Master Programme in Mathematical Finance

Lévy Processes and Applications

Exam - Época de Recurso

2 hours

February 1, 2018

1. Consider a Lévy process L_t with characteristic triplet (μ, c, ν).

(a) State what are the conditions that the Lévy measure ν and parameters μ, c must satisfy such that the process L_t has:

(i) trajectories with finite variation and with jumps of negative size (no jumps of positive size).

(ii) trajectories with infinite variation

(iii) infinite activity and with jumps of positive size (no jumps of negative size).

(iv) finite moments up to fourth order or finite kurtosis.

(b) Consider that $\mu = 0$, $c = 0$ and

$$\nu\left(dx\right) = x^{\alpha}\mathbf{1}_{\{x>1\}} + \left|x\right|^{\beta}\mathbf{1}_{\{-1<x<1\}} + \frac{e^{-\gamma x}}{x^2}\mathbf{1}_{\{x<-1\}}.$$

Determine for what values of the parameters α, β and γ is $\nu\left(dx\right)$ a well defined Lévy measure and L_t a well defined Lévy process.

2. Consider a Lévy process X_t with characteristic triplet $(0, \sigma^2, \nu)$.

(a) Assume that the Lévy exponent is given by

$$kiu - 2u^2 + \int_{\mathbb{R}\backslash\{0\}} \left(e^{iux} - 1\right) e^{-|x|} dx.$$

Deduce what are the values of k, σ and the Lévy measure $\nu\left(dx\right)$.

(b) Assume that the Lévy exponent is $\eta_X\left(u\right) = -\sigma^{\alpha}\left|u\right|^{\alpha}$, with $0 < \alpha \leq 2$. Say what kind of process is this one, show that the process is a self-similar process and calculate the Hurst parameter as a function of α.

3. Consider the probability distribution with probability density function given by

$$f(x) = ke^{-kx}\mathbf{1}_{\{x\geq 0\}},$$

where $k > 0$ is the parameter of the distribution. Calculate the characteristic function associated to this distribution and show that the distribution is infinitely divisible. (Hint: recall that the gamma distribution has characteristic function $\left(1 - \frac{iu}{\beta}\right)^{-\alpha}$ with $\alpha, \beta > 0$)

4. Let X be a Lévy process with Lévy measure

$$\nu\left(dx\right)=\frac{1}{x^{2}}\mathbf{1}_{\{-1<x<1\}}+\left|x\right|^{-6}\mathbf{1}_{\{x\leq-1\}}+e^{-3x}\mathbf{1}_{\{x\geq1\}},$$

(a) Calculate the expected value and the variance of the following compensated Poisson integral

$$\int_{1}^{+\infty}e^{x}\widetilde{N}(4,dx).$$

Is this process a martingale? Explain your reasons and say what the integral represents in terms of the sizes of the jumps of the Lévy process. (2 valores)

(b) Consider that the process Y is the solution of the stochastic differential equation

$$dY_{t}=\left(10-Y_{t-}\right)dt+4dB_{t}+\int_{x\geq1}x^{2}N\left(dt,dx\right),\qquad\text{with }Y\left(0\right)=3.$$

Solve this equation (hint: Consider the process $e^{t}Y_{t}$ and apply the Itô formula in an appropriate way).

5. Consider a financial market with one riskless asset and one risky asset with price process $S(t)$ modeled as an ordinary exponential of a Lévy process $U(t)$, that is

$$S(t)=S_{0}\exp(U\left(t\right)).$$

What is the general condition that allows us to conclude that the market model is arbitrage free? In which of the following cases can we ensure that the market model is arbitrage free (explain why in each case) ? :

(i) $U(t)$ is a jump-diffusion process with positive drift: $U(t)=2t+5B(t)+\sum_{k=1}^{N(t)}J_{k}$, where the random variables J_k have a Poisson distribution.

(ii) $U(t)$ is a pure jump process with Lévy measure $\nu(dx)=x^{-1}\mathbf{1}_{\{0<x<1\}}+x^{-4}\mathbf{1}_{\{x\geq1\}}$.

(iii) $U(t)$ is a compound Poisson process: $U(t)=\sum_{k=1}^{N(t)}J_{k}$, where the random variables J_k have uniform distribution on the interval $]-1,2[$.

(iv) $U(t)$ is a standard Poisson plus a compound Poisson process with drift: $U(t)=kt+N(t)+\sum_{k=1}^{N(t)}J_{k}$, where the random variables J_k have a exponential distribution.

6. Consider a Lévy process X and define a new process Y by

$$Y_{t}=\sum_{s\leq t,\Delta X_{s}\neq0}\left(\Delta X_{s}\right)^{6}.$$

Show that this process is a subordinator. (Hint: Start by showing, separating the “small jumps” and the “big jumps”, that the sum is finite).

Marks: 1(a): 2.0, 1(b): 2.25, 2(a): 2.0 , 2(b): 2.25, 3: 2.75, 4(a): 2.25, 4(b): 2.25, 5: 2.5, 6: 1.75.