CHAPTER 14
FORECASTING VOLATILITY II

Figure 14.1 Autocorrelograms of the Squared Returns

SP500 index (weekly data)

Sample: 1/12/1998 7/07/2008
Included observations: 547

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.236	0.236	30.605	0.000
		2	0.058	0.002	32.441	0.000
		3	0.083	0.073	36.282	0.000
		4	0.065	0.030	38.590	0.000
		5	0.041	0.017	39.525	0.000
		6	0.062	0.045	41.644	0.000
		7	0.072	0.044	44.538	0.000
		8	0.002	-0.032	44.541	0.000
		9	0.031	0.030	45.083	0.000
		10	0.057	0.035	46.911	0.000
		11	0.068	0.045	49.486	0.000
		12	0.054	0.023	51.110	0.000

Yen/US Dollor exchange rate (daily data)

Sample: 4251 9414
Included observations: 5164

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.198	0.198	201.84	0.000
		2	0.113	0.077	268.31	0.000
		3	0.088	0.054	307.91	0.000
		4	0.069	0.037	332.32	0.000
		5	0.103	0.076	387.32	0.000
		6	0.090	0.049	428.94	0.000
		7	0.090	0.049	470.50	0.000
		8	0.094	0.051	515.75	0.000
		9	0.089	0.044	556.38	0.000
		10	0.032	-0.019	561.60	0.000
		11	0.071	0.041	587.55	0.000
		12	0.041	-0.001	596.28	0.000

10-year Treasury Note (daily data)

Sample: 10000 12136
Included observations: 1960

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.089	0.089	15.563	0.000
		2	0.096	0.089	33.691	0.000
		3	0.067	0.052	42.535	0.000
		4	0.121	0.105	71.317	0.000
		5	0.186	0.163	139.06	0.000
		6	0.109	0.069	162.55	0.000
		7	0.111	0.068	186.85	0.000
		8	0.090	0.045	202.79	0.000
		9	0.126	0.072	234.20	0.000
		10	0.154	0.093	280.80	0.000
		11	0.129	0.067	313.52	0.000
		12	0.128	0.066	345.92	0.000

Slides adapted for this course. We thank Gloria González-Rivera and assume full responsibility for all possible errors due to our changes

Model:

$$r_t = \mu_{t|t-1} + \varepsilon_t = \mu_{t|t-1} + \sigma_{t|t-1} z_t$$

$$\varepsilon_t = \sigma_{t|t-1} z_t$$

where:

z_t is independent and identically distributed with $E[z_t] = 0$ and $E[z_t^2] = 1$

Conditional average: $E[r_t | I_{t-1}] = \mu_{t|t-1}$

Conditional variance:

$$\sigma^2{}_{t|t-1} = Var(r_t | I_{t-1}) = E\left[\left(r_t - \mu_{t|t-1}\right)^2 \middle| I_{t-1}\right] = E[\varepsilon_t{}^2 | I_{t-1}]$$

- The heteroscedasticity of ε_t is just driven by $\sigma^2{}_{t|t-1}$
- The unconditional variance is constant: $E[\varepsilon_t] = \sigma^2{}_\varepsilon$

14.1.1 ARCH(1)

$$r_t = \mu_{t|t-1} + \varepsilon_t = \mu_{t|t-1} + \sigma_{t|t-1} z_t$$

$$\sigma^2{}_{t|t-1} = \omega + \alpha \varepsilon^2{}_{t-1} \implies \sigma^2{}_{\varepsilon} = \frac{\omega}{1-\alpha} \qquad \omega > 0, \alpha \geq 0$$

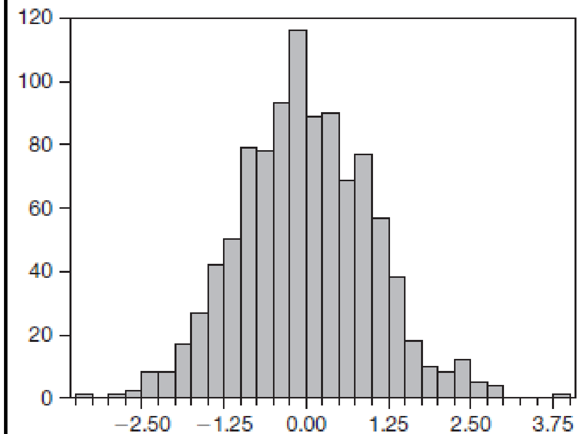
Table 14.1 Descriptive Statistics of ARCH(1) Process and Standardized Process

Panel A

Descriptive Statistics of an ARCH(1) process (returns)
Sample: 1 1000

	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$
Mean	1.975935	1.947474	1.881775
Median	1.930632	1.924136	1.906267
Maximum	11.78598	24.60907	55.02234
Minimum	-4.482281	-17.69567	-49.36718
Std. Dev.	1.766631	2.611512	4.810237
Skewness	0.208300	-0.065491	-0.863349
Kurtosis	4.847531	19.48612	56.65335
Jarque-Bera	149.4553	11325.38	120069.3
Probability	0.000000	0.000000	0.000000

Panel B



Series: Standardized process: z = r/st.dev
Sample 1 1000
Observations 1000

Mean	-0.010666
Median	-0.038664
Maximum	3.980160
Minimum	-3.319720
Std. Dev.	1.003468
Skewness	0.141614
Kurtosis	3.204294
Jarque-Bera	5.081409
Probability	0.078811

standardized error process

$$\frac{r_t - \mu_{t|t-1}}{\sigma_{t|t-1}} = z_t$$

Figure 14.2
Simulated
ARCH(1)
Processes

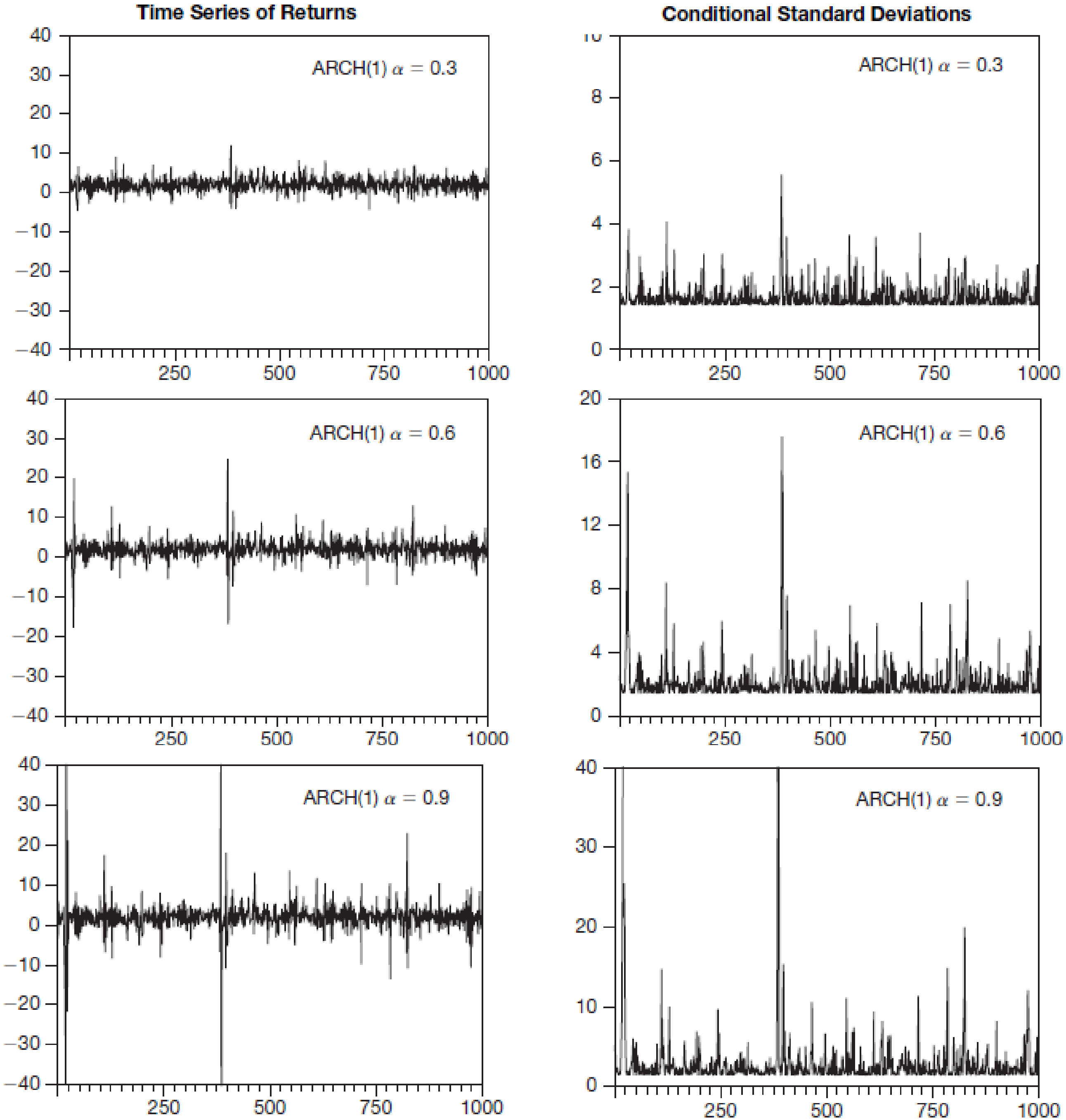


Figure 14.3 Autocorrelation Functions of simulated ARCH(1) Process

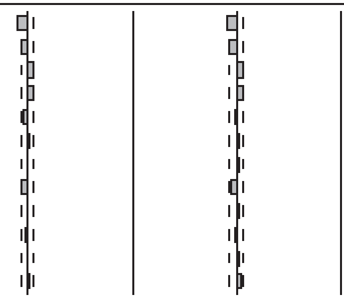
$$r_t = 2 + \varepsilon_t$$
$$\sigma^2_{t|t-1} = 2 + 0.3\varepsilon^2_{t-1}$$

Panel A

Times series r_t

Sample: 1 1000
Included observations: 1000

Autocorrelation | Partial Correlation

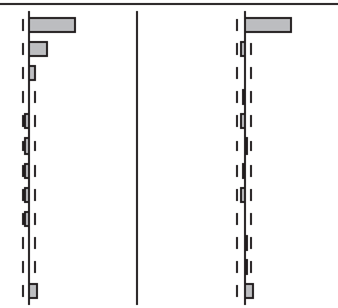


	AC	PAC
1	-0.090	-0.090
2	-0.062	-0.071
3	0.071	0.059
4	0.050	0.059
5	-0.033	-0.015
6	0.020	0.018
7	0.011	0.004
8	-0.054	-0.051
9	0.009	0.001
10	-0.000	-0.009
11	0.001	0.008
12	0.036	0.042

Times series r_t^2

Sample: 1 1000
Included observations: 1000

Autocorrelation | Partial Correlation



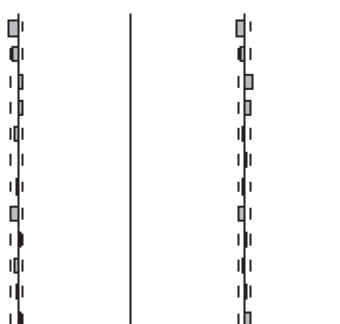
	AC	PAC
1	0.450	0.450
2	0.177	-0.032
3	0.063	-0.007
4	0.009	-0.015
5	-0.040	-0.044
6	-0.037	0.001
7	-0.030	-0.009
8	-0.040	-0.026
9	-0.033	-0.005
10	0.004	0.029
11	0.015	0.003
12	0.084	0.090

Panel B

Times series $r_t / \sigma_{t|t-1}$

Sample: 1 1000
Included observations: 1000

Autocorrelation | Partial Correlation

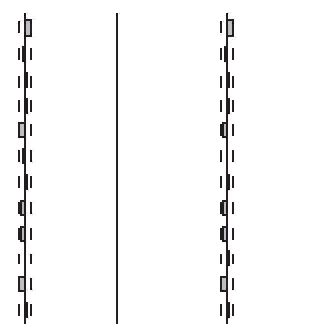


	AC	PAC
1	-0.072	-0.072
2	-0.034	-0.039
3	0.079	0.074
4	0.050	0.061
5	-0.027	-0.014
6	0.024	0.018
7	-0.005	-0.012
8	-0.059	-0.060
9	0.013	0.003
10	-0.011	-0.015
11	-0.002	0.008
12	0.046	0.051

Times series $r_t^2 / \sigma^2_{t|t-1}$

Sample: 1 1000
Included observations: 1000

Autocorrelation | Partial Correlation



	AC	PAC
1	0.067	0.067
2	-0.012	-0.017
3	0.018	0.020
4	0.007	0.004
5	-0.048	-0.049
6	-0.008	-0.001
7	0.033	0.032
8	-0.033	-0.036
9	-0.037	-0.031
10	0.012	0.012
11	-0.052	-0.055
12	0.018	0.031

Forecasting with ARCH(1)

1-step-ahead **variance forecast** $\sigma^2_{t+1|t} = \omega + \alpha\varepsilon^2_t,$

two-step-ahead forecast $\sigma^2_{t+2|t} = \omega + \alpha E(\varepsilon^2_{t+1} \,|\, I_t) = \omega + \alpha\sigma^2_{t+1|t}$

For any $h > 1$, using backward substitution we find that the h-step-ahead forecast of the conditional variance is

$$\sigma^2_{t+h|t} = \omega(1 + \alpha + \alpha^2 + \alpha^3 + \ldots\ldots\ldots + \alpha^{h-2}) + \alpha^{h-1}\sigma^2_{t+1|t}$$

$$\underset{h \longrightarrow \text{large}}{=} \frac{\omega}{1-\alpha} + \alpha^{h-1}\sigma^2_{t+1|t},$$

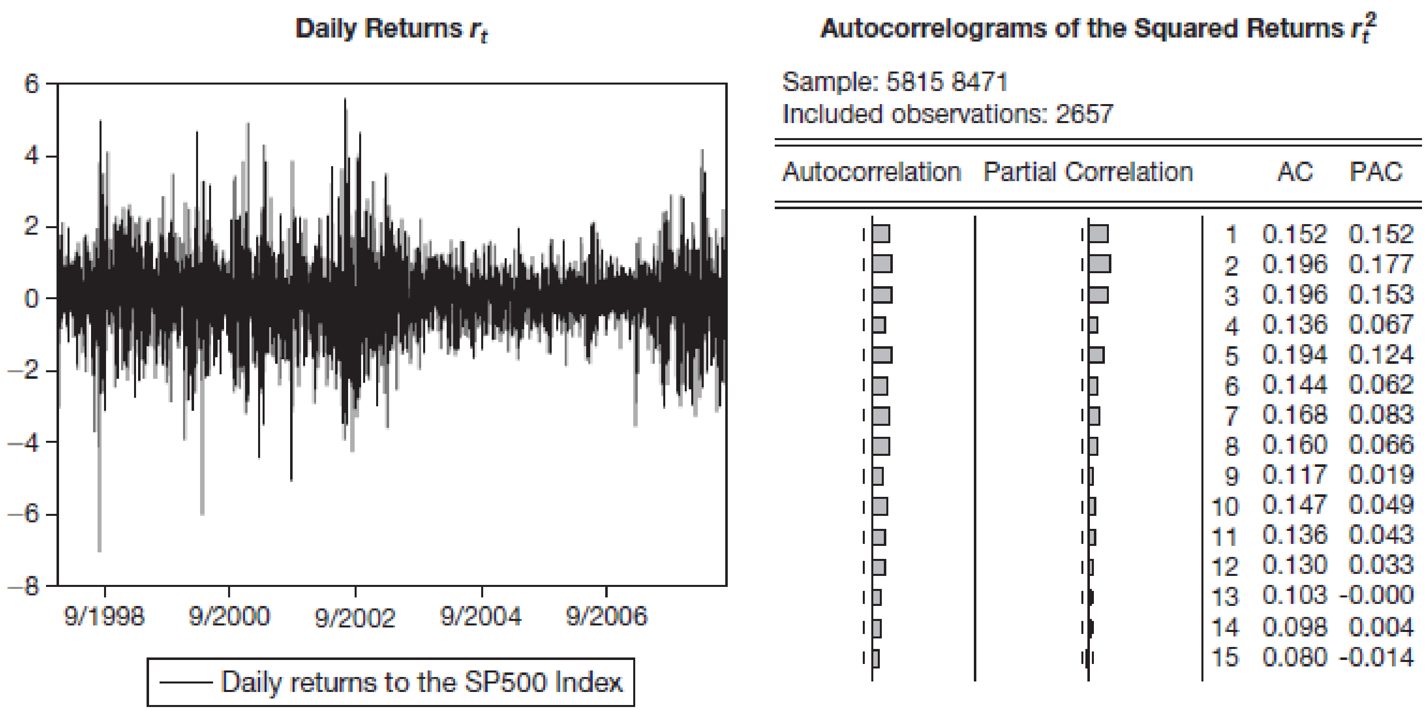
14.1.2 ARCH(p)

$$r_t = \mu_{t|t-1} + \varepsilon_t = \mu_{t|t-1} + \sigma_{t|t-1} z_t$$

$$\sigma^2_{t|t-1} = \omega + \alpha_1 \varepsilon^2_{t-1} + \alpha_2 \varepsilon^2_{t-2} + \cdots + \alpha_p \varepsilon^2_{t-p}$$

$$\omega > 0, \alpha_i \geq 0 \;\; i = 1, 2 \ldots p.$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_p < 1,$$



	AC	PAC
1	0.152	0.152
2	0.196	0.177
3	0.196	0.153
4	0.136	0.067
5	0.194	0.124
6	0.144	0.062
7	0.168	0.083
8	0.160	0.066
9	0.117	0.019
10	0.147	0.049
11	0.136	0.043
12	0.130	0.033
13	0.103	-0.000
14	0.098	0.004
15	0.080	-0.014

Figure 14.4 Daily SP500 Returns and Autocorrelations of Squared Returns

14.1.3 GARCH(1,1)

$$\sigma^2_{t|t-1} = \omega + \alpha\varepsilon^2_{t-1} + \beta\sigma^2_{t-1|t-2}$$

$$\sigma^2_{\varepsilon} = \frac{\omega}{1-\alpha-\beta}$$

with non-negative parameters $\omega > 0$, $\alpha \geq 0$, and $\beta \geq 0$.

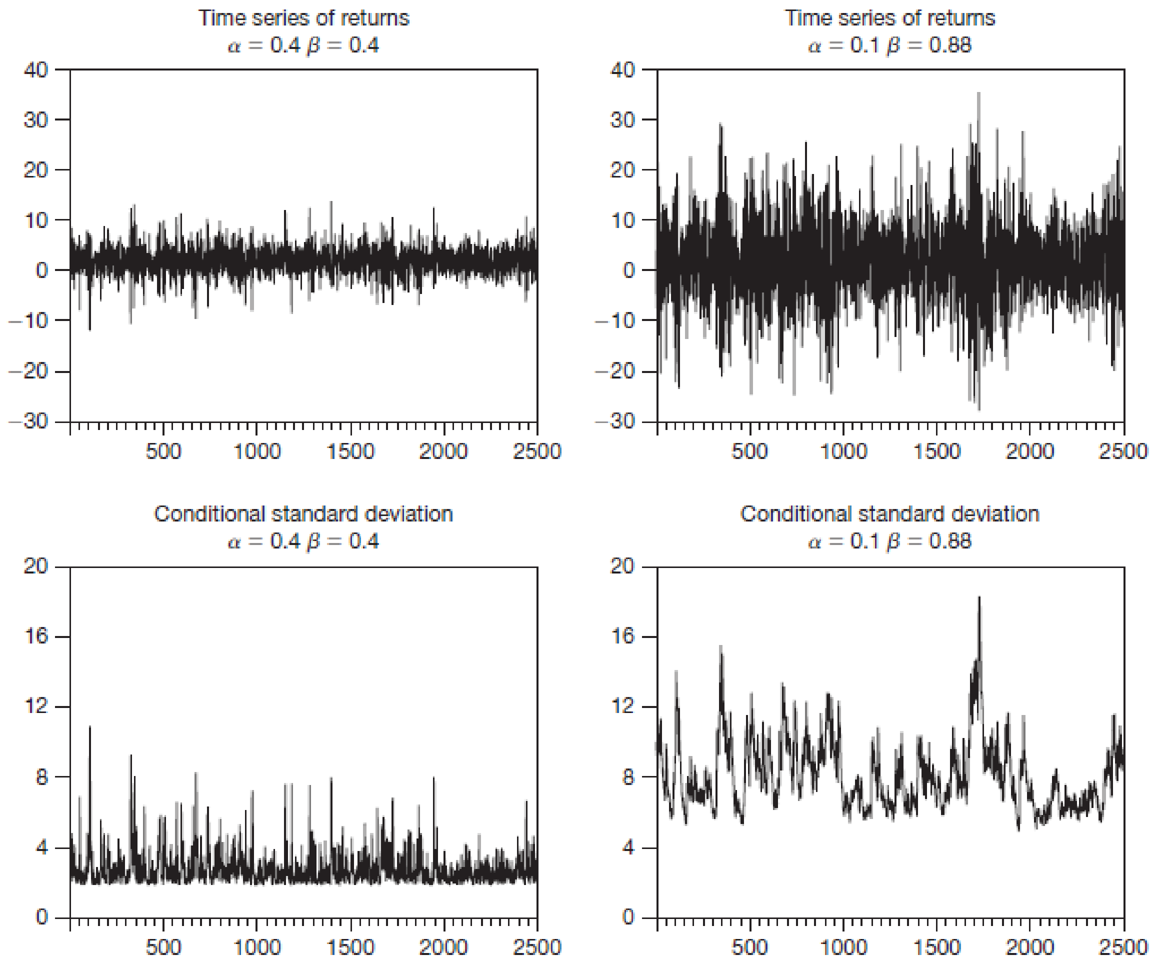


Figure 14.5 Low and High Persistence GARCH(1,1) Processes

Table 14.2 Descriptive Statistics of Low and High Persistence GARCH(1,1) Processes

Descriptive Statistics of a GARCH(1,1) process (returns)
Sample: 1 20000

	$\alpha = 0.4, \beta = 0.4$	$\alpha = 0.1, \beta = 0.88$
Mean	1.992061	2.019090
Median	1.998115	1.993319
Maximum	67.24605	84.91548
Minimum	-46.49060	-80.57394
Std. Dev.	3.281838	9.920762
Skewness	0.160152	0.116894
Kurtosis	26.86840	5.859899
Jarque-Bera	474835.8	6861.400
Probability	0.000000	0.000000

Figure 14.6 Autocorrelation Functions of Low and High Persistence GARCH(1,1) Processes

$$r_t = 2 + \varepsilon_t$$
$$\sigma^2_{t|t-1} = 2 + \alpha\varepsilon^2_{t-1} + \beta\sigma^2_{t-1|t-2}$$

Time series r_t^2

(1) $\alpha = 0.4$, $\beta = 0.4$ (low persistence)

Sample: 1 20000
Included observations: 20000

Autocorrelation	Partial Correlation		AC	PAC
		1	0.470	0.470
		2	0.383	0.208
		3	0.376	0.182
		4	0.249	-0.022
		5	0.167	-0.046
		6	0.139	-0.006
		7	0.113	0.019
		8	0.089	0.018
		9	0.086	0.024
		10	0.048	-0.029
		11	0.034	-0.014
		12	0.026	-0.006
		13	0.015	0.002
		14	0.019	0.017

Time series r_t^2

(2) $\alpha = 0.1$, $\beta = 0.88$ (high persistence)

Sample: 1 20000
Included observations: 20000

Autocorrelation	Partial Correlation		AC	PAC
		1	0.263	0.263
		2	0.301	0.249
		3	0.249	0.143
		4	0.254	0.129
		5	0.209	0.065
		6	0.255	0.119
		7	0.277	0.139
		8	0.248	0.079
		9	0.260	0.085
		10	0.215	0.025
		11	0.199	0.009
		12	0.212	0.041
		13	0.233	0.064
		14	0.245	0.073

Table 14.3 SP500 Daily Returns: Estimation of a GARCH(1,1) Model

Dependent Variable: R
Method: ML - ARCH (BHHH) - Normal distribution
Sample: 5815 8471
Included observations: 2657
Convergence achieved after 10 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.036267	0.017439	2.079665	0.0376
Variance Equation				
C	0.010421	0.005245	1.987099	0.0469
RESID(-1)^2	0.065649	0.011338	5.790038	0.0000
GARCH(-1)	0.927400	0.011045	83.96233	0.0000

R-squared	-0.000534	Mean dependent var	0.009761
Adjusted R-squared	-0.001666	S.D. dependent var	1.146761
S.E. of regression	1.147716	Akaike info criterion	2.888638
Sum squared resid	3494.671	Schwarz criterion	2.897498
Log likelihood	-3833.556	Durbin-Watson stat	2.079139

Figure 14.7 Autocorrelation Function of the Standardized Squared Residuals $\hat{\varepsilon}_t^2 / \hat{\sigma}_{t|t-1}^2$ from GARCH(1,1) for SP500 Daily Returns

Sample: 5815 8471
Included observations: 2657

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.031	-0.031	2.5244	0.112
		2	0.033	0.032	5.4669	0.065
		3	0.007	0.009	5.6164	0.132
		4	0.005	0.005	5.6914	0.223
		5	0.006	0.005	5.7756	0.329
		6	-0.016	-0.016	6.4731	0.372
		7	0.000	-0.001	6.4731	0.486
		8	0.023	0.024	7.8948	0.444
		9	0.001	0.002	7.8965	0.545
		10	0.024	0.023	9.4796	0.487
		11	0.009	0.010	9.7071	0.557
		12	-0.010	-0.012	9.9971	0.616
		13	0.002	0.000	10.011	0.693
		14	-0.004	-0.003	10.063	0.758

Table 14.4 Maximum Likelihood Estimation of ARCH and GARCH Processes

SP500 daily returns—ARCH(9)

Dependent Variable: R
Method: ML - ARCH (BHHH) - Normal distribution
Sample: 5815 8471
Included observations: 2657
Convergence achieved after 16 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-2)^2 + C(5)*RESID(-3)^2 + C(6)*RESID(-4)^2 + C(7)*RESID(-5)^2 + C(8)*RESID(-6)^2 + C(9)*RESID(-7)^2 + C(10)*RESID(-8)^2 + C(11)*RESID(-9)^2

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.037003	0.018214	2.031594	0.0422
Variance Equation				
C	0.271763	0.040891	6.645982	0.0000
RESID(-1)^2	0.029949	0.028081	1.066510	0.2862
RESID(-2)^2	0.149370	0.044623	3.347391	0.0008
RESID(-3)^2	0.095260	0.026377	3.611510	0.0003
RESID(-4)^2	0.101684	0.027620	3.681607	0.0002
RESID(-5)^2	0.082439	0.023397	3.523482	0.0004
RESID(-6)^2	0.060298	0.021251	2.837387	0.0045
RESID(-7)^2	0.090927	0.030511	2.980119	0.0029
RESID(-8)^2	0.142659	0.029601	4.819476	0.0000
RESID(-9)^2	0.082659	0.023815	3.470870	0.0005

R-squared	-0.000565	Mean dependent var	0.009761
Adjusted R-squared	-0.004346	S.D. dependent var	1.146761
S.E. of regression	1.149251	Akaike info criterion	2.910013
Sum squared resid	3494.776	Schwarz criterion	2.934377
Log likelihood	-3854.952	Durbin-Watson stat	2.079077

Table 14.4 Maximum Likelihood Estimation of ARCH and GARCH Processes (continued)

SP500 daily returns—GARCH(1,1)

Dependent Variable: R
Method: ML - ARCH (BHHH) - Normal distribution
Sample: 5815 8471
Included observations: 2657
Convergence achieved after 10 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.036267	0.017439	2.079665	0.0376
Variance Equation				
C	0.010421	0.005245	1.987099	0.0469
RESID(−1)^2	0.065649	0.011338	5.790038	0.0000
GARCH(−1)	0.927400	0.011045	83.96233	0.0000

R-squared	-0.000534	Mean dependent var	0.009761
Adjusted R-squared	-0.001666	S.D. dependent var	1.146761
S.E. of regression	1.147716	Akaike info criterion	2.888638
Sum squared resid	3494.671	Schwarz criterion	2.897498
Log likelihood	-3833.556	Durbin-Watson stat	2.079139

Forecasting with GARCH(1,1)

1-step-ahead **variance forecast**

$$\sigma^2_{t+1|t} = \omega + \alpha\varepsilon^2_t + \beta\sigma^2_{t|t-1}$$

two-step-ahead forecast

$$\sigma^2_{t+2|t} = \omega + \alpha E(\varepsilon^2_{t+1} \,|\, I_t) + \beta\sigma^2_{t+1|t} = \omega + (\alpha + \beta)\sigma^2_{t+1|t}$$

For any $h > 1$, using backward substitution we find that the h-step-ahead forecast of the conditional variance is

$$\sigma^2_{t+h|t} = \omega(1 + (\alpha + \beta) + (\alpha + \beta)^2 + \cdots + (\alpha + \beta)^{h-2}) + (\alpha + \beta)^{h-1}\sigma^2_{t+1|t}$$

If the forecast horizon is very large, and for $\alpha + \beta < 1$, the effect of $\sigma^2_{t+1|t}$ becomes negligible, and the h-step-ahead forecast becomes

$$\sigma^2_{t+h|t} = \omega(1 + (\alpha + \beta) + (\alpha + \beta)^2 + (\alpha + \beta)^3 + \cdots) \rightarrow \frac{\omega}{1 - (\alpha + \beta)} \equiv \sigma^2$$