Financial Markets and Instruments

Exercises Suggested Solutions

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December 2018

These are suggested solutions to (most of) the exercises in the **Booklet of Exercises**. They are not typo and/or error free. Solutions to some of the exercises are not yet available. Students are encouraged to compare their solutions with those of colleagues and if questions arise come to office hours.

I would very much appreciate if you would write down a list of typos and/or errors identified during study and hand it in at the exam, for future correction. The same applies to typos in the slides or any other material I have distributed during the course.

Contents

1 Mean–Variance Theory

1.1 Return and Diversification of Risk

Exercise 1.1.

(a) Expected return is the sum of each outcome times its associated probability. Expected returns:
$\bar{R}_1 = 16\% \times 0.25 + 12\% \times 0.5 + 8\% \times 0.25 = 12\%$
$\bar{R}_2 = 6\%$
$\bar{R}_3 = 14\%$
$\bar{R}_4 = 12\%$
Standard deviation of return is the square root of the sum of the squares of each outcome minus the mean times the associated probability. Standard deviations:
$\sigma_1 = \left[(16\% - 12\%)^2 \times 0.25 + (12\% - 12\%)^2 \times 0.5 + (8\% - 12\%)^2 \times 0.25\right]^{\frac{1}{2}} = 2.83\%$
$\sigma_2 = 1.41\%$
$\sigma_3 = 4.24\%$
$\sigma_4 = 3.27\%$

(b) Covariance of return between Assets 1 and 2
$\sigma_{12} = (16-12)\times(4-6)\times 0.25 + (12-12)\times(6-6)\times 0.5 + (8-12)\times(8-6)\times 0.25 = -4$
The variance/covariance matrix for all pairs of assets is:

$$V = \begin{pmatrix} 0.0008 & -0.0004 & 0.0012 & 0 \\ -0.0004 & 0.0002 & -0.0006 & 0 \\ 0.0012 & -0.0006 & 0.0018 & 0 \\ 0 & 0 & 0 & 0.00107 \end{pmatrix}$$

Correlation of return between Assets 1 and 2: $\rho_{12} = \frac{-4}{2.83 \times 1.41} = -1$.
The correlation matrix for all pairs of assets is:

$$\rho = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c)

Portfolio	Expected Return
A	$1/2 \times 12\% + 1/2 \times 6\% = 9\%$
B	13%
C	12%
D	10%
E	13%
F	$1/3 \times 12\% + 1/3 \times 6\% + 1/3 \times 14\% = 10.67\%$
G	10.67%
H	12.67%
I	$1/4 \times 12\% + 1/4 \times 6\% + 1/4 \times 14\% + 1/4 \times 12\% = 11\%$

We can conclude that assets A, B and D are nor efficient, as well as portfolios b and f. In all these cases we can find portfolios with lower or equal risk and higher or equal expected return.

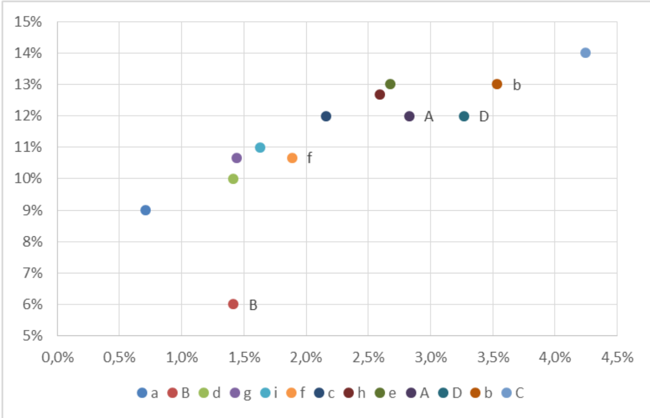

Figure 1: Exercise 1.1 – Representation of the assets and several portfolios in the space $(\sigma_p, \bar{R}_p)$.

Portfolio	Variance
A	$(1/2)^2 \times 0.0008 + (1/2)^2 \times 0.0002 + 2 \times 1/2 \times 1/2 \times (-0.0004) = 0.00005$
B	0.00125
C	0.00046
D	0.0002
E	0.0007
F	$(1/3)^2 \times 0.0008 + (1/3)^2 \times 0.0002 + (1/3)^2 \times 0.0018 + 2 \times 1/3 \times 1/3 \times (-0.0004) +$ $+2 \times 1/3 \times 1/3 \times 0.0012 + 2 \times 1/3 \times 1/3 \times (-0.0006) = 0.00036$
G	0.0002
H	0.00067
I	$(1/4)^2 \times 0.0008 + (1/4)^2 \times 0.0002 + (1/4)^2 \times 0.0018 + (1/4)^2 \times 0.00107 \times +$ $+2 \times 1/4 \times 1/4 \times (-0.0004) + 2 \times 1/4 \times 1/4 \times 12 + 2 \times 1/4 \times 1/4 \times 0 +$ $+2 \times 1/4 \times 1/4 \times (-0.0006) + 2 \times 1/4 \times 1/4 \times 0 + 2 \times 1/4 \times 1/4 \times 0 = 0.00027$

Exercise 1.2.

(a) The formula for the variance of an equally weighted portfolio (where $X_i = 1/N \ \forall i = 1, \ldots, N$ securities) is

$$\sigma_H^2 = \frac{1}{n}\left(\bar{\sigma}_j^2 - \bar{\sigma}_{kj}\right) + \bar{\sigma}_{kj} \tag{1}$$

where $\bar{\sigma}_j^2$ is the average variance across all securities, $\bar{\sigma}_{kj}$ is the average covariance across all pairs of securities, and N is the number of securities. Using the above formula with $\bar{\sigma}_j^2 = 50$ and $\bar{\sigma}_{kj} = 10$ we have:

n	5	10	20	50	100
σ_H^2	0.0018	0.0014	0.0012	0.00108	0.00104

(b) As the number of securities (N) approaches infinity, an equally weighted portfolio's variance (total risk) approaches a minimum equal to the average covariance of the pairs of securities in the portfolio, which is 10. Therefore the risk is $\sigma_{MV} = \sqrt{0.001} = 3.16\%$. Having a risk only 10% higher than the minimum variance portfolio means $\sigma_H \leq 3.16 \times 1.1 = 3.48\% \iff \sigma_H^2 = 0.00121$. To know how many securities a portfolio must have to respect this condition we need to solve the inequality:

$$\sigma_H^2 = \frac{1}{n}\left(\bar{\sigma}_j^2 - \bar{\sigma}_{kj}\right) + \bar{\sigma}_{kj} \leq 0.001211$$
$$\frac{1}{n}(0.005 - 0.001) \leq 0.001211 \Leftrightarrow n \geq 19.05$$

Thus, the portfolio must have, at least, 20 securities.

(c) No, the average covariance works as an asymptote to the variance of any portfolio. As n increases the variance of a portfolio converges to that limit, but would only reach it at infinity.

Exercise 1.3.

(a) If the portfolio contains only one security, then the portfolio's average variance is equal to the average variance across all securities, $\bar{\sigma}_j^2$. If instead an equally weighted portfolio contains a very large number of securities, then its variance will be approximately equal to the average covariance of all pairs of securities in the portfolio $\bar{\sigma}_{kj}$. Therefore, the fraction of risk that of an individual security that can be eliminated by holding a large portfolio is expressed by the following ratio:

$$D = \frac{\bar{\sigma}_i^2 - \bar{\sigma}_{kj}}{\bar{\sigma}_i^2}$$

The above ratio is equal to 0.6(60%) for Italian securities and 0.8(80%) for Belgian securities.

(b) Setting the above ratio equal to those values and solving for $\bar{\sigma}_{kj}$ gives $\bar{\sigma}_{kj} = 0.4\bar{\sigma}_i$ for Italian securities and $\bar{\sigma}_{kj} = 0.2\bar{\sigma}_i^2$ for Belgian securities.

If the average variance of a single security, $\bar{\sigma}_j^2$, in each country equals 0.005, then $\bar{\sigma}_{kj} = 0.4\bar{\sigma}_i^2 = 0.4 \times 0.0050 = 0.002$ for Italian securities and $\bar{\sigma}_{kj} = 0.2\bar{\sigma}_i^2 = 0.2 \times 0.005 = 0.001$ for Belgian securities. Using Equation (1) with $\bar{\sigma}_j^2 = 0.005$ and either $\bar{\sigma}_{kj} = 0.002$ for Italy or $\bar{\sigma}_{kj} = 0.001$ for Belgium we have:

Portfolio Size (n securities)	Italian σ_H^2	Belgian σ_H^2
5	0.0026	0.0018
20	0.00215	0.0012
100	0.00203	0.00104

Exercise 1.4.

(a) The diversification ratio measures, in percentage, how much of the average asset variance can be diversified away by building portfolios.

In this case we have

$$D = \frac{\bar{\sigma}_i^2 - \bar{\sigma}_{ij}}{\bar{\sigma}_i^2} = \frac{0.0046619 - 0.0007058}{0.0046619} = 84.86\%.$$

(b) The formula for an equally weighted portfolio's variance is

$$\sigma_H^2 = \frac{1}{n}\left(\bar{\sigma}_j^2 - \bar{\sigma}_{kj}\right) + \bar{\sigma}_{kj}$$

where $\bar{\sigma}_j^2$ is the average variance across all securities, $\bar{\sigma}_{kj}$ is the average covariance across all securities, and N is the number of securities. The average variance for the securities in the table is 0.0046619 and the average covariance is 0.0007058. Using the above equation with those two numbers, setting equal to 8, and solving for n gives:

$$8 \geq \frac{1}{n}(46.619 - 7.058) + 7.058$$
$$0.942n \geq 39.561$$
$$n \geq 41.997$$

Since the portfolio's variance decreases as n increases, holding 42 securities will provide a variance less than 0.0008, so 42 is the minimum number of securities requiered.

1.2 Investment Opportunity Sets and Efficient Frontiers

Exercise 1.5.

(a) We know that $\sigma_A = 9\%$ and $\sigma_B = 15\%$. We also know that securities A and B are combined in order to override the portfolio risk, which is only possible when $\rho = -1$. Therefore, the weight of each asset in portfolio of zero risk is given the equation system

$$\begin{cases} x_A + x_B = 1 \\ \sigma_A^2 x_A^2 + \sigma_B^2 x_B^2 + 2x_A x_B \sigma_{AB} = 0 \end{cases} \Leftrightarrow \begin{cases} x_B = 1 - x_A \\ \sigma_A^2 x_A^2 + \sigma_B^2 (1 - x_A)^2 + 2x_A (1 - x_A) \sigma_{AB} = 0 \end{cases}$$

$$\begin{cases} x_B = 1 - x_A \\ 0.0081 x_A^2 + 0.225 \left(1 - 2x_A + x_A^2\right)^2 + 2x_A (1 - x_A)(-0.0135) = 0 \end{cases} \Leftrightarrow$$

$$x_A = 0.625 \qquad x_B = 1 - 0.625 = 0.375$$

Therefore, $x_A = 62.5\%$ and $x_B = 37.5\%$.

(b) If the null portfolio has a return of 7.5%, we know its composition is

$$7.5\% = 5\% x_A + \bar{R}_B (1 - x_A)$$

and from (a) we also know that $x_A = 62.5\%$. Thus,

$$\bar{R}_B = \frac{0.075 - 0.05 \times 0.625}{0.375} = 11.67\%$$

(c) The statement is TRUE. Asset B is the one with the highest expected return and risk. From above we see the zero-risk portfolio requires a positive investment in asset B (of 37.5%). Any portfolio with lower weight in B has a negative Sharpe ratio (slope in mean-variance space). Thus, short selling of asset B to invest more than 100% in asset A is also necessarily inefficient.

Exercise 1.6.

(a)-(b) From Exercise 1.1 we know $\bar{R}_1 = 12\%$, $\bar{R}_2 = 6\%$, $\sigma_1 = 2.83\%$, $\sigma_2 = 1.41\%$ and $\rho_{12} = -1$.

We can get the IOS analytical expression to the equations by: (i) first finding the expected return of the combination with zero risk, and then (ii) using the basic assets 1 and 2 to find the slopes of the two lines.

Given the perfect negative correlation, we know that geometrically the investment opportunity set (IOS) is defined by two segments of lines each passing by each of the two risky securities and with a common y-cross at the zero risk portfolio. When shortselling is forbidden the drashed portions are not feasible.

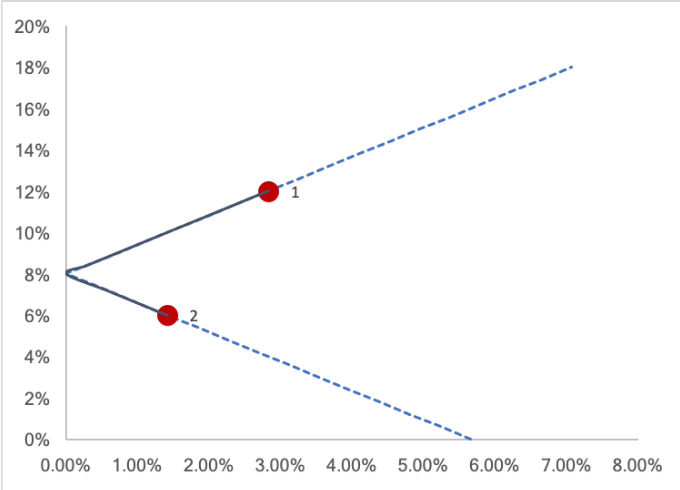

Figure 2: Exercise 1.6 – two risky assets $\rho = -1$. IOS with (full + dashed lines) and without (full) shortselling.

(i) The minimum variance portfolio is the one without risk, $\sigma_p = 0$. Analytically,

$$
\begin{aligned}
&0 = \sigma_1^2 X_1^2 + \sigma_2^2 (1 - X_1)^2 + 2X_1 (1 - X_1) \sigma_{12} \\
&X_1 = \frac{\sigma_p + \sigma_2}{\sigma_1 + \sigma_2} = \frac{\sigma_2}{\sigma_1 + \sigma_2} = \frac{\sqrt{2}}{\sqrt{8} + \sqrt{2}} = \frac{1}{3} \\
&\Rightarrow X_2 = \frac{2}{3}
\end{aligned}
$$

Thus, the portfolio has 33.33% of security 1 and 66.67% of security 2. The expected return is

$$\bar{R}_{MV} = \sum x_i \bar{R}_i = \frac{1}{3} \times 12\% + \frac{2}{3} \times 6\% = 8\%$$

(ii) The slopes of the two lines are given by $\dfrac{\bar{R}_1 - 8\%}{\sigma_1} = 1.41$ and $\dfrac{\bar{R}_2 - 8\%}{\sigma_2} = -1.41$, respectively.

So, the IOS is given by $\quad \bar{R}_p = \begin{cases} 8\% + 1.41\sigma_p & \sigma_p \leq 2,83\% \\ 8\% - 1.41\sigma_p & \sigma_p \leq 1.41\% \end{cases}$.

(c) All portfolios in the segment line with positive slope dominate those in the negative slope segment line, since risk averse investors will prefer from a set of two portfolios with the same risk, the one with highest return. Therefore, the efficient frontier in the positive slope segment line, i.e. $\bar{R}_p = 8\% + 1.41\sigma_p$ for $\sigma_p \leq 2.83\%$.

(d) If shortselling is allowed the derivations in (a)-(b) still stand, the only different is that in the representation of the IOS the entire lines should be considered. I.e. in the above figure the dashed segments would also be feasible.

The efficient set would, thus, be represented by the entire upper line.

$$\text{IOS:} \bar{R}_p = 8\% \pm 1.41\sigma_p \qquad \text{and} \qquad \text{EF:} \quad \bar{R}_p = 8\% + 1.41\sigma_p \ .$$

In particular, all combinations of 1 and 2 that require shortselling of asset 2 to invest more than 100% in 1 are efficient.

Exercise 1.7.

(a) We start by determining expected returns, variances and covariances of the two assets.

$$\bar{R}_1 = \mathbb{E}(R_1) = \frac{1}{3} \times (0.2 + 0.14 + 0.08) = 14\%$$
$$\bar{R}_2 = \mathbb{E}(R_2) = \frac{1}{3} \times (0.16 + 0.12 + 0.08) = 12\%$$
$$\sigma_1^2 = \mathbb{E}\left[\left(R_{1t} - \bar{R}_1\right)^2\right] = \frac{1}{3} \times \left[(0.2 - 0.14)^2 + (0.14 - 0.14)^2 + (0.08 - 0.14)^2\right] = 0.0024$$
$$\sigma_1 = \sqrt{0.0024} = 4.90\%$$
$$\sigma_2^2 = \mathbb{E}\left[\left(R_{2t} - \bar{R}_2\right)^2\right] = \frac{1}{3} \times \left[(0.16 - 0.12)^2 + (0.12 - 0.12)^2 + (0.08 - 0.12)^2\right] = 0.001067$$
$$\sigma_2 = \sqrt{0.001067} = 3.236\%$$
$$\sigma_{12} = \mathbb{E}\left[\left(R_{1t} - \bar{R}_1\right)\left(R_{2t} - \bar{R}_2\right)\right] = \frac{1}{3} \times \left[(0.2 - 0.14)(0.16 - 0.12) + (0.08 - 0.14)(0.08 - 0.12)\right]$$
$$= 0.0016$$
$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} = \frac{0.0016}{\sqrt{0.0024}\sqrt{0.001067}} = +1$$

Thus, the returns of the two securities are perfectly positively correlated, thus, the investment opportunity set (IOS), when shortselling is allowed, is given by two lines: one connecting the two risky securities, and the line with symmetric slope.

$$\text{IOS (i)}: \quad \bar{R}_p = \underbrace{\bar{R}_2 - \frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2}\sigma_2}_{\text{y-cross}} + \underbrace{\frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2}}_{\text{slope}}\sigma_p$$
$$= 0.12 - \frac{0.14 - 0.12}{0.049 - 0.03236}0.03236 + \frac{0.14 - 0.12}{0.049 - 0.03236}\sigma_p$$
$$= 0.08 + 1.2247\sigma_p$$

and

$$\bar{R}_p = 0.08 - 1.2247\sigma_p \; .$$

Although the negative slope line is not efficient it still belongs to the IOS.

When shortselling is not allowed, the IOS is only the segment of the line that passes by the two risky assets

$$\text{IOS (ii)}: \quad \bar{R}_p = 0.08 + 1.2247\sigma_p \quad \text{for} \quad 1.41\% \leq \sigma_p \leq 3.236\%$$

(b) The minimum variance portfolio, when shortselling is forbidden – scenario (ii) – involves placing all funds in the lower risk security (asset 2). Consequently, the expected return is $\bar{R}_{MV} = \bar{R}_2 = 12\%$ and risk is $\sigma_{MV} = \sigma_2 = 3.236\%$.

If short sales were allowed – scenario (i) – than $\sigma_p = 0$ and $\bar{R}_p = 8\%$. Moreover, the weights of the MV portfolio is,

$$x_1 = \frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2} = -200\% \Rightarrow x_2 = 1 - x_1 = 1 - (-2) = 300\%$$

(c) As known, the efficient frontier is the investment opportunity set and investor are considered to be risk averse. For the two scenarios we have:

$$\text{EF (i)}: \quad \bar{R}_p = 8\% + 1.2247\sigma_p$$
$$\text{EF (ii)} = \text{IOS (ii)}: \quad \bar{R}_p = 0.08 + 1.2247\sigma_p \quad \text{for} \quad 1.41\% \leq \sigma_p \leq 3.236\%$$

(d) If we have a riskless asset with $R_f = 10\%$ then The investment opportunity set becomes:

- IOS (i): When shortselling is allowed without any bound, the theoretical answer would be the entire space $(\sigma_p, \bar{R}_p)$.

  In a real life situation, there will be an extreme combination, E, where one takes the highest possible shortselling position in asset 2. In that case the IOS would be the *entire area bellow the straight line* $\bar{R}_p = 0.1 + \frac{\bar{R}_E - 0.1}{\sigma_E}\sigma_p$.

- IOS (ii): When shortselling is not allowed the IOS is *the cone limited by* the lines $\bar{R}_p = 0.1 \pm \frac{0.14 - 0.1}{0.049}\sigma_p$.

The efficient frontier (EF) becomes:

- When shortselling is allowed – scenario (i) – the efficient frontier would be the straight line that has y-cross at 10% and has the highest possible slope.

  In a real life situation ,where eventually there would be a limit to how much one can shortsell of asset 2, it would be combinations of the riskless asset with the portfolio with that extreme, E portfolio,

  EF (i): $\bar{R}_p = 0.1 + \frac{\bar{R}_E - 0.1}{\sigma_E}\sigma_p$.

- When shortseliing is not allowed – scenario (ii) the efficient frontier would be given by combinations of the riskless asset with asset 1,

  EF (ii): $\bar{R}_p = 0.1 + \frac{0.14 - 0.1}{0.049}\sigma_p$.

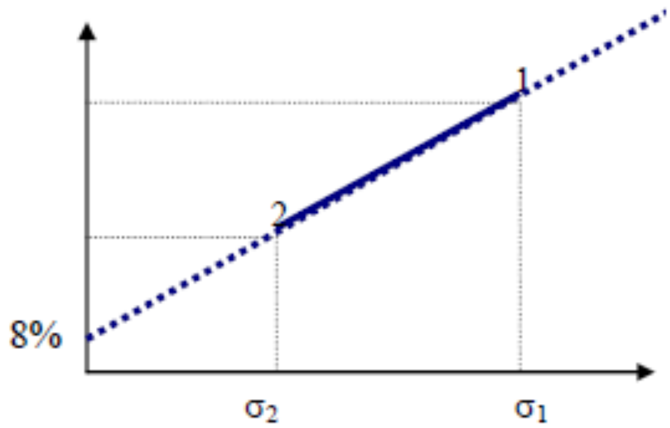

Figure 3: Exercise 1.7 – Two perfectly correlated assets. Efficient frontier with (full + drashed) and without (full) shortselling.

Exercise 1.8.

(a) Similar to (b) but with $\rho = -1$ (see slides).

(b) As discussed in Exercise 1.6, the investments opportunity set generated by two assets with perfect negative correlation is given by two line segments. An alternative to the solution presented there is to deduce directly the equation(s) $\bar{R}_p = f(\sigma_p)$.

Starting with some transformation in σ_p equation:

$$\begin{aligned}\sigma_p &= \sqrt{x_1^2\sigma_1^2 + x_2^2\sigma_2^2 - 2x_1x_2\sigma_1\sigma_2} \\ &= \sqrt{(x_1\sigma_1 - x_2\sigma_2)^2} \\ &= \pm\left|x_1\sigma_1 - x_2\sigma_2\right| \\ &= \pm\left|x_1\sigma_1 - (1-x_1)\,\sigma_2\right|\end{aligned}$$

With additional transformations, we get an equation to x_1

$$\sigma_p \pm \sigma_2 = \pm x_1\left(\sigma_1 - \sigma_2\right) \Leftrightarrow x_1 = \pm\frac{\sigma_p + \sigma_2}{\sigma_1 + \sigma_2}$$

Replacing x_1 in the expected return equation for a two assets portfolio, we get

$$\begin{aligned}\bar{R}_p &= x_1\bar{R}_1 + (1-x_1)\,\bar{R}_2 \\ &= \pm\frac{\sigma_p + \sigma_2}{\sigma_1 + \sigma_2}\bar{R}_1 + \left(1 \pm \frac{\sigma_p + \sigma_2}{\sigma_1 + \sigma_2}\right)\bar{R}_2 \\ &= \bar{R}_2 + \frac{\pm\sigma_p\bar{R}_1 + \sigma_2\bar{R}_1 \pm \sigma_p\bar{R}_2 - \sigma_2\bar{R}_2}{\sigma_1 + \sigma_2} \\ &= \left(\bar{R}_2 + \frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2}\sigma_2\right) \pm \left(\frac{\bar{R}_1 - \bar{R}_2}{\sigma_1 - \sigma_2}\right)\sigma_p\end{aligned}$$

The first term in the right side in the intersection in the y's axis and the second term is the slope, which can be positive or negative, giving origin to the two expected line segments.

(c) Equal to (d), replacing the generic ρ by 0 (see slides).

(d) Solved in class.

(e) Real life correlations are not perfect, so real life correlations are $\rho \neq -1$ or $\rho \neq +1$. Returns laso tend to be correlate with one another, so $\rho \neq 0$. All other values may occur, but for financial assets positive correlations are more common than negative.

Exercise 1.9.

(a) The investment opportunity sets are represented in the Figure 1.9 below.

(b) – When $\rho = +1$, the least risky "combination" of securities 1 and 2 is security 2 held alone (assuming no short sales). This requires $x_1^{MV} = 0$ and $x_2^{MV} = 1$, where the $x's$ are the investment weights. The standard deviation of this "combination" is equal to the standard deviation of security 2: $\sigma_{MV} = \sigma_2 = 2\%$.

– When $\rho = -1$, we can always find a combination of the two securities that will completely eliminate risk, and we this combination can be found by solving $x_1^{MV} = \frac{\sigma_2}{\sigma_1+\sigma_2}$. So, $x_1^{MV} = \frac{2\%}{5\%+2\%} = \frac{2}{7}$, and since the investment weights must sum to 1, $x_2^{MV} = 1 - x_1 = 1 - \frac{2}{7} = \frac{5}{7}$. So a combination of $\frac{2}{7}$ invested in security 1 and $\frac{5}{7}$ invested in security 2 will completely eliminate risk when ρ equals -1, and σ_{MV} will equal 0.

– When $\rho = 0$ e, the minimum-risk combination of two assets can be found by solving $x_1^{MV} = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$. So, $x_1^{MV} = \frac{4\%}{25\%+4\%} = \frac{4}{29}$, and $x_2^{MV} = 1 - x_1 = 1 - \frac{4}{29} = \frac{25}{29}$. When ρ equals 0, the expression for the standard deviation of a two-asset portfolio is

$$\sigma_p = \sqrt{x_1^2\sigma_1^2 + (1-x_1)^2\,\sigma_2^2}$$

Substituting $\frac{4}{29}$ for x_1 in the above equation, we have

$$\sigma_{MV} = \sqrt{\left(\frac{4}{29}\right)^2 \times 0.0025 + \left(\frac{25}{29}\right)^2 \times 0.0004} = 1.86\% \ .$$

(c) (i) Both for $\rho = -1$ nd $\rho = 0$ the minimum variance portfolios remain the same. However, for $\rho = +1$ if shortselling is allowed we can fully eliminate risk. Since, in the case, we have $\sigma_p = |x_1\sigma_1 + (1 - x_1)\sigma_2|$, setting $\sigma_{MV} = 0$, we obtain

$$0 = x_1^{MV} \times 0.05 + (1 - x_1^{MV})0.02 \quad \Leftrightarrow \quad x_1^{MV} = -\frac{0.02}{0.03} = -66.67\%, \ x_2^{MV} = 166.67\%.$$

(ii) When we have $\rho = \pm 1$ there is a combination of 1 and 2 that fully eliminates risk, thus there is a risk-free investment or a “ficticious” riskless asset. The return of the zero risk combinations give us the appropriate risk-free return R_f.

$$\begin{aligned}
\rho = -1: \qquad R_f &= x_1^{MV}\bar{R}_1 + x_2^{MV}\bar{R}_2 \\
&= \frac{2}{7} \times 10\% + \frac{5}{7} \times 4\% = 5.71\% \\
\rho = +1: \qquad R_f &= x_1^{MV}\bar{R}_1 + x_2^{MV}\bar{R}_2 \\
&= -0.6667 \times 10\% + 1.6667 \times 4\% = 0\% \ .
\end{aligned}$$

In the case of $\rho = 0$ the minimum risk combination portfolio has positive volatility $sigma_M V = 1.86\%$, thus there is no risk free investment.

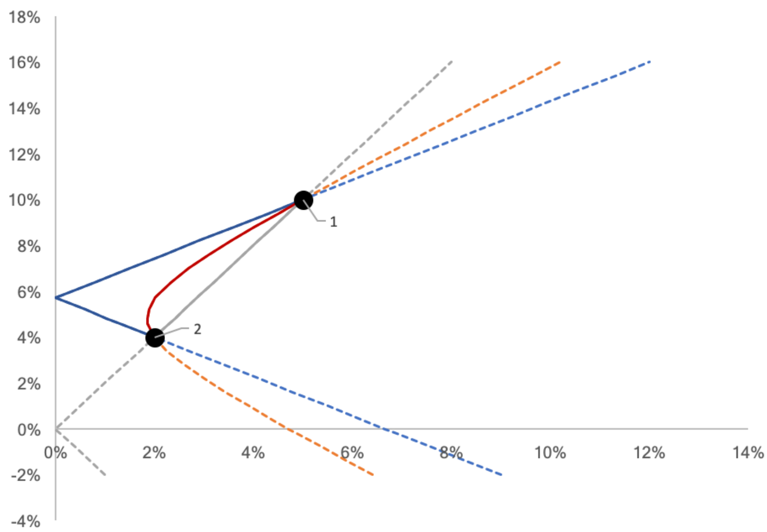

Figure 4: Exercise 1.9 – blue line $\rho = -1$, red line $\rho = 0$ and grey line $\rho + 1$, full lines (no shortselling), dashed lines (shortselling required).

Exercise 1.10. If the risk-less rate is 10%, then the risk-free asset dominates both risky assets both in terms of risk and return. It offers as much or higher return than each of the risky assets, for zero risk. Assuming the investor prefers more to less and is risk averse, the only efficient investment is 100% investment in the risk-free asset.

Exercise 1.11.

(a) When there is a risk-free asset that can be used for both lending and borrowing we know the efficient frontier is a straight line tangent to the investment opportunity set of risky assets. Thus, there is only one efficient portfolio made only of risky assets - the so called *tangent portfolio*. See Figure 1.11.

To find this unique efficient portfolio we need to maximize Sharpe's Ratio of all portfolios formed with assets A, B and C. From the first order conditions of this maximisation problem, result the following equation system:

$$\begin{cases} 0.11 - R_f = 0.0004z_A + 0.001z_B + 0.0004z_C \\ 0.14 - R_f = 0.0010z_A + 0.0036z_B + 0.003z_C \\ 0.17 - R_f = 0.0004z_A + 0.003z_B + 0.0081z_C \end{cases}$$

The Z-vector, for each given value for R_F and the unrestricted tangent portfolios are:

	$R_f = 6\%$	$R_f = 8\%$	$R_f = 10\%$
z_A	351.0067	185.2348	19.4631
z_B	-104.3624	-52.6845	-1.0070
z_C	34.8993	21.4765	8.0537
x_A	124.67%	120.26%	73.42%
x_B	-37.07%	-34.20%	-3.80%
x_C	12.40%	13.94%	30.38%
Tangent Portfolio			
Expected Return	10.63%	10.81%	12.71%
Standard Deviation	1.28%	1.35%	3.20%
Sharpe ratio	3.611	2.081	0.8474
Efficient Frontier	$\bar{R}_p = 0.06 + 3.611\sigma_p$	$\bar{R}_p = 0.08 + 2.081\sigma_p$	$\bar{R}_p = 0.1 + 0.8474\sigma_p$

(b) If there is no credit to invest in risky assets nothing changes in the efficient frontier for risk levels lower or equal to σ_T, however for $\sigma_p > \sigma_T$ the efficient thing to do are the combinations on the envelop hyperbola.

The hyperbola delimiting the IOS of the risky assets is given by

$$\sigma_p^2 = \frac{A\bar{R}_p^2 - 2B\bar{R}_p + C}{AC - B^2} \qquad \text{where} \qquad \begin{array}{c} A = \mathbb{1}'V^{-1}\mathbb{1} \\ B = \bar{R}'V^{-1}\mathbb{1} \\ C = \bar{R}'V^{-1}\bar{R} \end{array}$$

For our concrete example we get

$$\sigma_p^2 = 1.6450\bar{R}_p^2 - 0.3426\bar{R}_p + 0.018 \ .$$

The efficient frontier is, thus, given by

$$\begin{cases} \bar{R}_p = R_f + SR_T \times \sigma_p & \sigma_p \leq \sigma_T \\ \sigma_p^2 = 1.6450\ \bar{R}_p^2 - 0.3426\bar{R}_p + 0.018 & \sigma_p > \sigma_T \end{cases}$$

where in the expression above we should replace for the appropriate values of R_f, SR_T and σ_T, according to each scenario.

(c) If shortselling is forbidden we know we are not going to invest in asset B, since the optimal would be to short sell it.

We can solve the problem numerically, imposing $x_i \geq 0$ to all $i = A, B, C$, or, in this case the problem reduces to a two-asset case and find the two-asset tangent portfolios.

Either way, we get

	$R_f = 6\%$	$R_f = 8\%$	$R_f = 10\%$
x_A	93.77%	89.61%	68.83%
x_C	6.23%	10.39%	31.17%
Tangent Portfolio			
Expected Return	11.37%	11.62%	12.87%
Standard Deviation	2.07%	2.2%	3.39%
Sharpe ratio	2.592	1.648	0.8471
Efficient Frontier	$\bar{R}_p = 0.06 + 2.592\sigma_p$	$\bar{R}_p = 0.08 + 1.648\sigma_p$	$\bar{R}_p = 0.1 + 0.8471\sigma_p$

(d)

(i) We use the same Z-vectors as in the unrestricted case, to get the Lintner portfolios.

	$R_f = 6\%$	$R_f = 8\%$	$R_f = 10\%$
x_A	71.60%	71.41%	68.24%
x_B	-21.29%	-20.31%	-3.53%
x_C	7.11%	8.28%	28.24%
Lintner Portfolios			
$\sum x_i$	57.42%	59.38%	92.95%
x_f	42.58%	40.62%	7.05%
Expected Return	8.66%	9.67%	12.52%
Standard Deviation	0.74%	0.80%	2.97%
Sharpe ratio	3.611	2.081	0.8474

Note that Lintner portfolios have the same Sharpe ratios as unrestricted tangent portfolios. They can always be interpreted as a combination of deposit with the (unrestricted) tangent portfolio.

(ii) Since, none the original tangent portfolios requires more than 50% shortselling, they all satisy this restriction.

(ii) For the case of $R_f = 10\%$ this limit is satisfied and nothing changes.

For $R_f = 6\%$ and $R_f = 8\%$ the limit is not satisfied by the original tangent portfolios, thus, we know that we will now get $x_B = -25\%$. The remaining weight we can get numerically (for instance using excel solver).

The table below show the results.

	$R_f = 6\%$	$R_f = 8\%$	$R_f = 10\%$
x_A	115.27%	112.82 %	68.24%
x_B	-25.00%	-25.00%	-35.29%
x_C	9.73%	12.18%	28.24%
Tangent Portfolios (limited 25% shortselling)			
Expected Return	10.83%	10.98%	12.52%
Standard Deviation	1.42%	1.47%	2.97%
Sharpe ratio	3.413	2.021	0.8474

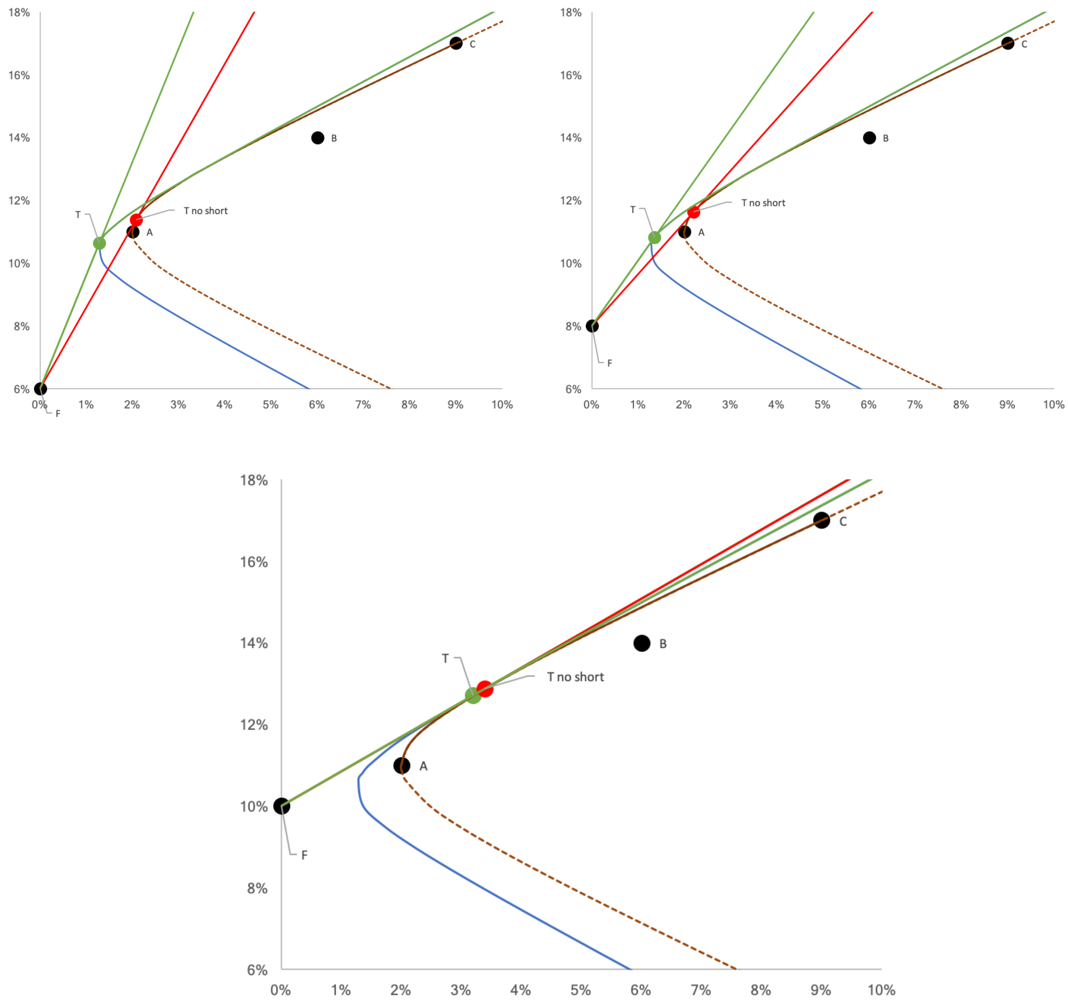

Figure 5: Exercise 1.11 – Efficient Frontiers (green) when shortselling is allowed (with and without borrowing). Outer hyperbola (blue) is the envelop hyperbola when we consider investment without constraints in the three assets A, B, C. Inner hyperbola is the two-assets hyperbola for assets A and C where the full line represents the no shortselling segment and the dashed line the portfolios that require shortselling. Top left image: $R_f = 6\%$. Top right: $R_f = 8\%$. Bottom image: $R_f = 10\%$

Exercise 1.12.

(a) Since the given portfolios, A and B, are on the efficient frontier, the MV portfolio can be obtained by finding the minimum-risk combination of the two portfolios:

$$x_A^{MV} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} = \frac{0.0016 - 0.0002}{0.0036 + 0.0016 - 2 \times 0.002} = -\frac{1}{3}$$

$$x_B^{MV} = 1 - X_A^{MV} = 1 - \left(-\frac{1}{3}\right) = \frac{4}{3}$$

This gives $\bar{R}_{MV} = 7.33\%$ and $\sigma_{MV} = 3.83\%$.

Also, since the two portfolios are on the efficient frontier, the entire efficient frontier can then be traced by using various combinations of the two portfolios, starting with the MV portfolio and moving up along the efficient frontier (increasing the weight in portfolio A and decreasing the weight in portfolio B).

Since $x_B = 1 - x_A$ the efficient frontier equations are:

$$\begin{cases} \bar{R}_p = x_A \bar{R}_A + (1 - x_A)\,\bar{R}_B \\ \sigma_p^2 = x_A^2 \sigma_A^2 + (1 - x_A)^2\,\sigma_B^2 + 2x_A\,(1 - x_A)\,\sigma_{AB} \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} \bar{R}_p = 10x_A + 8 \times (1 - x_A) \\ \sigma_p^2 = 0.0036x_A^2 + 0.0016\,(1 - x_A)^2 + 2x_A\,(1 - x_A)\,0.002 \end{cases}$$

$$\Leftrightarrow$$

$$\sigma_p^2 = 3\bar{R}_p^2 - 0.44\bar{R}_p + 0.0176 \qquad \text{(hyperbola equation)}$$

Since short sales are allowed, the efficient frontier will extend beyond portfolio A and out toward infinity. The efficient frontier appears as shown in Figure 1.12 (full blue line).

(b) If there is a risk-free asset that can be used for both deposit and borrowing, then we know the efficient frontier is a straight line passing by the risk-free asset and tangent to the hyperbola given by combinations of any two efficient portfolios. So, Its equations is given by $\bar{R}_p = R_f + SR_T\sigma_p$ where SR_T is the Sharpe ratio of the tangent portfolio.

The tangent portfolio is the combination of the two efficient portfolios that has the highest Sharpe ratio. From the FOC we find

$$Z = V^{-1}1'(\bar{R} - R_f\mathbb{1}) = \begin{pmatrix} 4.5455 \\ 31.8182 \end{pmatrix} \qquad \Rightarrow \qquad X_T = \begin{pmatrix} 12.5\% \\ 87.5\% \end{pmatrix}$$

and we get

$$\bar{R}_T = X_T'\bar{R} = \begin{pmatrix} 12.5\% & 87.5\% \end{pmatrix} \begin{pmatrix} 8\% \\ 6\% \end{pmatrix} = 8.25\%$$

$$\sigma_T^2 = X_T' V X_T = \begin{pmatrix} 12.5\% & 87.5\% \end{pmatrix} \begin{pmatrix} 0.0036 & 0.002 \\ 0.002 & 0.0016 \end{pmatrix} \begin{pmatrix} 12.5\% \\ 87.5\% \end{pmatrix} = 0.001467$$

$$\Rightarrow \sigma_T = 3.83\%$$

$$SR_T = \frac{\bar{R}_T - R_f}{\sigma_T} = \frac{0.0825 - 0.02}{0.0415} = 1.5076\ .$$

So the efficient frontier in this case is given by

$$\bar{R}_p = 0.02 + 1.5076\sigma_p\ ,$$

the straight green line in Figure 1.12.

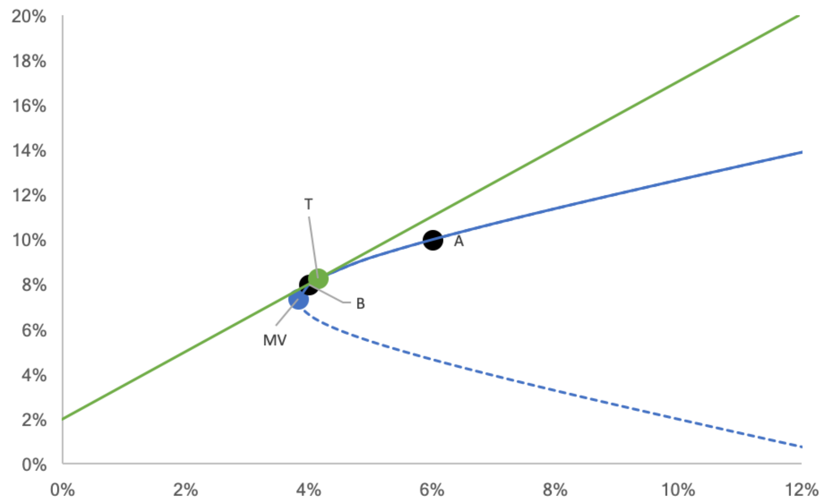

Figure 6: Exercise 1.12 – Efficient Frontier with (green line) and without the risk-free asset (full blue line on the upper part of the hyperbola.

(c) Since the tangent portfolio does not require shortselling the Lintner portfolio is the tangent portfolio itself and noting changes. When shortselling is limited *a la* Lintner nothing changes in term

   (i) If A and B are still feasible this means they are portfolios without any shortselling position. In addition, since the tangent portfolio does not require shortselling of A nor B, we have the guarantee it remains feasible. For the same reason the minimum variance portfolio is also feasible. So in this case nothing substantial chanages expect that when there is no risk-free asset eventually the hyperbola stops at the point where we would need to shortsell an asset.

   (ii) If one of the original efficient portfolios is not longer feasible, that means that portfolio would require shortselling of some risky asset. Without two efficient portfolios we would not be able to derive the envelop hyperbola equation. Since we have no information about the basic risky assets in this market, we cannot derive the new efficient frontier, but it would be contained in the interior of the previously derived hyperbola.