\%} \times 1.01^{-2} = 2840.64$$
$$FV_8 = 500\, s_{\overline{6}|1\%} = 3076.01$$

Definition (Perpetuity)

A Perpetuity is an annuity with infinite term.

Ordinary Perpetuity, Present Value

$$a_{\overline{\infty}|} = \lim_{n\uparrow\infty} a_{\overline{n}|i} = \lim_{n\to\infty} \frac{1-(1+i)^{-n}}{i} = \lim_{n\to\infty} \frac{1}{i}\left(1-\frac{1}{(1+i)^n}\right) = \frac{1}{i}$$

Perpetuity-due, *PV*

$$\ddot{a}_{\overline{\infty}|} = 1 + \frac{1}{i}$$

Example

Perpetua will start studying at ULisboa, once in Lisbon she intends to stay and find a job afterwards. $i_A^{(12)} = 6\%$. She has two options:

a) Rent a Flat: €500/month; b) Buy a flat: €120,000.

$$i_M = 5\%/12 = 0.5\%$$

$$PV = 500 a_{\overline{\infty}|i_M} = 100\,000.00€ < 120\,000.00€$$

PV of a R-payment annuity

$$A_n = R\, a_{\overline{n}|i} = R\left(\frac{1-(1+i)^{-n}}{i}\right)$$

Calculation of Payment R, knowing A_n, n & i

$$R = \frac{A_n}{a_{\overline{n}|i}}$$

Calculation of Time n, knowing A_n, R & i

$$\begin{aligned} i\, A_n &= R\left(1-(1+i)^{-n}\right) \\ (1+i)^{-n} &= 1 - i\, A_n / R \\ -n \ln(1+i) &= -\ln\left(1 - i\, A_n / R\right) \\ n &= -\ln\left(1 - i\, A_n / R\right) / \ln(1+i) \end{aligned}$$

Calculation of Rate i, knowing A_n, R & n

Use equation

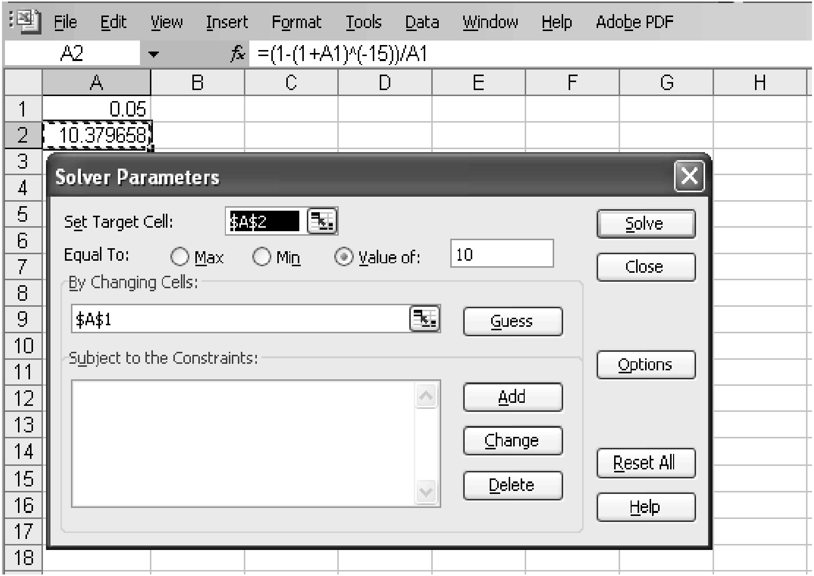
$$
\begin{aligned}
i\,A_n &= R\left(1-(1+i)^{-n}\right) \\
(1+i)^{-n} &= 1 - i\,A_n/R
\end{aligned}
$$

It can be solved numerically with software: With Excel (function *Rate*, Solver)

Excel function: RATE(np,1,pv,type,guess)

np $= n$,
pv $= -A$,
type $= 0$ (or omitted) for annuity-immediate, 1 for annuity-due
guess $=$ starting value, set to 0.1 if omitted
Output $= i$, rate of interest per payment period of the annuity

Calculation of Installments, Time& Rate



Example (Payments in Increasing Arithmetic form)

O John is buying a computer and iPhone payable in three instalments: €950, 1000 e 1050 (interest included, 9% annual).Present Value?

$$
\begin{aligned}
\text{P.V.} &= 950.00(1.09)^{-1} + 1000.00(1.09)^{-2} + 1050.00(1.09)^{-3} \\
&= 2524.03€
\end{aligned}
$$

$$
\begin{array}{llllll}
= & 950(1.09)^{-1} & + & 1000(1.09)^{-2} & + & 1050(1.09)^{-3} \\
\hline
= & 900(1.09)^{-1} & + & 900(1.09)^{-2} & + & 900(1.09)^{-3} \\
+ & 50(1.09)^{-1} & + & 50(1.09)^{-2} & + & 50(1.09)^{-3} \\
 & & + & 50(1,09)^{-2} & + & 50(1.09)^{-3} \\
 & & & & + & 50(1.09)^{-3} \\
\hline
= & 900a_{\overline{3}|9\%} & + & & & \\
+ & 50a_{\overline{3}|9\%} & + & 50_{1|}a_{\overline{2}|9\%} & + & 50_{2|}a_{\overline{1}|9\%} \\
\hline
= & \multicolumn{5}{l}{€2524.03 = (950-50)a_{\overline{3}|} + 50\,(Ia)_{\overline{3}|}}
\end{array}
$$

Result (P.V. Payments in Increasing Arithmetic form)

Present Value (P.V.)

$$PV = (C-h)\, a_{\overline{n}|} + h\left(a_{\overline{n}|} + {}_{1|}a_{\overline{n-1}|} + {}_{2|}a_{\overline{n-2}|} + \ldots + {}_{n-1|}a_{\overline{1}|}\right) = $$
$$= (C-h)\, a_{\overline{n}|i} + \mathbf{h}.\,(\mathbf{Ia})_{\overline{n}|\mathbf{i}}$$

where, simplifying,

$$(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i} = \frac{\ddot{a}_{\overline{n}|i} - n(1+i)^{-n}}{i}$$

- *If* $C = h$ *then* $PV = h \cdot (Ia)_{\overline{n}|i}$
- *If* $C = h$ *and* $h = 1$ *then* $PV = (Ia)_{\overline{n}|i}$

Example (Payments in Decreasing Arithmetic form)

Edward is buying an *iPod* payble in 4 monthly instalments of 45, 35, 25 e 15€. $i_M = 0,012$, $h^* = -10$. P.V.?

$$PV = 45(1.012)^{-1} + 35(1.012)^{-2} + 25(1.012)^{-3} + 15(1.012)^{-4}$$

Looking backwards,

$$\begin{array}{lllll}
PV = & 45v & +35v^2 & +25v^3 & +15v^4 \\
\hline
= & 5v & +5v^2 & +5v^3 & +5v^4 \\
+ & 10v & +10v^2 & +10v^3 & +10v^4 \\
+ & 10v & +10v^2 & +10v^3 & \\
+ & 10v & +10v^2 & & \\
+ & 10v & & & \\
\hline
= & 5a_{\overline{4|}} & & & \\
+ & 10\left(a_{\overline{4|}}\right. & +a_{\overline{3|}} & +a_{\overline{2|}} & \left.+a_{\overline{1|}}\right) \\
\hline
= & \multicolumn{4}{l}{(15-10)\, a_{\overline{4|}} + 10\, (Da)_{\overline{4|}}} \\
= & \multicolumn{4}{l}{5a_{\overline{4|}} + 10\frac{4-a_{\overline{4|}1,2\%}}{0,012}}
\end{array}$$

Result (P.V. Payments in Decreasing Arithmetic form, last pay D)

Last Payment D, look backwards,

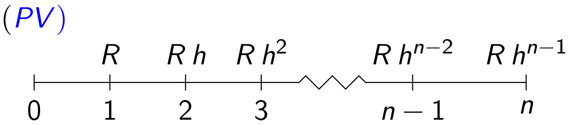
$$PV = (D-h)\, a_{\overline{n}|i} + h \cdot (Da)_{\overline{n}|i}$$

$$\begin{array}{rcl}
(Da)_{\overline{n}|i} & = & nv + (n-1)v^2 + (n-2)\,v^3 + \ldots + 2v^{n-1} + v^n \\
\hline
 & & \begin{array}{ccccccc}
+v & +v^2 & +v^3 & +\ldots & +v^{n-1} & +v^n & + \\
+v & +v^2 & +v^3 & +\ldots & +v^{n-1} & & + \\
+v & +v^2 & +v^3 & +\ldots & & & + \\
 & & & \ldots & \ldots & \ldots & + \\
+v & +v^2 & & & & & + \\
+v & & & & & & =
\end{array} \\
 & = & \\
\hline
 & = & a_{\overline{n}|i} + a_{\overline{n-1}|i} + \ldots + a_{\overline{3}|i} + a_{\overline{2}|i} + a_{\overline{1}|i} \\
(Da)_{\overline{n}|i} & = & \dfrac{n - a_{\overline{n}|i}}{i}, \text{ P.V. with } D = h = 1)
\end{array}$$

Example

John is buying a computer, paying in three instalments: 1st €950, others with 25% increase, $i = 9\%$.

$$\begin{aligned} PV &= 950(1,09)^{-1} + 950 \times 1.25(1,09)^{-2} + 950 \times 1.25^2(1,09)^{-3} \\ &= 3017,26€ \end{aligned}$$



- 1st Payment: R; Rate (Increasing or Decreasing): h.
- If $h > 1$, annuity is increasing;
- If $0 < h < 1$, annuity is decreasing.
- P.V.: Geometric series with rate hv:

$$\begin{aligned} \text{P.V.} &= Rv + Rhv^2 + Rh^2v^3 + \ldots + Rh^{n-1}v^n \\ &= R\left(\frac{v - h^{n-1}v^n \times hv}{1 - hv}\right) = R\left(\frac{v\,(1 - h^n v^n)}{v\,(1/v - h)}\right) \\ &= R\left(\frac{1 - h^n\,(1+i)^{-n}}{1 + i - h}\right) \end{aligned}$$

In many applications we set $h = 1 + g$, where g is a growth rate, we get

$$PV = R\left(\frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{i-g}\right)$$

$$= R\left(\frac{1-(1+x)^{-n}}{i-g}\right)$$

$$x = \frac{i-g}{1+g}$$

In a Loan Repayment, two different payments are due:

- Interest
- Amortization payment

Many different ways to do the **Amortization**, we summarize:

Bullet loan: The entire principal of the loan is due at the end of the loan term. This is an interest-only loan. The payments prior to maturity are only to offset the interests.
These are short term and/or small loan sizes

Constant-payment loan: A sequence of equal-size payments, each of which is composed of the interest due plus a portion of the principal. Ex: Home mortgage

Constant-principal loan The principal amortizations is constant throughout the life of the loan.

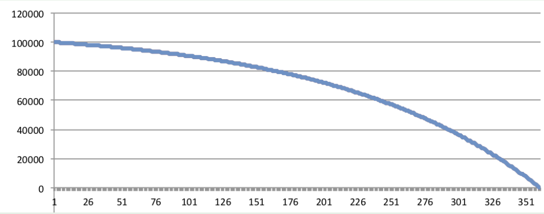
Number of payments is determined by the **Term**, and **Payment Frequency**.

Constant Payment Loan:

- Often, Long Term loans, like home mortgage;
- Interest at each payment slowly decreases;
- Total payment is constant;
- Amortization, portion going to the Principal slowly increases.

Example (Interest rate $8_A^{(12)}$% for 30 years.)

Payment No.	Payment	Interest Paid	Principal Paid	Balance
0				100.000,00 €
1	733,77€	666,67€	67,10 €	99.932,90 €
2	733,77€	666,22€	67,55 €	99.865,35 €
3	733,77€	665,77€	68,00 €	99.797,35 €



Balance

Constant Principal Loan:

- Interest at each payment decreases;
- Total payment is decreasing;
- Amortization, portion going to the Principal is constant.

Example ($i = 2\%$)

t	Payment	Interest Due	Principal Repaid	Outstanding Balance
0	–	–	–	$L = OB_0 = 3000$
1	$K_1 = 310$	$I_1 = OB_0 \times i = 60$	$PR_1 = 250$	$OB_1 = OB_0 - PR_1 = 2750$
2	305	55	250	2500
3	300	50	250	2250
4	295	45	250	2000
5	290	40	250	1750
6	285	35	250	1500
7	280	30	250	1250
8	275	25	250	1000
9	270	20	250	750
10	265	15	250	500
11	260	10	250	250
12	255	5	250	0

Find the Outstanding Balance after m paymts, term n, OB_m:

Prospective method, after m payments

$$OB_m = R\, a_{\overline{n-m|}\, i}$$

Retrospective method, after m payments

$$OB_m = OB_0\,(1+i)^m - R\, s_{\overline{m|}\, i}$$

Sometimes, it is better to rent an asset than buy it:

Definição (Leasing)

A contractual arrangement to grant the use of specific fixed assets for a specific time in exchange for payment, usually in the form of a rent. It has an optional clause for buying later, by a Residual Value.

Theree are different sorts of Leasing contracts, like:

Operating l. Generally a short-term cancellable arrangement;

Financial l. (or capital) is a long-term non cancellable agreement.

Lessee

One that receives the use of assets under a lease. He uses, not owns, pays a rent.

Lessor

One that conveys the use of assets under lease. He owns, not uses, receives a rent.

Many examples of situations to do a Lease, insted of buying a property or an equipment. Like:

- An individual looking for a car or apartment;
- A business looking at office space or a photocopier.

Example

Suppose ABC Ltd is looking for a new car. Leasing is essentially the same as renting the car for a set period of time. If ABC leases, ABC will have the right to drive the car, will have the responsibility for maintaining it and insuring it, but she will not actually be the owner of the car.
At the end of the lease, ABC's payments cease and the car must be returned to the leasing company, or buy it for a residual value.

Example (Car Loan)

Suppose that ABC decides to finance the entire cost of the car with a 5-year loan at an 8.4% annual interest rate compounded monthly. Calculate ABC monthly payment, R, knowing that the entire cost of the car is €$19,875$.

$$19,875 = R\, a_{\overline{60}|0.7\%} \Leftrightarrow R = €406.81$$

Example (Car Lease)

If ABC Ltd signs a lease, she doesn't directly borrow any money, but the car itself:

- With the loan, ABC monthly payments repay all of the principal borrowed (+ interest).
- With the lease, ABC partly repays with the monthly lease payments. Another part is repaid at the end of the lease by returning the car to the leasing company.

Example (Car Lease)

Suppose that the leasing company has determined that, after two years of normal use and proper maintenance (Lessee's duty), the value of this car should be €14,055. This is its Residual Value:

- Of the €19,875 that ABC borrowed in the form of the car, ABC will repay €14,055 by returning the car.
- ABC lease payments need to cover the difference together (+ interest) on this debt of €19,875.

With $i^{(12)} = 8.4\%$, find the the monthly payments:

$$\begin{aligned} 19,875 &= R\, a_{\overline{24}|0.7\%} + 14055\,(1.084)^2 \\ R &= €362.67 \end{aligned}$$

Example (Car Lease)

Alternatively, to calculate the Lease Payment:

$$\begin{aligned} 5,820 &= R^* \, a_{\overline{24|}0.7\%} \\ R^* &= 264.29 \leftarrow \underline{\text{Payment on Loss}} \\ I &= 14055 \times 0.007 \leftarrow \underline{\text{Interest on Residual}} \\ R &= 264.29 + 98.38 = €362.67 \leftarrow \underline{\underline{\text{Lease Payment}}} \end{aligned}$$

In many situations, in ordinary annuity payments, and Residual Value paid in one sole amount at the end. **Ex.**:

Leasing Formula for calculation

$$AC = DP + R\, a_{\overline{n}|i} + RV(1+i)^{-(n+k)}$$

- AC: Asset Cost;
- DP: Down Payment;
- R: Lease Payment;
- RV: Residual Value;
- n: Term;
- k: (Time) Deferment of RV payment, from time n.
- i: Effective Rate of Interest

Definition (Bond)

A Bond is debt instrument that pays interest at regular intervals and that matures at some specific, given date in the future. Issued directly to the public, borrowing money (usually, big loan amounts). Pays interest, it can be traded in the financial markets

Issuers (Money Borrowers):

- Public Companies (with the authorization of the Financial Market Regulator);
- Governments: Treasury Bonds (Public Debt).

Buyers (Money Lenders):

- Public, individuals and/or companies;
- Funds, Pension funds, Investment funds, Insurance...

Important Concepts

- **Coupon**: (Guaranteed) Interest payment;
- **Term**: Can be short or long, perpetual even;
- **Maturity** or **Redemption** Value: Value at the end of its term;
- **Issue Value**: Can be "at the par" (at the **Face Value**), above or below the par;
- **Redeeming the bond**: Process called when the bond holder receives his principal back at the maturity date.
  It can also be Can be *at the par*, *above* or *below* the par;
- **Premium**: It is paid when a bond is redeemed avove the par;
- **Market value**: Value traded in the market;
- **Yield**: Annual interest / bond value;

There are many types of Bonds, like:

- Coupon Bonds: Unregistered bonds;
- Mortgage bonds: Attached fixed assets as securities;
- Convertible Bonds: Given option to convert the loan into a Share;
- Zero Coupon or Discount Bond: Pays no interest;
- Indexed Bond: Interest indexed to inflation;

Bond prices fluctuate according to day-to-day trading at the securities secondary market, as they are negotiable.

Example

Omega PLC, issued a bond loan with the following terms:

- Date of issue: 01/01/yy.
- Nominal Value: €10.00.
- No. of bonds issued: 20,000.
- Issue value at par;
- Loan term: 3 years.
- Semi-annual coupon rate: 3.0%.
- Payment of semiannual interest. The first payment will occur one semester after issuance.
- Mode of Redemption (above the par): Repayments semi-annually of equal number of bonds, starting one year after the issuance date;
- Redemption premium: €0.50 per bond during the first two repayments and €1.00 per bond after that.

1. Compute the total value of the bond loan;
2. Fill out the bond amortization table (Euros).

Example (Cont'd, Amortization table)

Sem.	Balance before	Interest	Bonds paid	Amortization	Premium	Total Payment	Balance after
1	200,000	6,000				6,000	200,000
2	200,000	6,000	4,000	40,000	2,000	48,000	160,000
3	160,000	4,800	4,000	40,000	2,000	46,800	120,000
4	120,000	3,600	4,000	40,000	4,000	47,600	80,000
5	80,000	2,400	4,000	40,000	4,000	46,400	40,000
6	40,000	1,200	4,000	40,000	4,000	45,200	0

Yield

Quotient between the annual interest paid (Coupon) and the bond value (Purchase, Market value)

Rate of Return / Return Rate

The Return Rate of an Investment it's the rate (of interest) that equates the Investment Outflow (Cost) to the Inflows (Income) that it generates, at some focal date. It uses an Equation of Value.

Yield to Maturity (YTM)

Annual Rate of return the buyer gets if he buys and holds the bond until its maturity date when it is redeemed. Otherwise, it is the interest rate that equates interest and principal payments to be received in the future to the present cost. Also called the Effective Rate of Return.

Formula (Calculation of $YTM = j$)

$$P = \sum_{k=1}^{n} \frac{C}{(1+YTM)^k} + \frac{S}{(1+YTM)^n}$$
$$= F\, r\, a_{\overline{n}|j} + S\, v_j^n$$

P: Purchase value (Market value); S: Redemption value; $C = F\,r$: Annual coupon payment; F: Face value; r: Coupon Rate; $j = YTM$; $v_j = (1+j)^{-n}$; n: Term.

Formula (Calculation of $YTM = j$, with $S = F$)

When Redemption is at the par

$$P = F\, r\, a_{\overline{n}|j} + F\, v_j^n$$
$$= F + F(r-j)a_{\overline{n}|j}$$

Proof:

Because $a_{\overline{n}|j} = (1 - v_j^n)/j \Leftrightarrow v_j^n = 1 - j\, a_{\overline{n}|j}$,

$$\begin{aligned} P &= F\, r\, a_{\overline{n}|j} + F\, v_j^n \\ &= F\, r\, a_{\overline{n}|j} + F(1 - j\, a_{\overline{n}|j}) \\ &= F + F(r - j) a_{\overline{n}|j} \end{aligned}$$

Makeham's Formula (with $S = F$)

In the above formula, set $K = F\, v_j^n$ and $a_{\overline{n}|j} = (1 - v_j^n)/j$ to get

$$P = K + \frac{r}{j}(F - K)\,.$$

Price equals K (Present Value of the bond redemption) plus $\frac{r}{j}(F - K)$ (Present Value of the coupons received).

Current Yield

Rate of return for a given year. It is simply the annual interest earned (Coupon) divided by the bond current market value (or Purchase, Market price):

$$\frac{C}{P}$$

Example (Yield calculation)

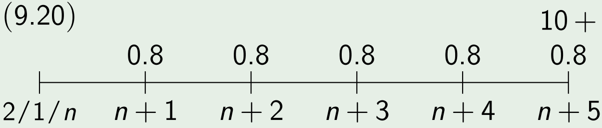
On the $02/01/n$ Delta PLC issued a bond loan of $200,000$ bonds, as follows.

- Face value, F: €10.00;
- Issuance value (under the par), P: €9.20;
- Redemption: annually at equal value, at the par $R = F$;
- No. of Repayments: 4;
- Date of first redemption: $02/01/n+2$
- Coupon payment C: annually
- Coupon (annual) rate: 8%
- Date of first coupon payment: $02/01/n+1$

An investor bought one bond at issuance with redemption in the last group ($02/01/n+5$). Calculate Investor's yield.

Example (Yield calculation (cont'd))

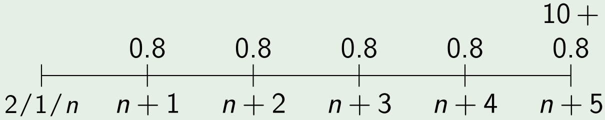
Calculate Investor's yield.



$$
\begin{aligned}
C &= 10,00 \times 0,08 = 0,80€ \\
P &= C\, a_{\overline{k}|r} + \frac{R}{(1+r)^k} \\
9,20 &= 0,8 \cdot a_{\overline{5}|r} + \frac{10}{(1+r)^5} \Leftrightarrow r = 10,117\% \quad \text{(Excel)}
\end{aligned}
$$

Example (Ex. 6.7 (cont.). Alternative invesment)

What is the investment accumulated value, S, at the end of term?



$$r \simeq 10,117\%$$
$$S = 0,8 \cdot s_{\overline{5}|r} + 10 \simeq 14.89$$

Alternatively: If he had invested €9.20 in a savings account, under rate $r = 10,117\%$:

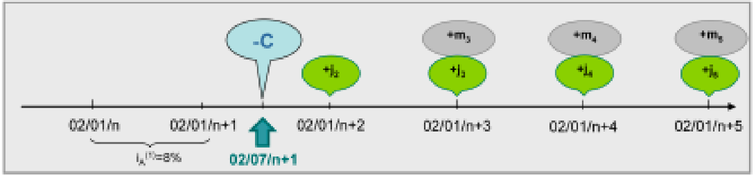
$$S = 9.2(1+r)^5 \simeq €14.89$$

Example (Calculation of Purchase Value (ex. cont'd))

An investor intends to buy, on $02/07/n+1$, 200 of these bonds and wants an yield rate of 9%. Its redemption plan is as follows:

- 25 in the 2nd group;
- 75 in the 3rd;
- 100 in the last one.

How much should the investor pay for each bond?



Example (Calculation of Purchase Value (cont'd))

Time	No. of Bonds	Interest	Bonds Redeemed No.	Bonds Redeemed *Value*
$2/1/n+2$	200	$10(0.08) = 160.00$		
$2/1/n+3$	200	160.00	25	$25\,(10) = 250.00$
$2/1/n+4$	175	140.00	75	$75\,(10) = 750.00$
$2/1/n+5$	100	80.00	100	$100(10) = 1000.00$

$$C = \frac{160.00}{1.09^{0.5}} + \frac{410.00}{1.09^{1,5}} + \frac{890.00}{1,09^{2.5}} + \frac{1080.00}{1.09^{3.5}} = 2043.62$$

$$P = \frac{C}{200} \simeq €10.22$$

Shares/Stocks

Share (UK) or Stocks (US) are equity securities (claim of the owners of the firm). Each share entitles its holder to an equal share in the ownership of the firm. Usually, each share entitled to the same amount of profits and its entitled to one vote on matters of corporate governance.

Common shares represent a residual claim on the assets of a firm (assets that are left over after meeting all of the firm's other financial obligations).

Share prices

- The share has a **Nominal Value** (issue value) corresponding to its percentage on the firm's capital;
- The **market value** is dependent on the cash flows investors are expecting to receive with the acquisition of the share.

Value of a share

It's calculated as the Present Value of the firm's Expected Future Cash Flows.

To value a share we needed to

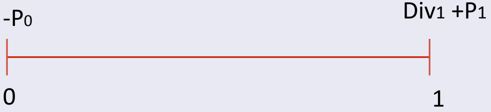
1. Estimate future cash flows:
   1. Size (amount); and
   2. Timing (when).
2. Discount future cash flows at an appropriate interest rate. The discounting rate should be related with the share risk (not just an interest rate).

Potential sources of Cash flows

- Future dividends to shareholders;
- Market value trading, selling.

Share value. One year transaction

Investor buys and sells in one year. Futre Value Estimate



Share value (buying), r - Discount rate, cost of capital:

$$
\begin{aligned}
P_0 &= \frac{Div_1 + P_1}{1+r} \\
r &= \frac{Div1}{P_0} + \frac{P_1 - P_0}{P_0} \\
&= \text{Dividend yield} + \text{Capital gain}
\end{aligned}
$$

Example (Share value. One year transaction)

Expectations are that Alfa PLC is going to pay a dividend of €0.56/share next year and that the share can be sold at €45.5 at the end of the year.

1. If another investment of similar risk has an expected return of 6.80%, what value would you expect to pay for this share?
2. At that transation value calculate the Dividend Yield the Capital Gain ganho expected.

Sol.:

$$
\begin{aligned}
P_0 &= (0.56 + 45.5)/1.068 \simeq 43.13 \\
\text{D Yield} &= 0.56/43.13 \simeq 1.298480\% \\
\text{C Gain} &= (45.5 - 43.13)/43.13 \simeq 5.550152\%
\end{aligned}
$$

Share value. For a share kept for many years

- Share is kept for n years:

$$P_0 = \frac{Div_1}{1+r} + \frac{Div_2}{(1+r)^2} + \cdots + \frac{Div_2}{(1+r)^n}$$

- Share is kept undefinitely:

$$P_0 = \sum_{k=1}^{\infty} \frac{Div_k}{(1+r)^k}$$

Easy way of estimating future dividends:

1. Current or coming dividend is known;
2. Estimate a growth rate for the following ones;
3. Discount rate / Cost of capital (estimate) is known.

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Share value. For a share kept for many years

$$
\begin{aligned}
P_0 &= D\left(\frac{1}{1+r} + \frac{1+g}{(1+r)^2} + \frac{(1+g)^2}{(1+r)^3} + \dots\right) \\
&= D \lim_{n\to\infty} \frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-k} \\
&= \frac{D}{r-g}, \quad \text{if } r > g
\end{aligned}
$$

Example

The ABC, PLC Share is expected to pay a dividend of €2 per share a year from now, and its dividends are expected to grow by 6% per year thereafter. If its price is now €20 per share, what must be the discount rate?

Sol.:

$$20 = \frac{2}{r - 0.06} \Leftrightarrow r = 0.16 \rightarrow r = 16\%$$

Example (Valuation of a share)

The ABCD Co. is expecting earnings of €10 per share, an earnings retention rate of 75%, an expected rate of return on future investments of 18% per year, and a discount rate of 15% per year. Compute an estimate of the ABCD share price. (R: $P_0 = €166.67$)

Let $r = 0.15$ and $g = 0.75 \times 0.18$ be the discount and the growth rate, respectively. Then

$$\begin{aligned} Div_1 &= 10.00 \times 0.25 \\ Div_{n+1} &= Div_n \left(1 + 0.75(0.18)\right), \quad n = 1, 2, \ldots \end{aligned}$$

Discounting

$$\begin{aligned} P_0 &= 2.5\frac{1}{1+r} + 2.5\frac{(1+g)}{(1+r)^2} + 2.5\frac{(1+g)^2}{(1+r)^3} + \ldots \\ &= 2.5\sum_{k=0}^{\infty}\frac{1.135^k}{1.15^{k+1}} = \frac{2.5}{0.015} = \frac{2.5}{0.15 - 0.75(0.18)} = 166.6(6) \end{aligned}$$