PART III
CREDIT RISK MODELS

1 - INTRODUCTION

Credit Risk

"Default risk is the risk that an obligor does not honour his payment obligati-ons."

Typically,

- Default events are rare.
- They may occur unexpectedly.
- Default events involve significant losses.
- The size of these losses is unknown before default.

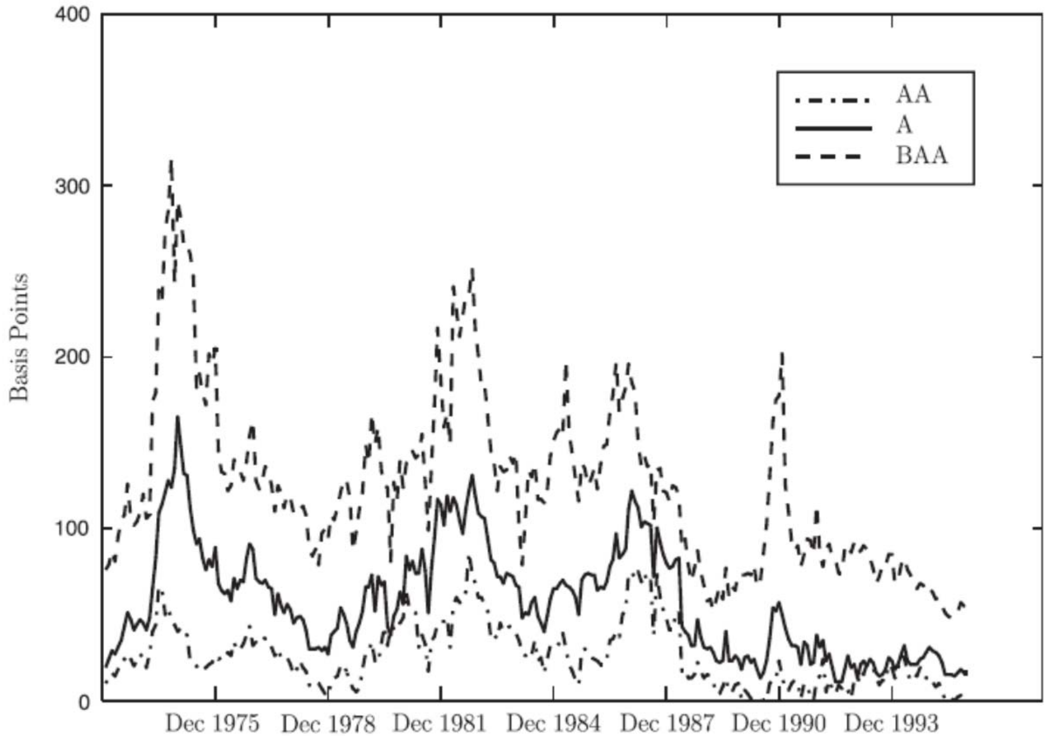
All payment obligations represent some sort of default risk.

DETERMINANTS OF CREDIT RISK

- "Credit risk is the risk of default or of reductions in market value caused by changes in the credit quality of issuers or counterparties", Duffie, Darrell and Kenneth J. Singleton (2003), "Credit Risk", Princeton University Press.
- Credit Risk is associated to the PD of the debtor, as well as the LGD.
- Regarding the credit risk of the debtor, it is relevant not only to quantify the PDs but also the rating transition frequencies, which also impact on bond prices.
- Nonetheless, the expected loss is usually calculated taking only default into consideration: EL = PD x LGD
- Given the diversity of the counterparties, the market usually distinguishes sovereign, banking, corporate and individual/household credit risk.

DETERMINANTS OF CREDIT RISK

- The bond spreads usually provide relevant information on credit risk.



Source: Duffie, Darrell and Kenneth J. Singleton (2003), “Credit Risk”, Princeton University Press.

COMPONENTS OF CREDIT RISK

Arrival risk is a term for the uncertainty whether a default will occur or not. To enable comparisons, it is specified with respect to a given time horizon, usually one year. The measure of arrival risk is the *probability of default*. The probability of default describes the distribution of the indicator variable *default before the time horizon*.

Timing risk refers to the uncertainty about the precise time of default. Knowledge about the time of default includes knowledge about the arrival risk for all possible time horizons, thus timing risk is more detailed and specific than arrival risk. The underlying unknown quantity (random variable) of timing risk is the *time of default*, and its risk is described by the *probability distribution function of the time of default*. If a default never happens, we set the time of default to infinity.

COMPONENTS OF CREDIT RISK

Recovery risk describes the uncertainty about the severity of the losses if a default has happened. In recovery risk, the uncertain quantity is the actual payoff that a creditor receives after a default. It can be expressed in several ways which will be discussed in a later chapter. Market convention is to express the recovery rate of a bond or loan as the fraction of the notional value of the claim that is actually paid to the creditor. Recovery risk is described by the *probability distribution of the recovery rate*, i.e. the probabilities that the recovery rate is of a given magnitude. This probability distribution is a conditional distribution, conditional upon default.

If we consider the risk of joint defaults of several obligors, an additional risk component is introduced. **Default correlation risk** describes the risk that several obligors default together. Again here we have *joint arrival risk* which is described by the joint default probabilities over a given time horizon, and *joint timing risk* which is described by the joint probability distribution function of the times of default.

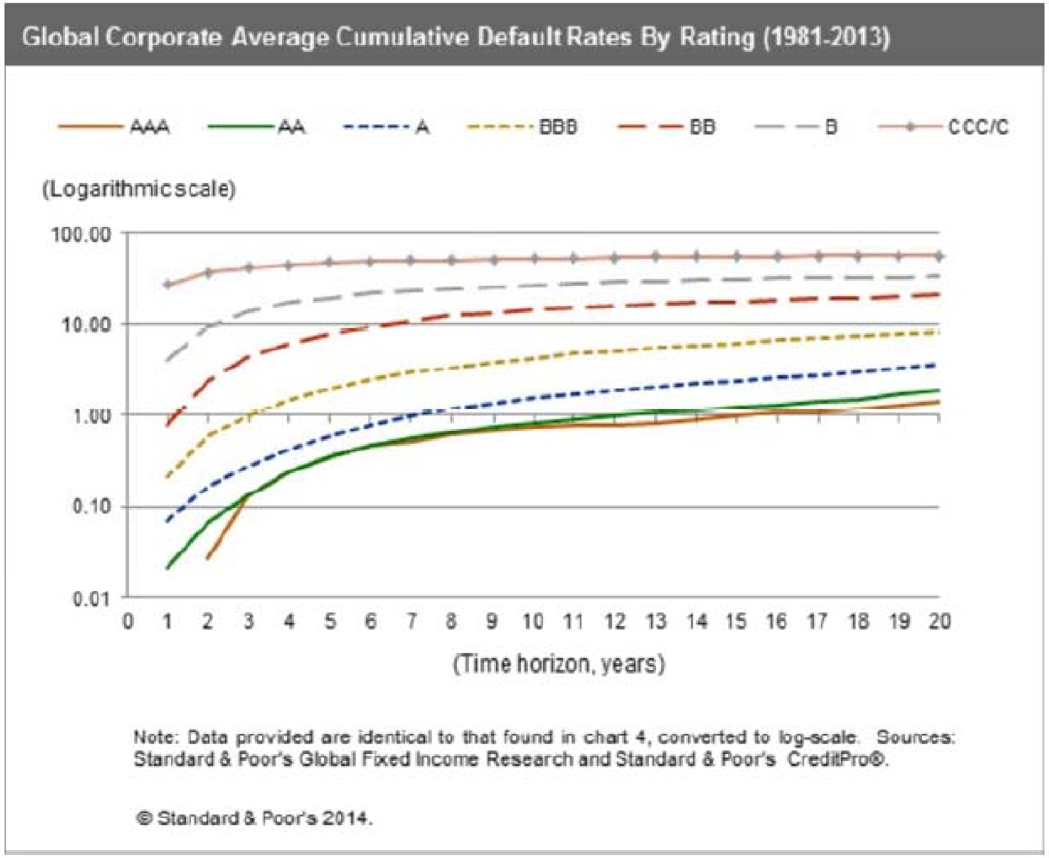
PDs

- Ratings are a ranking of credit risk and do not explicitly provide any PD measure.
- However, one can obtain historical frequencies of default for each rating classification, as well as the historical frequencies of transition between ratings.
- The long term ratings of the main agencies (S&P and Moody's) split by 7 classes, each of them (excluding AAA) with rating modifiers +/ /- (S&P) or 1/2/3 (Moody's).

	S&P	Moody's
Investment Grade	AAA	Aaa
	AA	Aa
	A	A
	BBB	Baa
Speculative Grade	BB	Ba
	B	B
	CCC	Caa
	CC	Ca
	C	C

PDs

- Simplest measure of credit risk – default frequencies from rating agencies:



Source: S&P (2014), "Default, Transition and Recovery: 2013 Annual Global Corporate Default Study and Rating Transitions".

PDs

- Transition matrices illustrate the significant stability of rating classifications, being this stability higher for higher ratings.

Average One-Year Letter Rating Migration Rates, 1920-2016

From/To:	Aaa	Aa	A	Baa	Ba	B	Caa	Ca-C	WR	Default
Aaa	86.746%	7.848%	0.784%	0.193%	0.030%	0.002%	0.000%	0.000%	4.397%	0.000%
Aa	1.059%	84.158%	7.642%	0.729%	0.160%	0.046%	0.012%	0.004%	6.129%	0.060%
A	0.070%	2.740%	84.952%	5.597%	0.646%	0.119%	0.036%	0.008%	5.747%	0.084%
Baa	0.036%	0.239%	4.261%	82.661%	4.632%	0.741%	0.129%	0.017%	7.027%	0.257%
Ba	0.006%	0.072%	0.496%	6.148%	73.923%	6.880%	0.669%	0.089%	10.553%	1.164%
B	0.005%	0.044%	0.162%	0.620%	5.574%	71.711%	6.175%	0.476%	11.940%	3.292%
Caa	0.000%	0.010%	0.028%	0.125%	0.567%	6.897%	67.342%	2.944%	13.675%	8.413%
Ca-C	0.000%	0.016%	0.108%	0.038%	0.616%	2.975%	8.034%	48.426%	18.719%	21.068%

Source: Moody's (2017), "Corporate Default and Recovery Rates, 1920-2016".

PDs

- Default frequencies also tend to change along time, namely for lower ratings.

Exhibit 30

Annual Issuer-Weighted Corporate Default Rates By Letter Rating, 1920-2016*

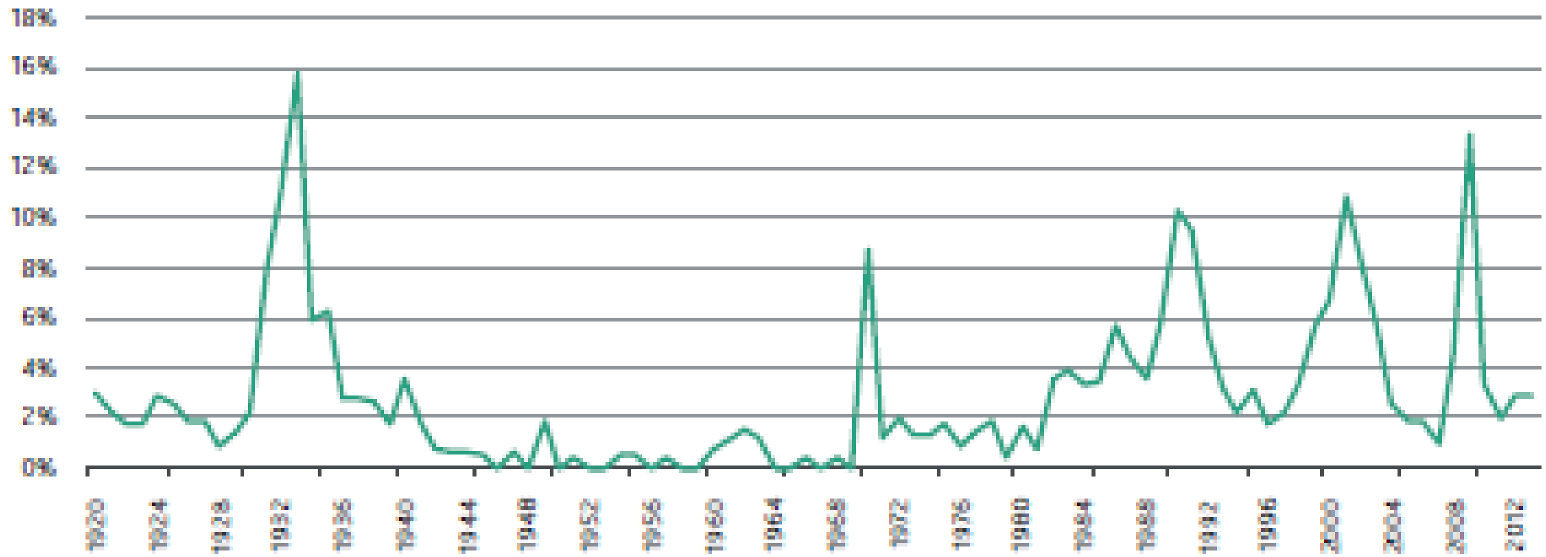
Year	Aaa	Aa	A	Baa	Ba	B	Caa-C	Inv Grade	Spec Grade	All rated
1920	0.000	0.000	0.323	0.942	2.153	4.382	0.000	0.427	3.009	1.234
1921	0.000	0.189	0.353	0.648	0.444	2.683	13.332	0.387	2.150	1.068
1922	0.000	0.185	0.165	1.100	1.078	1.705	7.629	0.506	1.762	1.007
1923	0.000	0.000	0.000	0.622	0.929	2.270	5.932	0.244	1.705	0.804
1924	0.000	0.367	0.000	0.126	2.065	2.705	12.835	0.140	2.852	1.152
1925	0.000	0.000	0.141	0.707	1.745	2.585	14.397	0.321	2.562	1.171
1926	0.000	0.395	0.147	0.113	1.387	2.900	3.704	0.188	1.909	0.768
1927	0.000	0.000	0.212	0.000	1.300	1.980	12.842	0.069	1.831	0.736
1928	0.000	0.000	0.000	0.000	0.164	1.320	10.477	0.000	0.877	0.363
1929	0.000	0.293	0.000	0.446	0.825	0.918	9.733	0.242	1.401	0.715
1930	0.000	0.000	0.000	0.402	0.917	3.163	7.720	0.151	2.204	1.040
1931	0.000	0.000	0.269	1.085	3.005	9.523	31.670	0.502	7.897	3.805
1932	0.000	0.670	1.099	0.929	6.097	13.978	24.062	0.861	10.989	5.503
1933	0.000	0.000	0.258	1.771	11.550	16.147	25.921	0.790	15.709	8.489
1934	0.000	0.617	0.306	0.857	2.529	4.224	16.504	0.586	5.897	3.405
1935	0.000	0.000	1.429	1.923	5.134	4.275	13.024	1.285	6.253	3.935
1936	0.000	0.847	0.543	0.327	1.234	2.385	7.795	0.482	2.720	1.634
1937	0.000	0.000	0.505	1.043	0.997	2.669	9.074	0.619	2.749	1.723
1938	0.000	0.855	1.639	1.990	0.991	1.467	12.808	1.550	2.599	2.110
1939	0.000	0.000	0.000	0.995	0.623	1.744	6.073	0.412	1.774	1.224
1940	0.000	0.000	0.000	1.370	0.433	3.307	11.829	0.592	3.562	2.472
1941	0.000	0.000	0.000	0.000	0.973	0.813	5.071	0.000	1.713	1.085
1942	0.000	0.000	0.000	0.000	0.000	0.791	2.004	0.000	0.736	0.456
1943	0.000	0.000	0.000	0.000	0.000	1.359	0.000	0.000	0.615	0.370
1944	0.000	0.000	0.000	0.000	0.000	0.495	2.551	0.000	0.666	0.389
1945	0.000	0.000	0.000	0.000	0.000	0.000	3.571	0.000	0.565	0.306
1946	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1947	0.000	0.000	0.000	0.000	0.000	0.719	2.778	0.000	0.636	0.315
1948	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1949	0.000	0.000	0.000	0.000	1.360	1.031	8.571	0.000	1.926	0.837
1950	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1951	0.000	0.000	0.000	0.000	0.000	0.000	4.762	0.000	0.433	0.176
1952	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1953	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1954	0.000	0.000	0.000	0.000	0.000	0.000	7.143	0.000	0.467	0.166
1955	0.000	0.000	0.000	0.000	0.000	1.613	0.000	0.000	0.518	0.166
1956	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1957	0.000	0.000	0.000	0.000	0.000	1.266	0.000	0.000	0.448	0.143
1958	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1959	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1960	0.000	0.000	0.000	0.000	1.251	0.000	0.000	0.000	0.750	0.245
1961	0.000	0.000	0.000	0.000	0.599	0.000	8.696	0.000	1.072	0.354
1962	0.000	0.000	0.000	0.000	1.749	1.471	0.000	0.000	1.516	0.471
1963	0.000	0.000	0.000	0.000	1.162	1.471	0.000	0.000	1.152	0.352

Source: Moody's (2017), "Corporate Default and Recovery Rates, 1920-2016".

PDs

- Actually, the volatility of default frequencies for lower ratings (speculative grade) is significant.

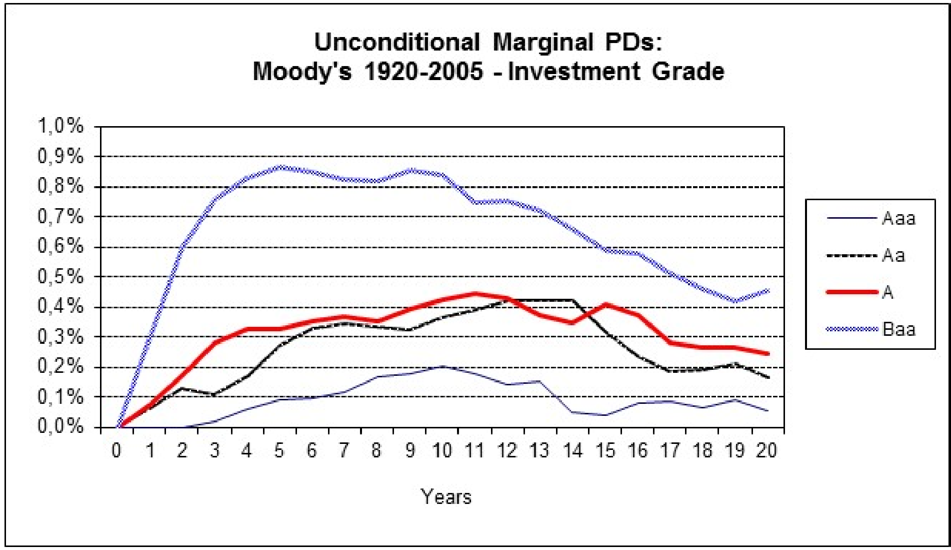
Global Speculative-Grade Default Rate Remained Low in 2013

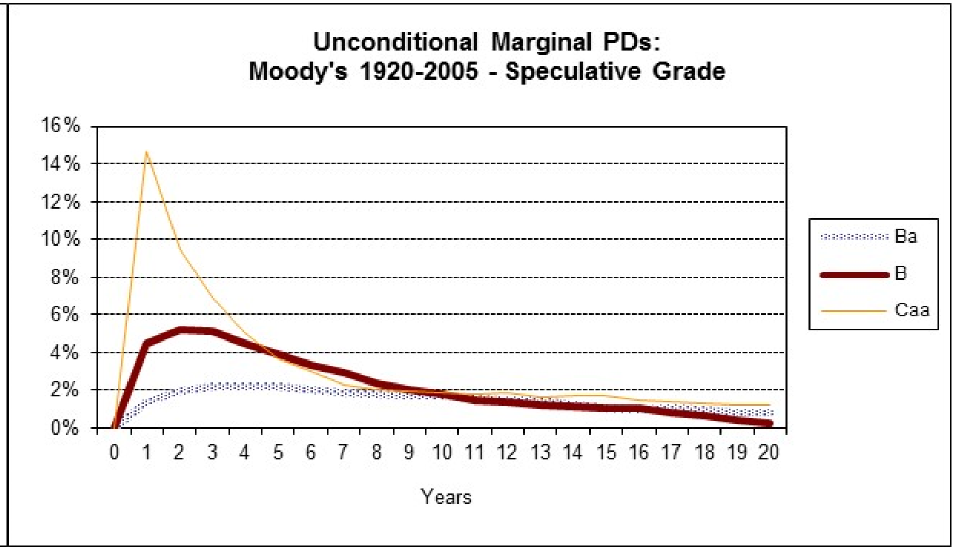


Source: Moody's (2014), "Corporate Default and Recovery Rates, 1920-2013".

PDs

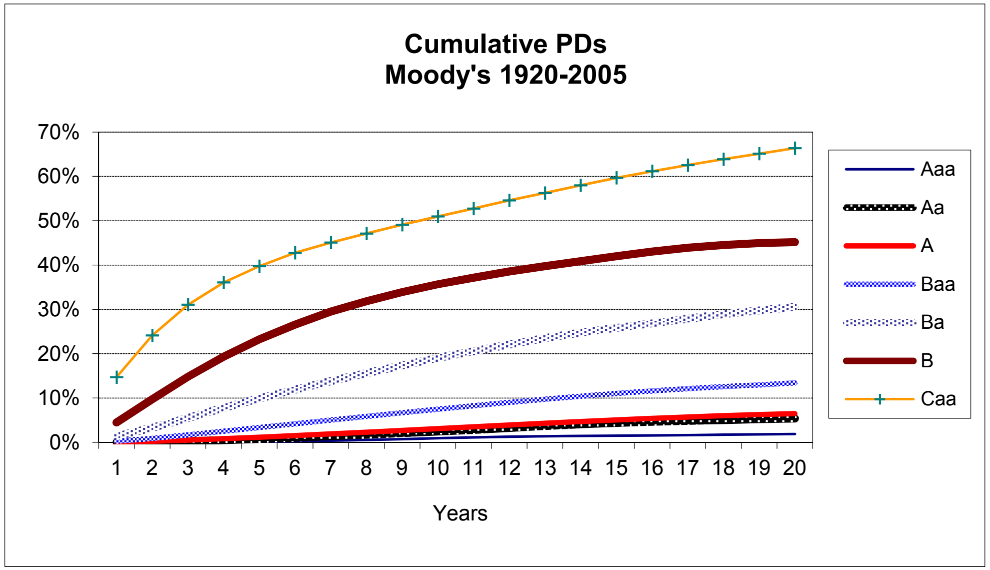
- Marginal frequencies obtained from the cumulative figures tend to exhibit a very irregular shape.
- It can be observed that marginal PD curves have different inflection points, depending on the rating class, with the lower inflection points for the higher risk classes.





PDs

- The irregular shape of marginal PD curves occurs even when cumulative PD curves exhibit an apparently smooth behavior.
- Therefore, in order to ensure a smother behavior of marginal PD curves, it is recommended to smooth the cumulative PD curves, as the marginal curves as a measure of the 1st derivative of the cumulative curves.
- The cumulative PD curves can be smoothed by methods like the Nelson-Siegel-Svensson, with the cumulative PD curves corresponding to the spot curves and the marginal PD curves to the instantaneous forward curves.



PDs

- *P(t)* – cumulative probability of surviving *t* years

- **Unconditional default probability between t and s - probability of default between any times *t* and *s* ≥ *t*:** difference between the probability of default until s and the same probability until t:

$$d'(s) = [1-P(s)]-[1-P(t)] = P(t) - P(s) = D(s) - D(t)$$

difference between 2 cumulative probabilities of default (D) seen today

(being D_0=0)

PDs

- **Probability of surviving to time s (P(s))** = probability of surviving until t (P(t)) x probability of surviving between t and s, given that it has survived until t (p(s|t)):

$$P(s) = P(t) \times p(s|t)$$

↓

- **Conditional probability of surviving to time *s*, given survival to time *t*** (or forward default probability):

$$p(s|t)=P(s)/P(t)$$

PDs

Table 24.1 Average cumulative default rates (%), 1970–2012, from Moody's.

Term (years):	*1*	*2*	*3*	*4*	*5*	*7*	*10*	*15*	*20*
Aaa	0.000	0.013	0.013	0.037	0.106	0.247	0.503	0.935	1.104
Aa	0.022	0.069	0.139	0.256	0.383	0.621	0.922	1.756	3.135
A	0.063	0.203	0.414	0.625	0.870	1.441	2.480	4.255	6.841
Baa	0.177	0.495	0.894	1.369	1.877	2.927	4.740	8.628	12.483
Ba	1.112	3.083	5.424	7.934	10.189	14.117	19.708	29.172	36.321
B	4.051	9.608	15.216	20.134	24.613	32.747	41.947	52.217	58.084
Caa–C	16.448	27.867	36.908	44.128	50.366	58.302	69.483	79.178	81.248

- For the Caa rating, the unconditional default probability (d') seen today for the 3rd year is equal to the difference between the cumulative probabilities of default for 3 (s) and 2 (t) years:
- d'(3) = D(3) – D(2) = 39,908% - 27,867% = 9,041%

PDs

- **The unconditional probability of default measured today** is also the product between the cumulative probability of survival until *t* and the probability of default between t and s, given survival until *t*:

d'(s) = P(t) x d(*s*|*t*)

- Therefore, any unconditional probability of survival may be measured as:

$$d_i^{'} = d_i \prod_{j=1}^{i} \left(1 - d_{j-1}\right)$$

being d_i = d(s|t) and $(1-d_{j-1})$ = P(t)

- The unconditional marginal PD in *i* is the product between the probabilities of survival until *I* and the probability of default in *i*, given that it has survived until then, being obtained from d_i taking out the effect of the condition of having survived in the previous periods (being d_0' = 0).

PDs

- From $d'(s) = P(t) \times d(s|t)$, the **conditional probability of default between s and t,** given survival until t ($d(s|t)$) is:

$$d(s|t) = d'(s) / P(t)$$

Also called **default intensity or hazard rate**.

- In our example for the Caa rating in the 3rd year:

$d(3|2) = d'(3) / p(2) = 9{,}041\%/(100\%-27{,}867\%) = 9{,}041\%/72{,}133\% = 12{,}53\%$

PDs

- *C*umulative default frequencies are the sum of unconditional marginal default frequencies.

- However, cumulative default frequencies can also be calculated as is 1 - the joint (cumulative) probability of surving until *i-1* and the probability of surving in *i*:

$$D_i = 1 - (1 - d_i)(1 - D_{i-1})$$

DEFAULT INTENSITY

- The conditional marginal default probability to the rating Caa previously calculated (12,53%) was for a 1-year period.

- If one considers a very short period of time Δt, denoting the hazard rate at *t* by *λ(t)*, the **probability of default between *t* and *t + Δt* conditional on no previous default (until *t*) is $\lambda(t) \times \Delta t$.**

- Many models of PDs are based on the notion of the arrival intensity of default.

DEFAULT INTENSITY

- The simplest version of such a model defines **<u>default as the 1st arrival time τ</u>** <u>of a Poisson process with some constant mean arrival rate – average default intensity or hazard rate (λ)</u>:

$p(t) = e^{-\lambda t}$ - probability of survival for *t* years (to be shown afterwards)

$1/\lambda$ - expected time to default

$\lambda(t)\Delta t$ – default intensity in t over a small period of length Δ (between *t* and $t+\Delta t$), given survival until *t*.

- Example: default intensity (λ) = 0.04 =>

=> 1-year PD ($1-p(1)$) = $1-e^{-0.04x1}$ =3,9% => expected time to default ($1/\lambda$) = 1/0.04 = 25 (years).

DEFAULT INTENSITY

- As it was shown before, d'(s) = P(t) x d(*s*|t*)* ⇔ d(*s*|*t*) =d'(s) / P(t).
- For a very short period of time Δt, this result comes:

d(t+Δt|*t*) = d'(t+Δt)/P(t) = [P(t) – P(t+Δt)]/P(t)

- As the conditional marginal probability of default for a very short period of time is *λ(t)*Δt, we have:

[P(t) – P(t+Δt)]/P(t) = *λ(t)*Δt ⇔ [P(t+Δt) - P(t)] = -*λ(t)* P(t) Δt

- Taking limits:

dP(t)/dt = -*λ(t)* P(t) => P(t) = $e^{-\int_0^t \lambda(\tau)d\tau}$ => D(t) = $1 - e^{-\int_0^t \lambda(\tau)d\tau}$ = $1 - e^{-\bar{\lambda}(t)t}$

where $\bar{\lambda}(t)$ is the average hazard rate between time 0 and time t.

Default Intensity

- Default time ⇔ 1st time that a coin toss results in "heads," given independent tosses of coins, one each period, with each toss having a probability λ of heads and $1-\lambda$ of tails ⇔ default is unpredictable ⇔ when default does occur, it is a "surprise." ⇔ default time is inaccessible.

The following assumption describes the way in which default arrival risk is modelled in all intensity-based default risk models:

Assumption 5.1 (intensity model default arrivals) *Let $N(t)$ be a counting process[1] with (possibly stochastic) intensity $\lambda(t)$. The time of default τ is the time of the first jump of N, i.e.*

$$\tau = \inf\{t \in \mathbb{R}_+ \mid N(t) > 0\}. \tag{5.1}$$

The survival probabilities in this setup are given by:

$$P(0, T) = \mathbf{P}[N(T) = 0 \mid \mathcal{F}_0]. \tag{5.2}$$

Poisson Processes

A Poisson process $N(t)$ is an increasing process in the integers $0, 1, 2, 3, \ldots$. More important than its unexciting set of values are the *times of the jumps* $\tau_1, \tau_2, \tau_3, \ldots$ and the probability of a jump in the next instant.

We assume that the probability of a jump in the next small time interval Δt is proportional to Δt:

$$\mathbf{P}[N(t+\Delta t) - N(t) = 1] = \lambda \Delta t, \quad (5.3)$$

Probability of default in a small period of time Δt

that jumps by more than 1 do not occur, and that jumps in disjoint time intervals happen independently of each other. This means, conversely, that the probability of the process remaining constant is

$$\mathbf{P}[N(t+\Delta t) - N(t) = 0] = 1 - \lambda \Delta t,$$

Probability of survival in a small period of time Δt

There is only 1 default, i.e. the default is an absorbing state.

Poisson Processes

Probability of survival in 2 small periods is the joint probability of default in each of them (given that the hazard rate is the same for all periods of the same magnitude)

and over the interval $[t, 2\Delta t]$ this probability is

$$\mathbf{P}[N(t+2\Delta t) - N(t) = 0] \\ = \mathbf{P}[N(t+\Delta t) - N(t) = 0] \cdot \mathbf{P}[N(t+2\Delta t) - N(t+\Delta t) = 0] = (1 - \lambda\Delta t)^2.$$

Now we can start to construct a Poisson process. We subdivide the interval $[t, T]$ into n subintervals of length $\Delta t = (T-t)/n$. In each of these subintervals the process N has a jump with probability $\Delta t\lambda$. We conduct n independent binomial experiments each with a probability of $\Delta t\lambda$ for a "jump" outcome.

The probability of *no* jump at all in $[t, T]$ is given by:

$$\mathbf{P}[N(T) = N(t)] = (1 - \Delta t\lambda)^n = \left(1 - \frac{1}{n}(T-t)\lambda\right)^n.$$

Δt

Probability of no jumps in the *n* periods

POISSON PROCESSES

$$\mathbf{P}[N(T) = N(t)] = (1 - \Delta t\lambda)^n = \left(1 - \frac{1}{n}(T - t)\lambda\right)^n$$

x

Because $(1 + x/n)^n \to e^x$ as $n \to \infty$, this converges to:

$$\mathbf{P}[N(T) = N(t)] \to \exp\{-(T - t)\lambda\}$$

The Poisson process

POISSON PROCESSES

Next we look at the probability of exactly *one* jump in $[t, T]$. There are n possibilities of having exactly one jump, giving a total probability of

Probability of a jump

Probability of no jumps in n-1 periods

$$
\begin{aligned}
\mathbf{P}[N(T) - N(t) = 1] &= n \cdot \Delta t \lambda (1 - \Delta t \lambda)^{n-1} \\
&= n \cdot \frac{(T - t)}{n} \lambda \left(1 - \frac{1}{n}(T - t)\lambda\right)^n \Big/ \left(1 - \frac{1}{n}(T - t)\lambda\right) \\
&= \frac{(T - t)\lambda}{1 - \frac{1}{n}(T - t)\lambda} \left(1 - \frac{1}{n}(T - t)\lambda\right)^n \\
&\to (T - t)\lambda \exp\{-(T - t)\lambda\} \quad \text{as } n \to \infty,
\end{aligned}
$$

Δt

Probability of exactly one jump in [t,T]

POISSON PROCESSES

- For 2 jumps, there will be n/2 chances => probability of having 2 jumps:

$$\mathbf{P}[N(T) - N(t) = 2] = \frac{1}{2}(T - t)^2 \lambda^2 \exp\{-(T - t)\lambda\}$$

- Probability of *n* jumps:

$$\mathbf{P}[N(T) - N(t) = n] = \frac{1}{n!}(T - t)^n \lambda^n \exp\{-(T - t)\lambda\}$$

- When a Poisson process with constant intensity λ is used, the term structure of spreads will be flat and constant over time.

Stochastic dynamics in the credit spreads are necessary for several reasons. We need them if we want to price credit derivatives whose payoff is directly affected by volatility (e.g. credit spread options), if the credit derivative has a payoff which might be correlated with the spread movements (e.g. an option on the currency of an emerging market), and in general if we want to have a model which enables us to measure, manage and hedge this type of risk.

These stochastic intensity dynamics can be modelled with a generalisation of the Poisson process $N(t)$, the *Cox process*.

VARIABLE DEFAULT INTENSITY

Roughly speaking, Cox processes are Poisson processes with stochastic intensity.

$$\mathbf{P}[N(t+\Delta t) - N(t) = 1] = \lambda(t)dt$$

Now λ is time-varying

- If λ changes over time, with $\lambda(1)$ and $\lambda(2)$ known beforehand => the cumulative probability of survival for 2 years is:

$$p(2) = p(1)p(2\,|\,1) = e^{-\lambda(1)}e^{-\lambda(2)} = e^{-[\lambda(1)+\lambda(2)]}$$

- Carrying out the same calculation over t years, recursively, the cumulative probability of survival for t years is:

$$p(t) = e^{-[\lambda(1)+\cdots+\lambda(t)]}$$

VARIABLE DEFAULT INTENSITY

- With deterministic continual variation in default intensity, we get:

$$p(t) = e^{-\int_0^t \lambda(t)\, dt}$$

- Deterministic variation in intensity implies that the only information relevant to default risk that arrives over time is the mere fact of survival to date.
- However, in reality, as time passes, one should have new information, beyond simply survival, that would bear on the credit quality of an issuer.

↓

- The default intensity would generally vary at random as this additional information arrives.

VARIABLE DEFAULT INTENSITY

- For example, one may assume that the intensity varies with an underlying state variable (driver), such as the credit rating, distance to default, equity price, or the business cycle.
- If intensities are updated with new information at the beginning of each year and are constant during the year => Probability of survival to time *t* given survival to *t* − 1, and given all other information available at time *t* − 1:

$$p(t-1, t) = e^{-\lambda(t)}$$

- Survival probability in the 2-year when default intensity in the 2nd year ($\lambda(2)$), assuming the firm survives the first and takes 2 possible levels, $\lambda(2,H)$ and $\lambda(2, L)$, with conditional probabilities q and $1 - q$:

$$qe^{-\lambda(2,H)} + (1-q)e^{\lambda(2,L)} = E(e^{-\lambda(2)})$$

DEFAULTABLE ZERO COUPON BONDS

The implied survival probability from t to $T \geq t$ as seen from time t is the ratio of the defaultable to the default-free ZCB prices:

$$P(t,T) = \frac{\overline{B}(t,T)}{B(t,T)}$$

Zero Coupon Defaultable bond (with zero recovery rate)

Zero Coupon Risk-free bond → $B(t,T) = \mathbf{E}[e^{-\int_t^T r(s)ds} \cdot 1]$

$$\text{Payoff} = \mathbf{1}_{\{\tau > T\}} = \begin{cases} 1 & \text{if default after } T\text{, i.e. } \tau > T, \\ 0 & \text{if default before } T\text{, i.e. } \tau \leq T \end{cases}$$

For the Zero Coupon Defaultable bond, the pay-off will be 1 only if the debtor is still alive at *T*.

$$\overline{B}(t,T) = \mathbf{E}[e^{-\int_t^T r(s)ds} \cdot I(T)]$$

$$\overline{B}(t,T) = \mathbf{E}[e^{-\int_t^T r(s)ds} \cdot I(T)] = \mathbf{E}[e^{-\int_t^T r(s)ds}]\mathbf{E}[I(T)]$$
$$= B(t,T)\mathbf{E}[I(T)] = B(t,T)P(t,T),$$

DEFAULTABLE ZERO COUPON BONDS

- If the time of default is the time of the 1st jump of a Poisson process N(t) and it's independent from the default-free interest rate, the price of a defaultable bond with zero recovery becomes:

$$\overline{B}(0,T) = \mathbf{E}\left[e^{-\int_0^T r(s)ds}\right]\mathbf{E}\left[\mathbf{1}_{\{N(T)=0\}}\right],$$
$$\overline{B}(0,T) = B(0,T)e^{-\int_0^T \lambda(s)ds}.$$

The price of a defaultable bond is determined by the risk-free interest rate and the probability of paying the redemption value.

$$\overline{B}(t,T) = B(t,T)e^{-\int_t^T \lambda(s)ds}.$$

$$\overline{B}(0,T) = \mathbf{E}\left[e^{-\int_0^T r(s)+\lambda(s)ds}\right]$$

The defaultable bond price corresponds to the NPV of the cash-flows of the corresponding risk-free bond, using as discount rate the yield of the defaultable bond.

CREDIT DERIVATIVES

Traditionally, a bank could only manage its credit risks at origination. Once the risk was originated, it remained on the books until the loan was paid off or the obligor defaulted. There was no efficient and standardised way to transfer this risk to another party, to buy or sell protection, or to optimise the risk–return profile of the portfolio. Consequently, the pricing of credit risks was in its infancy, spreads on loans only had to be determined at origination and were often determined by non-credit considerations such as the hope of cross-selling additional business in the corporate finance sector. There was no need to become more efficient because the absence of a transparent market meant that the mode of operation was more like an oligopoly than an efficient competition. Whether a loan was mispriced or not was impossible to determine with certainty, it all depended on the individual subjective assessment of the obligor's default risk. The main "cost" of extending a loan was the cost of the regulatory risk capital as prescribed by the rules of the Basel I capital accord, and this is the point where credit derivatives came in.

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

CREDIT DERIVATIVES

Definition:

(a) A credit derivative is a derivative security that is primarily used to transfer, hedge or manage credit risk.

(b) A credit derivative is a derivative security whose payoff is materially affected by credit risk.

Narrower definition:

- A **credit derivative** is a derivative security that has a payoff which is conditioned on the occurrence of a **credit event**.

➡ **We need to define what are credit events.**

Credit Events

- bankruptcy
- failure to pay
- obligation default
- obligation acceleration
- repudiation/moratorium
- restructuring
- ratings downgrade below given threshold
- changes in the credit spread

Standardized by ISDA (International Swap Dealers Association), even though they may also be freely negotiated.

- The credit event is defined with respect to a **reference credit**, and the **reference credit asset(s)** issued by the reference credit.

Reference Credit: Firm, institution or person who may default.

Reference Credit Assets

- loans
  - floating or fixed rate
  - may include optionality (interest rate caps, credit facilities)
  - not traded, thus recovery rate may be hard to determine
- bonds
  - fixed–coupon or floater
  - zero coupon
  - convertible
- counterparty risk

MARKET TERMINOLOGY

- Credit derivatives can be defined on single-name or multi-name.
- The most popular single-name credit derivative is the CDS.

  - Buying a credit derivative typically means **buying credit protection**, which is economically equivalent to **shorting the credit risk**.
  - So **selling** credit protection means going **long** the credit risk.
  - Alternatively, one may speak of protection buyers/sellers as the payers/receivers of the premium.

USE OF CREDIT DERIVATIVES

- Traditional uses of derivatives: hedging, speculation, arbitrage
- Reduction of regulatory capital — this in particular applies to synthetic securitisations
- An unfunded way to diversify revenue

Examples

- Asset Swaps
- Total Return Swaps
- Credit Default Swaps
- Exotic Credit Derivatives
- Default Digital Swaps

ASSET SWAPS

An *asset swap package* is a combination of a defaultable bond (the asset) with an interest-rate swap contract that swaps the coupon of the bond into a payoff stream of Libor plus a spread. This spread is chosen such that the value of the whole package is the par value of the defaultable bond. Usually, the bond is a fixed-coupon bond and the interest-rate swap a fixed-for-floating interest-rate swap.

Example 2.2 *The payoffs of the asset swap package are as follows.* **A** *sells to* **B** *for 1 (the notional value of the* **C***-bond):*

- *A fixed coupon bond issued by* **C** *with coupon* $\bar{c}$ *payable at coupon dates* $T_i, \ i = 1, \ldots, N$*;*
- *A fixed-for-floating swap (as below).*

As payments of the swap the following payments are made: at each coupon date $T_i, \ i \leq N$ *of the bond*

- **B** *pays to* **A***:* $\bar{c}$*, the amount of the fixed coupon of the bond;*
- **A** *pays to* **B***: Libor* $+ s^A$.

s^A *is called the* asset swap spread *and is adjusted to ensure that the asset swap package has indeed the initial value of 1.*

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

ASSET SWAPS

The asset swap is not a credit derivative in the strict sense, because the swap is unaffected by any credit events, so it is only a portfolio of a bond and a swap contract. Its main purpose is to transform the pre-default payoff streams of different defaultable bonds into the same form: *Libor + asset swap spread*. **B** still bears the full default risk and if a default should happen, the swap will still have to be serviced or to be unwound at market value.

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

- Asset swaps also serve frequently as underlying assets for other derivatives, e.g. options on asset swaps (**swaptions**) – gives the holder the right to enter an asset swap package at some future date T at a predetermined asset swap spread s^A.

Total Return Swaps

The aim is to swap the actual return of a defaultable bond into a cashflow of LIBOR plus a spread

In a *total return swap* (**TRS**) (or *total rate of return swap*) **A** and **B** agree to *exchange all cash flows* that arise from two different investments. Usually one of these two investments is a defaultable investment, and the other is a default-free Libor investment. This structure allows an exchange of the assets' payoff profiles without legally transferring ownership of the assets.

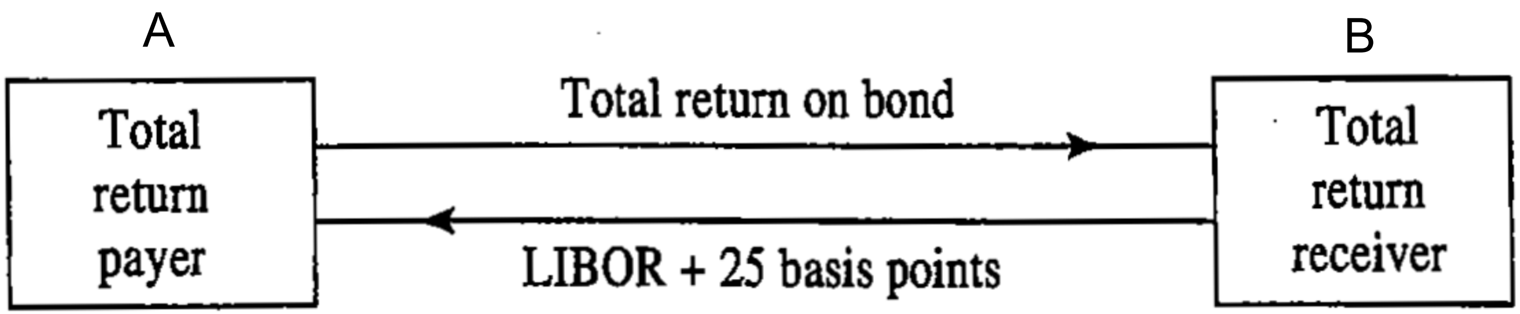
The payoffs of a total rate of return swap are as follows. Counterparty **A** pays to counterparty **B** at regular payment dates T_i, $i \leq N$

- The coupon $\bar{c}$ of the bond issued by **C** (if there was one since the last payment date T_{i-1});
- The price appreciation $(\overline{C}(T_i) - \overline{C}(T_{i-1}))^+$ of bond **C** since the last payment;
- The principal repayment of bond **C** (at the final payment date);
- The recovery value of the bond (if there was a default).

→ A pays while the bond price increases (afterwards, with the pull-to-par, it will receive again)

B pays at the same intervals:

- A regular fee of Libor $+ s^{TRS}$;
- The price depreciation $(\overline{C}(T_{i-1}) - \overline{C}(T_i))^+$ of bond **C** since the last payment (if there was any);
- The par value of the bond (if there was a default in the meantime).



Total Return Swaps

B has almost the same payoff stream as if he had invested in the bond **C** directly and funded this investment at $Libor + s^{TRS}$. The only difference is that the total rate of return swap is marked to market at regular intervals. Price changes in the bond **C** become cash flows for the TRS immediately, while for a direct investment in the bond they would only become cash flows when the bond matures or the position is unwound. This makes the TRS similar to a futures contract on the **C**-bond, while the direct investment is more similar to a forward. The TRS is not exactly equivalent to a futures contract because it is marked to market using the *spot* price of the underlying security, and not the *futures* price. The resulting price difference can be adjusted using the spread s^{TRS} on the floating payment of **B**.

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

TOTAL RETURN SWAPS

Advantages:

- Counterparty **B** is long the reference asset without having to fund the investment up front. This allows counterparty **B** to leverage his position much higher than he would otherwise be able to. Usually, depending on his credit quality, **B** will have to post collateral, though.
- If the reference asset is a loan and **B** is not a bank then this may be the only way in which **B** can invest in the reference asset.
- Counterparty **A** has hedged his exposure to the reference credit if he owns the reference asset (but he still retains some counterparty risk).
- The transaction can be effected without the consent or knowledge of the reference credit **C**. **A** is still the lender to **C** and keeps the bank–customer relationship.
- If **A** does not own the reference asset he has created a short position in the asset. Because of its long maturity, a short position with a TRS is less vulnerable to short squeezes than a short repo position. Furthermore, directly shorting defaultable bonds or loans is often impossible.

Fundamentally, a TRS can be viewed as a synthetic form of funding the investment into the **C**-bond, where the **C**-bond is used as collateral. Thus, the TRS spread s^{TRS} should not only be driven by the default risk of the underlying asset but also by the credit quality of **B** as a counterparty.

If there is only one payment date, the TRS is equivalent to a forward contract on the **C**-bond.

CREDIT DEFAULT SWAPS

The aim is to transfer **ONLY** the default risk from A to B.

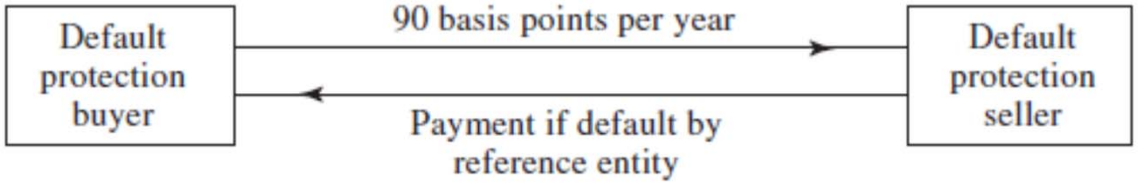
The protection seller **B** agrees to pay the default payment

$$\text{notional} \times (1 - \text{recovery rate})$$

to **A** if a default has happened.

For this, **A** pays a periodic fee $\overline{s}$ to **B** (until maturity of the CDS or until default, whichever comes first)

CREDIT DEFAULT SWAPS



Source: Hull, John (2015), “Options, futures and other derivatives”, Pearson.

The total return swap achieves the goal of a transferral of the **C**-risk from **A** to **B**, but it has some disadvantages. The default risk of **C** is not isolated but mixed up with the market risk of the reference asset. Furthermore, basis risk may remain as the default risk of the obligor **C** is not transferred, only the risk of one bond issued by **C**. The obligor might not default on the reference asset of the TRS but on other obligations. A credit default swap on the other hand enables the investors to isolate the default risk of the obligor. The basic structure is as follows.

In a single-name *credit default swap* (**CDS**) (also known as a *credit swap*) **B** agrees to pay the default payment to **A** *if a default has happened*. The default payment is structured to replace the loss that a typical lender would incur upon a credit event of the reference entity. If there is no default of the reference security until the maturity of the default swap, counterparty **B** pays nothing.

Source: Schonbucher, Philipp J. (2003), “Credit Derivatives Pricing Models – Models, Pricing and Implementation”, Wiley.

CREDIT DEFAULT SWAPS

A pays a fee for the default protection. The fee can be either a regular fee at intervals until default or maturity (the most common version, we speak of a default *swap*) or a lump-sum fee up front (less common, a default *put*). If a default occurs between two fee payment dates, **A** still has to pay the fraction of the next fee payment that has accrued until the time of default (Table 2.2).

Table 2.2 Payoff streams of a credit default swap to protection seller **B** (the payoffs to the protection buyer **A** are the converse of these). Payoffs marked with an asterisk cease at default

Time	Defaultable bond	CDS
$t = 0$	$-\overline{C}(0)$	0
$t = T_i$	$\overline{c}^*$	$+\overline{s}^*$
$t = T_N$ (Maturity date)	$(1 + \overline{c})^*$	$+\overline{s}^*$
Default	Recovery	−(1 − Recovery) (minus the default payment)

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

CREDIT DEFAULT SWAPS

Default swaps can differ in the specification of the default payment. Possible alternatives are:

- Physical delivery of one or several of the reference assets against repayment at par;
- Notional minus post-default market value[3] of the reference asset (cash settlement);
- A pre-agreed fixed payoff, irrespective of the recovery rate (default digital swap).

An example of a default swap with a fixed repayment at default was given in Example 2.1 (default digital swap on Brazil) but the fixed payment at default is a less common specification.

Example 2.1 *Default digital swap on the United States of Brazil. Counterparty **B** (the insurer) agrees to pay USD 1m to counterparty **A** if and when Brazil misses a coupon or principal payment on one of its Eurobonds. Here:*

- *The* reference credit *is the United States of Brazil;*
- *The* reference credit assets *are the Eurobonds issued by Brazil (in the credit derivative contract there would be an explicit list of these bonds);*
- *The* credit event *is a missed coupon or principal payment on one of the reference assets;*
- *The* default payment *is USD 1m.*

*In return for this, counterparty **A** pays a fee to **B**.*

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

CREDIT DEFAULT SWAPS

The vast majority of credit default swaps specify physical delivery of one of the underlying securities against a payment of its par value as default payment. Sometimes substitute securities may be delivered, or a payoff that depends on other market variables may be specified (e.g. to hedge counterparty exposure in derivatives transactions).

To identify a credit default swap, the following information has to be provided:

1. The reference obligor and his reference assets;
2. The definition of the credit event that is to be insured (default definition);
3. The notional of the CDS;
4. The start of the CDS, the start of the protection;
5. The maturity date;
6. The credit default swap spread;
7. The frequency and day count convention for the spread payments;
8. The payment at the credit event and its settlement.

Source: Schonbucher, Philipp J. (2003), “Credit Derivatives Pricing Models – Models, Pricing and Implementation”, Wiley.

CREDIT DEFAULT SWAPS

The event that is to be insured against is a default of the reference obligor, but because of the large payments involved the definition of what constitutes a default has to be made more precise, and a mechanism for the determination of the default event must be given. The standard definition of default includes:

- bankruptcy, filing for protection,
- failure to pay,
- obligation default, obligation acceleration,
- repudiation/moratorium,
- restructuring.[4]

There is a debate whether restructuring should be included as a default event in the specification or not, and some market makers even quote different prices for CDSs with and without restructuring in the default definition. Sometimes (in particular in default definitions for CDOs), a slightly different default definition is used which is based upon rating agencies' definitions of default. Despite the growing standardisation of the default definition, one advantage of a CDS is that both parties can agree to an event definition that can be completely different from the standard ISDA specification.

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

CREDIT DEFAULT SWAPS

Example 2.3 *Credit default swap on Daimler Chrysler.*

The trade

*At time $t = 0$, **A** and **B** enter a credit default swap on Daimler Chrysler, **A** as protection buyer and **B** as protection seller. They have agreed on:*

(i) The reference credit: Daimler Chrysler AG.
(ii) The term of the credit default swap: 5 years.
(iii) The notional of the credit default swap: 20m USD.
(iv) The credit default swap fee: $\bar{s} = 116$ bp.

*The credit default swap fee $\bar{s} = 116$ bp is quoted per annum as a fraction of the notional. **A** pays the fee in regular intervals, semi-annually. To make our life easier, we simplify the day count fractions to 1/2 such that **A** pays to **B**:*

$$116\ bp \times 20m/2 = 116\,000\ USD \quad at \quad T_1 = 0.5,\ T_2 = 1, \ldots, T_{10} = 5$$

These payments are stopped and the CDS is unwound as soon as a default of Daimler Chrysler occurs.

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

CREDIT DEFAULT SWAPS

Because the payments are done each semester

The default payment

*First, **A** pays the remaining accrued fee. If the default occurred two months after the last fee payment, **A** will pay 116 000 × 2/6. The next step is the determination of the default payment. If physical settlement has been agreed upon, **A** will deliver Daimler Chrysler bonds to **B** with a total notional of USD 20m (the notional of the CDS). The set of deliverable obligations has been specified in the documentation of the CDS. As liquidity in defaulted securities can be very low, this set usually contains more than one bond issue by the reference credit. Naturally **A** will choose to deliver the bond with the lowest market value, unless he has an underlying position of his own that he needs to unwind. (Even then he may prefer to sell his position in the market and buy the cheaper bonds to deliver them to **B**.) This* delivery option *enhances the value of his default protection. **B** must pay the full notional for these bonds, i.e. USD 20m in our example.*

If cash settlement has been agreed upon, a robust procedure is necessary to determine the market value of the bonds after default. If there were no liquidity problems, it would be

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

Credit Default Swaps

sufficient to ask a dealer to give a price for these bonds, and use that price, but liquidity and manipulation are a very real concern in the market for distressed securities. Therefore not one, but several, dealers are asked to provide quotes, and an average is taken after eliminating the highest and lowest quotes. This is repeated, sometimes several times, in order to eliminate the influence of temporary liquidity holes. Thus the price of the defaulted bonds is determined, e.g. 430 USD for a bond of 1000 USD notional. Now, the protection seller pays the difference between this price and the par value for a notional of 20m USD, i.e.

$$(1000 - 430)/1000 \times 20m\ USD = 11.4m\ USD$$

Because the price determination in cash settlement is so involved, most credit default swaps specify physical delivery in default. Cash settlement is only chosen when there may not be any physical assets to deliver (i.e. the reference entity has not issued enough bonds) or if the CDS is embedded in another structure where physical delivery would be inconvenient, e.g. a credit-linked note.

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

Valuation

- Maturity = 3 years;
- Notional = € 100.000
- Payment in case of default = 70% of the notional
- *Risk-neutral* marginal probability of default in t, given that it didn't default in t-1:

Period (t)	Probability (λ)
1	0.59%
2	1.00%
3	1.27%

Valuation

- Spot rates for maturity t:

Maturity (t)	Spot Rate s(t)	Discount Factor F(t)
1	4.0%	0.9615
2	4.2%	0.9210
3	4.4%	0.8788

- In order to obtain the *swap premium*, it is necessary (as usual) to calculate the NPV of the future cash-flows, which will be done recursively from the last cash-flows.

Valuation

- E_2[*pay-off*(3)|ND] = *pay-off* in case of no default *(1-PD) + *pay-off* in default *PD

  =0 x (1– 0.0127) + 0.7 x 0.0127 = 0.00889

  PD(3)

- E_2[*pay-off*(3|D] = 0.7

- E_1[*pay-off*(3)|ND] = E_2[*pay-off*(3)|ND]*(1-PD)+ E_2[*pay-off*(3)|D] *PD= 0.00889 x (1 – 0.01) + 0.7 x 0.01 = 0.0158011

  PD(2)

- E_1[*pay-off*(3)|D] = 0.7

Valuation

- $E_0[X(3)]$ = E_1[*pay-off*(3)|ND]*(1-PD)+ E_1[*pay-off*(3)|D] *PD= 0.0158011 x (1 – 0.0059) + 0.7 x 0.0059 = 0.0198

- V(0,3) = F(0,3) $E_0[X(3)]$ *nocional = 0.8788 * 0.0198*100000 = €1740.

- If the premium is paid on an annual basis, we'll have:

  1740 = 0.9615 x p +0.9210 x p + 0.8788 x p

  p = 630.14.

First to default Swaps

A first-to-default swap (FtD) is the extension of a credit default swap to portfolio credit risk. Its key characteristics are the following:

- Instead of referencing just a single reference credit, an FtD is specified with respect to a **basket of reference credits** $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_I$.
- The set of reference credit assets (the assets that can trigger default events) contains assets by all reference credits.
- The protection buyer **A** pays a regular fee of $\bar{s}^{\text{FtD}}$ to the protection seller **B** until the default event occurs or the FtD matures.
- The default event is the **first default** of any of the reference credits.
- The FtD is terminated after the first default event.
- The default payment is "1 − recovery" on the defaulted obligor. If physical delivery is specified, the set of deliverable obligations contains only obligations of the defaulted reference credit.

The basket of a FtD tipically comprises 4 to 12 reference credits.

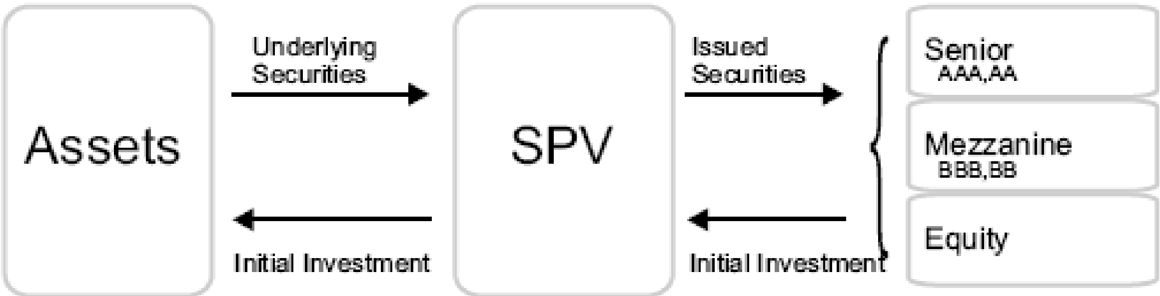
A natural extension of the first-to-default concept is the introduction of second-to-default (StD) and nth-to-default (ntD) basket credit derivatives. Such credit derivatives only differ in the specification of the default event, the basic structure remains the same. While FtD credit derivatives are a common structure, second- and higher-order ntD structures are rarer.

COLLATERALIZED BOND OBLIGATIONS

- underlying portfolio of defaultable bonds
- the portfolio is transferred to an SPV
- the SPV issues notes
  - an equity (or first loss) tranche
  - several mezzanine tranches
  - a senior tranche

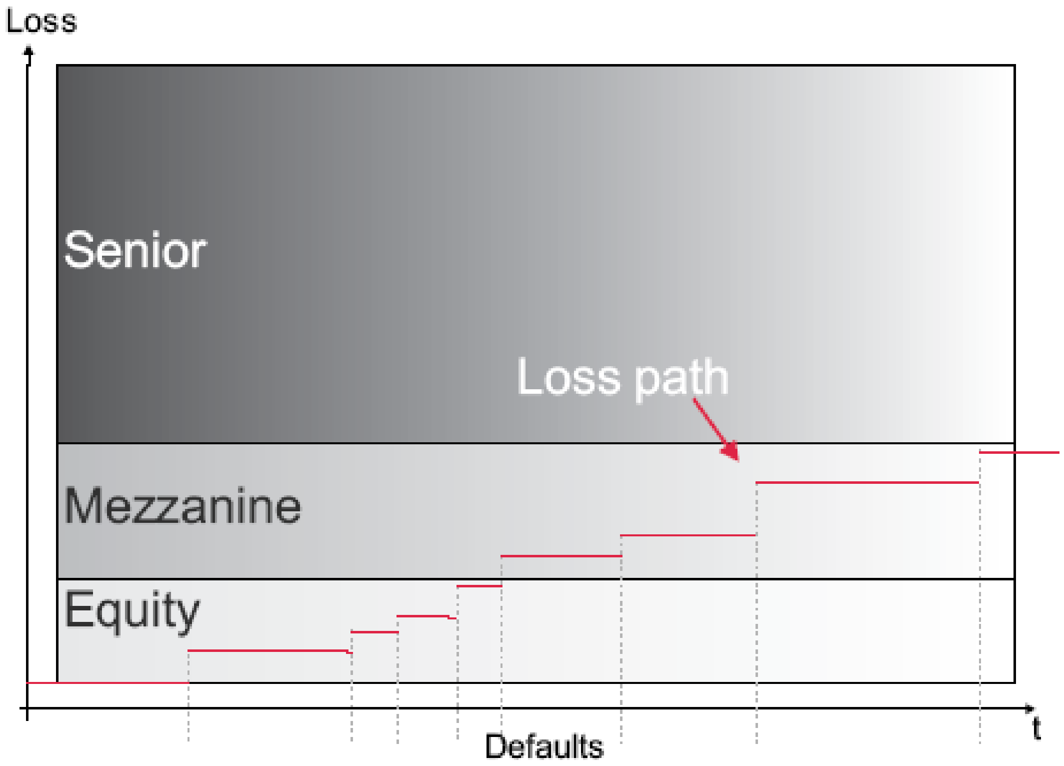
These notes are collateralized by the bonds sold to the SPV

Similar to RMBS but with bonds instead of residential mortgage loans



- if during the life of the CBO one of the bonds defaults, the recovery payments are reinvested in default–free securities

- at maturity of the CBO, the portfolio is liquidated and the proceeds distributed to the tranches, according to their seniority ranking



illustration

In this case, no losses will be suffered by the senior bonds, while equity bonds will get a total loss.

COLLATERALIZED DEBT OBLIGATIONS

- Designed exactly in the same way as CBOs. The main difference is that the underlying assets can be defaultable bonds or any other credit related instruments.
- The most well-known CDOs are the so called synthetic CDOs – when the underlying defaultable bonds are replaced by credit derivatives, e.g.:
  - CLOs – when the underlying assets are loans.
  - CBOs - when the underlying assets are bonds.

In particular, even if the underlying portfolio is mostly unrated or speculative-grade, it is possible to enhance the credit rating of most of the notes to the high investment-grade ratings by concentrating the default risk in a small first loss layer. This enables investors to invest in these notes who otherwise would not be allowed to invest in the underlying assets themselves. The notes of a well-designed CDO can sometimes be sold for a cumulative price that is higher than the sum of the market value of the underlying assets.

CREDIT-LINKED NOTES

Credit-linked notes (CLNs) are a combination of a credit derivative with a medium-term note. The underlying note pays a coupon of Libor plus a spread and is issued by a high-quality issuer. The issuer of the note buys protection on the risk referenced in the credit derivative. In addition, having effectively sold protection on the underlying credit exposure, the investors also face the counterparty risk of the issuer.

Example 2.8 (Wal-Mart credit-linked note) *Issuer: JPMorgan, September 1996 (via an AAA trust). The buyers of the CLN receive:*

- *Coupon (fixed or floating);*
- *Principal if no default of reference credit (Wal-Mart) until maturity;*
- *Only the recovery rate on the reference obligation as final repayment if a default of reference credit occurs.*

The buyers of the note now have credit exposure to Wal-Mart which is largely equivalent to the direct purchase of a bond issued by Wal-Mart. They also have some residual exposure to the credit risk of the AAA-rated trust set up to manage the note. From JPMorgan's point of view the investors of the CLN have sold them a CDS and posted 100% collateral.

2 – Structural Models of Credit Risk

- The drawbacks of traditional credit risk models and rating updates by agencies in the recent past led to the development of new credit risk models, based on the prices of financial assets issued by the company.
- The rationale is that **market prices are the best assessment available on the companies' capital or debt value**.
- The first attempt to incorporate market prices in a credit risk model was done in the Z-Score model. Later, in 1974, Merton developed a corporate valuation approach based on financial options.

2 – STRUCTURAL MODELS OF CREDIT RISK

- **In structural models, the default time is determined endogenously by the behavior of the company's asset** ⇔ default occurs when the asset value falls so much that makes it impossible to ensure the debt service - **1st passage of assets to a default boundary**.

- Therefore, structural models are based on the company's asset values and use information from the stock market to measure the market value of own funds.

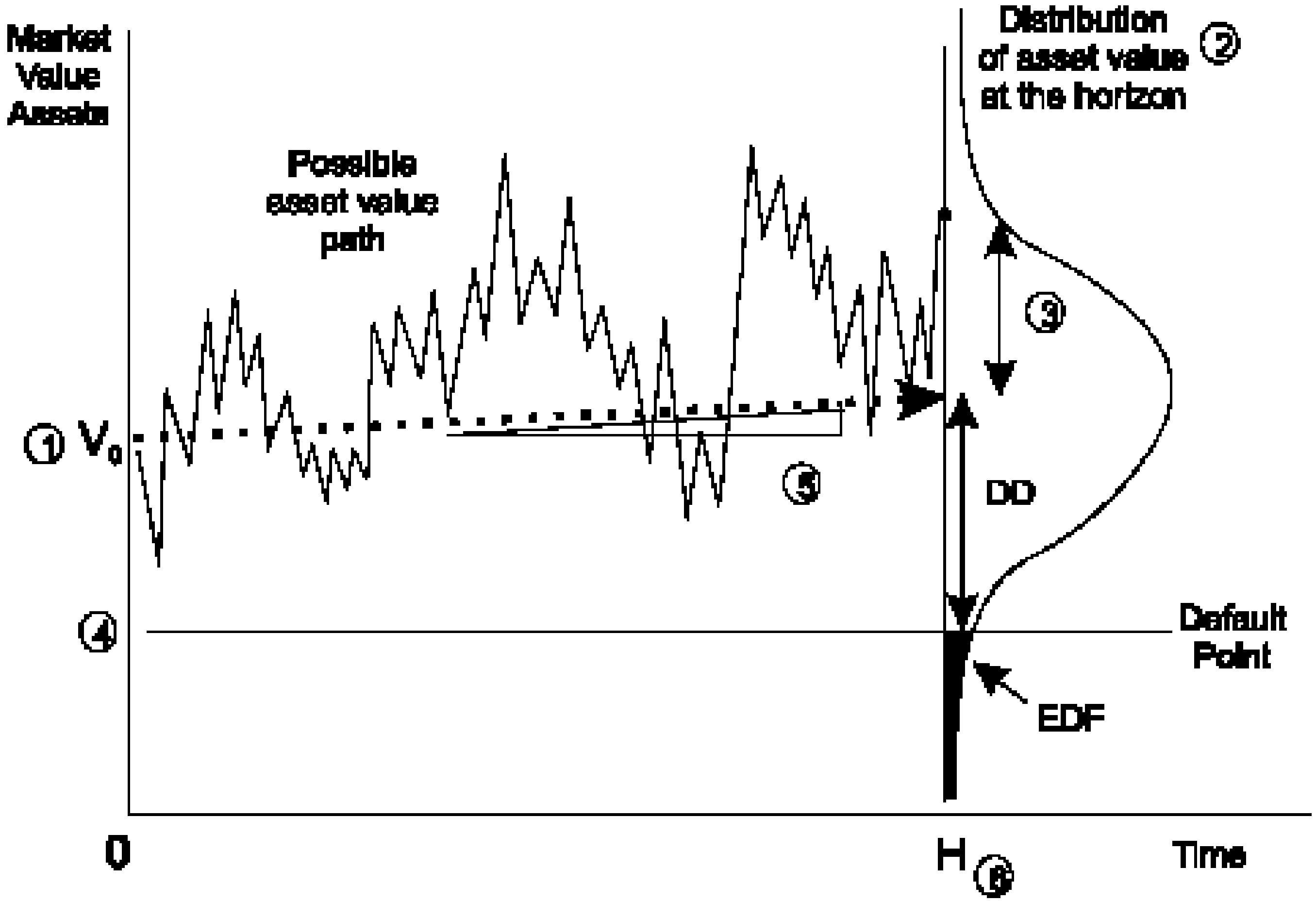
- The main problem with these models corresponds to the false alarms.

Merton Model

- The model is based on the assumption that, when the company issues debt, the shareholders transfer the control of the company to the creditors.
- However, they retain an option of recovering that control if the company reimburses the debt.
- Therefore, the value of capital may be seen as the price of a call-option on the company assets, with a strike equal to the debt value.
- If on the redemption date, the market value of assets is lower than the debt value, the shareholders don't exercise the call option, i.e. the debt is not repaid =>

  - PD= P[Market Value of Assets < Debt value].

Merton Model



Source: Crosbie and Bohn (2002), “Modeling Default Risk”, KMV.

Merton Model

- Consequently, if the call-option can be valued, **the PD will be obtained from the distribution function resulting from the stochastic process of the company's asset market value**.
- Assuming that the option is European and the asset market value may be taken as the price of non-paying dividend asset, **one can use the Black-Scholes formula and calculate the PD from the implied volatility of the company's asset value** and an estimate for the respective growth rate.
- Hence, the Merton model is based on the assumption of the growth rates of the company's market value of assets (V_A) being normally distributed:

$$dV_A = \mu V_A dt + \sigma_A V_A dz \Leftrightarrow \qquad dV_A / V_A = \mu dt + \sigma_A dz$$

where V_A is the company's market value of assets, μ *and* σ_A the respective trend and instantaneous volatility and *dz* is a Wiener process (random shocks normally distributed).

Merton Model

- Given that this is exactly the stochastic process of the underlying asset of an European option under the assumptions taken in the Black-Scholes pricing formula, the pricing formula for the European *call-option* on the company's market value of assets that corresponds to the stock price is:

$$V_E = V_A N(d1) - e^{-rT} X N(d2)$$

where

V_E is the market value of the company's own funds

N is the cumulative normal distribution function

r is the risk-free interest rate for the maturity T

X is the nominal value of the company's total debt payable in maturity T.

$$d1 = \frac{\ln\left(\frac{V_A}{X}\right) + \left(r + \frac{\sigma_A^2}{2}\right)T}{\sigma_A \sqrt{T}} \qquad d2 = d1 - \sigma_A \sqrt{T}.$$

Merton Model

- In the pricing formula, there are two unknowns - V_A and σ_{A}.
- Consequently, an additional equation is required, in order to determine the values for those two variables.
- This equation will result from the relationship between the volatility of assets and the volatility of capital:

(1) $$\sigma_E = \frac{V_A}{V_E} N(d1)\sigma_A$$ (from $\sigma_E = \frac{\partial V_E}{\partial V_A}\frac{V_A}{V_E}\sigma_A$)

- In Jarrow and Rudd (1983), it is shown that the stock volatility is a multiple of the volatility of the market value of assets:

(2) $$\sigma_E = \eta\sigma_A$$

Merton Model

- Given (1) and that one gets (from (1) and (2):

(3) $$\delta = \frac{\partial C(X)}{\partial V_A} = N(d_1)$$

(4) $$\frac{V_A}{V_E} N(d1)\sigma_A = \eta\sigma_A \Leftrightarrow$$

$$\Leftrightarrow \frac{V_A}{V_E}\delta = \eta \Leftrightarrow$$

$$\Leftrightarrow \delta = \frac{\eta}{V_A/V_E} \Leftrightarrow \delta = \frac{\sigma_E/\sigma_A}{V_A/V_E} \Leftrightarrow$$

$$\Leftrightarrow \delta = \frac{\sigma_E}{\sigma_A}\frac{V_E}{V_A} \Leftrightarrow V_A = \frac{\sigma_E}{\sigma_A}\frac{V_E}{\delta}$$

- Therefore, from inputs V_E, σ_E , X, r and T, the equation system including the option pricing formula and (4) allows to estimate V_A and σ_A .

Merton Model

- The PD is thus the probability of the market prices of assets falling below the nominal value of debt at the expiry date:

$$p_t = \Pr\left[V_A^t \le X_t \mid V_A^0 = V_A\right] = \Pr\left[\ln V_A^t \le \ln X_t \mid V_A^0 = V_A\right]$$

- Given that the market value of assets follows a log-normal distribution, one gets (with μ = expected asset returns):

$$\ln V_A^t = \ln V_A + \left(\mu - \frac{\sigma_A^2}{2}\right)t + \sigma_A \sqrt{t}\varepsilon$$

- Therefore, the PD is:

$$p_t = \Pr\left[\ln V_A + \left(\mu - \frac{\sigma_A^2}{2}\right)t + \sigma_A\sqrt{t}\varepsilon \le \ln X_t\right] = \Pr\left[-\frac{\ln\frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}} \ge \varepsilon\right] \Leftrightarrow p_t = N\left[-\frac{\ln\frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}}\right]$$

- Risk-neutral PD (μ = r): $p_t = N\left[-d_2\right]$

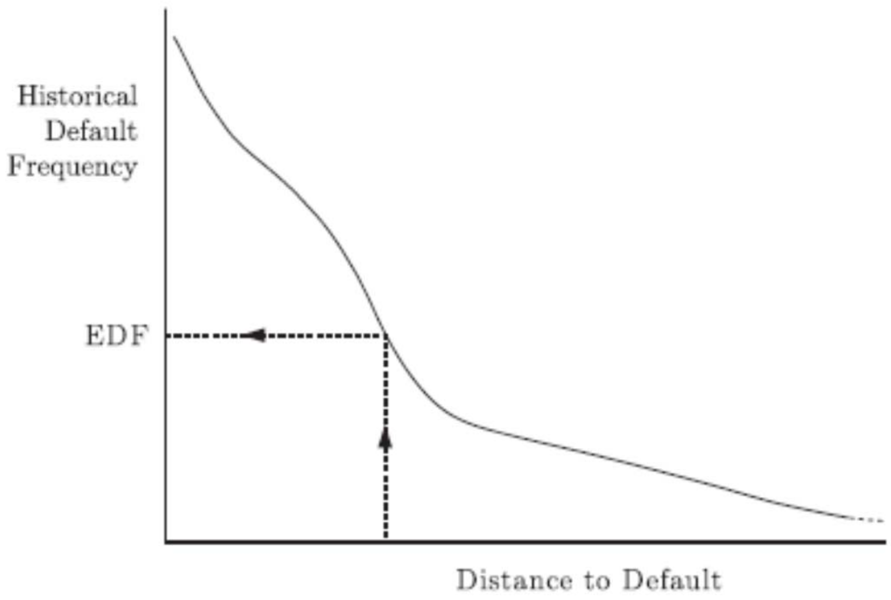
Merton Model

- Open issues:
  - How to obtain values for μ and σ_E?
  - How to deal with complex debt structures, with different maturities, seniority degrees and installments?
  - How to deal with the sensitivity of PDs to the leverage ratio?
  - How to solve the kurtosis problem in the market value of assets?
  - How to use the PD estimates as a leading indicator of rating changes?
- Estimation – non-linear least squares, minimizing the sum of the squared differences between the market value and the estimated value of the stocks (through the option pricing formula) and the assets.

Moody's KMV Model

- Moody's KMV overcomes the distribution problems by using a database of loans providing empirical PDs as a function of the distance-to-default measure.

$$DD = \left[\frac{\ln \frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2} \right) t}{\sigma_A \sqrt{t}} \right]$$



Source: Duffie, Darrell and Kenneth J. Singleton (2003), "Credit Risk", Princeton University Press.

Moody's KMV Model

- In this model, σ_A **is a linear combination of a modeled and an empirical volatility** (the latter weighting 70%, 80% for Financial Institutions).
- **Empirical vols -** calculated as the annualized standard deviation of the growth rates of the nominal value of assets, using 3 years of weekly observations for US companies (5 years of monthly data for European companies), excluding extreme values and adjusting for effects of M&A.
- **Modeled vols -** obtained from a regression between the observed vol and size, revenues, profitability, sector and region variables.

Moody's KMV Model

- For FIs, the PD is harder to estimate, given the diversity and uncertainty of the liabilities' maturities.
- On the other hand, by definition, banks are highly leveraged companies.

↓

- Moody's KMV proposes the *default point* (the value of the payable liabilities in the maturity considered) to be calculated as a % of the total liabilities, being that % differentiated according to the type of institution.

Moody's KMV Model

- In Chan-Lau and Sy (2006), it is proposed an adjustment to the Moody's KMV model, in order to accommodate the possibility of a bail-out.
- Consequently, the "*Distance-Risk measure*" concept is created, with L_t being the bank's liabilities (λ=1 => DR=DD) and PCAR the planned capital ratio:

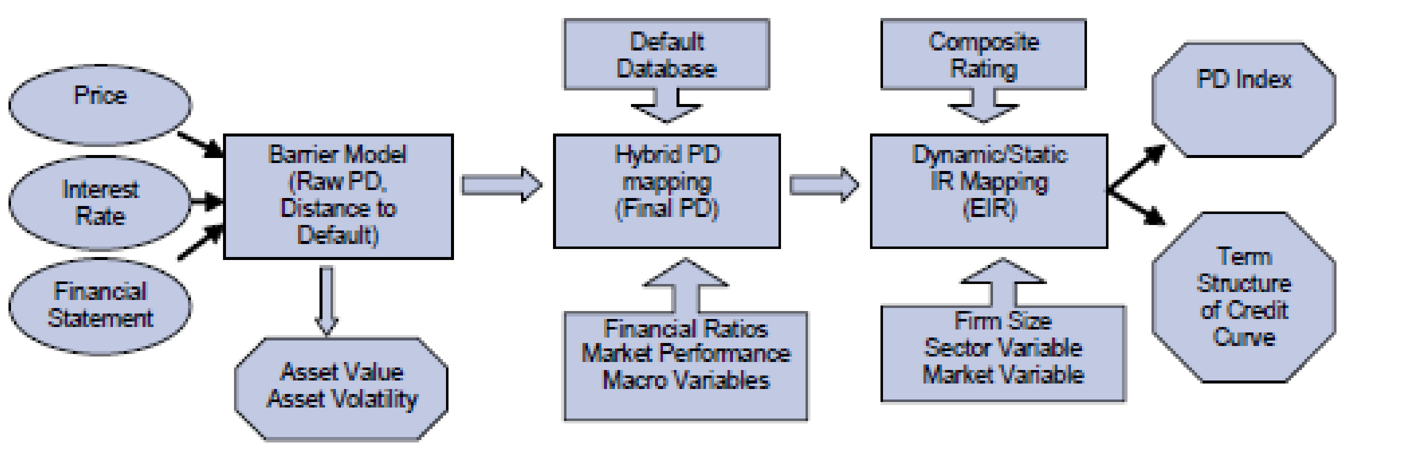
$$DR_T = \frac{\ln\left(\frac{V_t}{\lambda L_t}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \qquad \lambda = \frac{1}{1 - PCAR_i}$$

- With a very low PCAR, λ gets higher and the DR lower ⇔ with a lower capital target, the bank gets closer to default and can reach this stage at a lower level of liabilities.
- According to Oderda et al. (2002), Moody's KMV model anticipates defaults with a lead of around 15 months, but also produces false alarms in 88% of the cases.

Fitch EIR

- In order to smooth the excessive volatility of PDs obtained from equity prices, hybrid models were developed, being the PD obtained from corporate financial, market and macroeconomic information.
- One of these models was developed by Fitch, the **Equity Implied Rating (EIR),** relating the DD to a set of financial ratios and macroeconomic variables:

Figure 8: Modeling Step of EIR



Source: Fitch (2007).

Bondscore

- Another model is the Bondscore, developed by CreditSights:

$$p = -9.593 + 7.366X_1 - 3.989X_2 - 5.308X_3$$
$$-6.333X_4 - 2.501X_5 + 3.807X_6 + 5.469X_7$$

being:
X_1 = Total Liabilities/Market Value of Capital
X_2 = EBITDA/Sales
X_3 = Sales/ Total Assets
X_4 = Working Capital / Total Assets
X_5= log(Assets)
X_6= Vol of EBITDA/Sales
X_7= Vol of Market Value of Capital

3 – Reduced Form Models

- A structural model of credit risk provide a link between the prices of equity and debt instruments issued by a given firm.
- **A reduced-form model does not give any fundamental reason for the arrival of the defaults**, assuming that hazard rates for the different companies are stochastic processes correlated with macroeconomic variables.
- Given that *credit spreads* can be decomposed in default risk (PD, or λ) and recovery risk (LGD, or ϕ), the **PD can be modeled from the credit spreads and LGDs**.
- Taking several maturities, one can obtain a term structure of PDs.
- However, we must have in mind that spreads are not only a function of PDs and LGDs, but also of liquidity, taxation and risk premia charged by investors => **PDs implied by spreads are risk neutral**.

Credit spreads

- 2 equivalent ways to calculate the price of a risky zero coupon bond (assuming one-period maturity and redemption value of one monetary unit):

(i) Expected value of the future cash-flows, discounted at the risk-free rate:

$$P = \frac{E_0(X_1)}{1+r} = \frac{\lambda\phi + (1-\lambda)\cdot 1}{1+r}$$

(ii) Future cash-flows, discounted at the risk-free rate plus the credit spread:

$$P = \frac{1}{1+r+s}$$

Credit spreads

- Equalizing both expressions =>

$$s = \frac{\lambda(1-\phi)(1+r)}{1-\lambda(1-\phi)} \cong \lambda(1-\phi)$$

=> Credit spread:

- Increases with the probability of default λ;
- Decreases with the recovery rate ϕ;
- Increases with the risk-free rate r;
- In reality , these spreads may also be impacted by risk premium due to uncertainty about risk-free interest rates, PDs and LGDs.

Credit spreads

- This relationship can be generalized for any maturity:

(i) Expected value of the future cash-flows, discounted at the risk-free rate:

$$P_0 = \sum_{i=1}^{T} \frac{E_0(X_i)}{(1+r)^i} = \sum_{i=1}^{T} \frac{[\lambda\phi + (1-\lambda)]C_i}{(1+r)^i} + \frac{[\lambda\phi + (1-\lambda)]NV}{(1+r)^T}$$

(ii) Future cash-flows, discounted at the risk-free rate plus the credit spread:

$$P_0 = \sum_{i=1}^{T} \frac{E_0(X_i)}{(1+r+s)^i} = \sum_{i=1}^{T} \frac{C_i}{(1+r+s)^i} + \frac{NV}{(1+r+s)^T}$$

Credit spreads

- Consequently, the (risk-neutral) PD can be obtained by modeling the risk-free and the recovery rate, instead of the market value of the company's assets.
- From the spreads of similar bonds for different maturities, one can obtain the PD term structure, that can be compared to the statistics of rating agencies (the "true" PDs).
- The initial and most popular reduced form models were presented in Artzner ad Delbaen (1995), Jarrow and Turnbull (1995) and Duffie and Singleton (1995).

REDUCED FORM MODELS

Advantages:

✔ Credit spreads are directly modelled: the intensity of the default process *is* the credit spread.

✔ Can be fitted to observed credit spreads.

✔ Realistic credit spreads through discontinuous dynamics.

✔ Suitable for the pricing of credit derivatives.

Disadvantages:

X There is no explicit link to the company's fundamentals

REDUCED FORM MODELS

Features of specifications of default intensity and risk-free rate for derivatives pricing:

- Both r and λ are stochastic. Stochastic default-free interest rates are indispensable for fixed-income analysis, and a stochastic default intensity is required to reach stochastic credit spreads, necessary for meaningful prices for credit spread options and to capture spread change risks.
- The dynamics of r and λ are rich enough to allow for a realistic description of the real-world prices. Duffie and Singleton (1997) and Duffee (1999) come to the conclusion that in many cases a multifactor model for the credit spreads is necessary.
- There should be scope to include correlation between credit spreads and default-free interest rates.
- It is desirable to have processes for interest rates and credit spreads that remain positive at all times. Although negative credit spreads or interest rates represent an arbitrage opportunity, relaxing this requirement in favour of a Gaussian specification is still acceptable because of the analytical tractability that is gained. The Gaussian specification should then be viewed as a local approximation to the real-world dynamics rather than as a fully closed model. Furthermore, many important effects are more easily understood in the Gaussian setup.

REDUCED FORM MODELS

- **Zero recovery defaultable bond price:**

$$d_0(t, T) = E_t^*\left[e^{-\int_t^T (r(u)+\lambda^*(u))\, du}\right]$$

being λ^* the risk-neutral hazard rate

- **Risk-free interest and hazard rates depend on a set of macroeconomic variables (*X*(*t*)):**

$$r(t) = a_r(t) + b_r(t) \cdot X(t)$$

$$\lambda^*(t) = a_{\lambda^*}(t) + b_{\lambda^*}(t) \cdot X(t)$$

As both depend on X(*t*), the hazard rate becomes correlated with the interest rates.

REDUCED FORM MODELS

- From the equations in the previous slide, we get **prices for the defaultable and the risk-free bond**, respectively:

$$d_0(t, T) = e^{\alpha_d(t,T)+\beta_d(t,T)\cdot X(t)} \qquad \delta(t, T) = e^{\alpha_\delta(t,T)+\beta_\delta(t,T)\cdot X(t)}$$

- **Credit risk spread**:

$$s(t, T) = -\frac{\log d_0(t, T) - \log \delta(t, T)}{T - t} \Leftrightarrow s(t, T) = -\frac{\alpha_s(t, T) + \beta_s(t, T) \cdot X(t)}{T - t}$$

for α_s and β_s obtained by subtraction of the respective α's and β's

The introduction of stochastic default intensities allows us to capture an important risk component: the risk of a *change in the credit quality* of the obligor. This additional realism comes at the price of additional complexity in the model.

REDUCED FORM MODELS

- 3 alternative setups:

1. A two-factor *Gaussian short rate/intensity* setup. This setup suffers from the possibility of reaching negative credit spreads and interest rates with positive probability, but a high degree of analytical tractability is retained. The model can be calibrated to full term structures of bond prices and spreads, and the correlations between interest rates and default intensities are unconstrained.
2. A *multifactor Gaussian* setup. This setup is very similar to the two-factor Gaussian setup, but now the model is specified directly in terms of the initial term structures of defaultable and default-free zerobonds and their volatilities. This setup has a similar degree of tractability, and some of the formulae are even more intuitively accessible than the formulae of the Gaussian short-rate model.
3. A *multifactor Cox–Ingersoll–Ross (CIR)* (Cox *et al.*, 1985b) setup, following mainly Jamshidian (1996).[2] This model setup gives us the required properties of non-negativity and ease of calibration while still retaining a large degree of analytical tractability. Furthermore, models of credit spreads of the CIR square root type have been estimated by Duffie and Singleton (1997) and Duffee (1999). Unfortunately, the correlation between interest rates and intensities in this model is restricted to essentially non-negative correlations.

REDUCED FORM MODELS

- 2-factor Gaussian model:

(i) *The dynamics of the default-free short rate are given by the extended Vasicek (1977) model:*

$$dr(t) = (k(t) - ar)dt + \sigma(t)dW(t). \tag{7.1}$$

(ii) *Similarly, the dynamics of the default intensity λ are:*

$$d\lambda(t) = (\bar{k}(t) - \bar{a}\lambda)dt + \bar{\sigma}(t)d\overline{W}(t). \tag{7.2}$$

(iii) *$W(t)$ and $\overline{W}(t)$ are Brownian motions with correlation:*

$$dWd\overline{W} = \rho dt. \tag{7.3}$$

The dynamics (7.1) (and analogously (7.2)) have the following interpretation: interest rates move stochastically with an absolute volatility of $\sigma(t)$. The drift term is positive if $r(t)$ is below $k(t)$ at time t, otherwise it is negative. The drift always has a tendency to move r towards k. This effect is known as *mean reversion*, and $k(t)$ is the *level of mean reversion*. The strength of this effect is measured by the parameter $a \geq 0$ which is known as the *speed of mean reversion*.

REDUCED FORM MODELS

- Results:

(i) *The price of a default-free zero-coupon bond with maturity T:*

$$B(t,T) = e^{\mathcal{A}(t,T;a,k,\sigma)-\mathcal{B}(t,T;a)r(t)}. \tag{7.12}$$

(ii) *The survival probability from t until T:*

$$P(t,T) = e^{\mathcal{A}(t,T;\overline{a},\overline{k},\overline{\sigma})-\mathcal{B}(t,T;\overline{a})\lambda(t)}. \tag{7.13}$$

(iii) *The price of a defaultable zero-coupon bond under zero recovery with maturity T, given survival until t:*

$$\overline{B}(t,T) = B(t,T)e^{\mathcal{A}(t,T;\overline{a},\widetilde{k},\overline{\sigma})-\mathcal{B}(t,T;\overline{a})\lambda(t)} \tag{7.14}$$

where

$$\widetilde{k}(t) = \overline{k}(t) - \rho\overline{\sigma}(t)\sigma(t)\mathcal{B}(t,T). \tag{7.15}$$

REDUCED FORM MODELS

- Multifactor Gaussian model:

- *default-free bond prices* $B(t,T)$

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + a(t,T)dW,$$

- *defaultable bond prices* $\overline{B}(t,T)$

$$\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} = (r(t) + q\lambda(t))dt + \overline{a}(t,T)dW - qdN(t).$$

REDUCED FORM MODELS

- Multifactor CIR model:

Assumption 7.3 *Interest rates and default intensities are driven by n independent factors* x_i, $i = 1, \ldots, n$ *with dynamics of the CIR square root type:*

$$dx_i = (\alpha_i - \beta_i x_i)dt + \sigma_i \sqrt{x_i} dW_i. \tag{7.25}$$

The coefficients satisfy $\alpha_i > \frac{1}{2}\sigma_i^2$ *to ensure strict positivity of the factors, and the Brownian motions* W_i *are mutually independent.*

The default-free short rate r and the default intensity λ *are positive linear combinations of the factors* x_i *with weights* $w_i \geq 0$ *and* $\overline{w}_i$ $(0 \leq i \leq n)$ *respectively:*

$$r(t) = \sum_{i=1}^{n} w_i x_i(t), \tag{7.26}$$

$$\lambda(t) = \sum_{i=1}^{n} \overline{w}_i x_i(t). \tag{7.27}$$

Reduced Form Models

- <u>Term structure of credit spreads:</u>

In forward-rate-based models not only the short-term interest rate r is evolved stochastically, but the whole term structure of forward rates is taken as state variable and is directly equipped with stochastic dynamics.

- The *instantaneous risk-free forward rate* at time t for date $T > t$ is defined as:

$$f(t,T) = -\frac{\partial}{\partial T} \ln B(t,T). \tag{7.116}$$

- The *instantaneous defaultable forward rate* at time t for date $T > t$ is defined as:

$$\overline{f}(t,T) = -\frac{\partial}{\partial T} \ln \overline{B}(t,T). \tag{7.117}$$

- <u>The implied hazard rate of default at time $T > t$</u> as seen from time t is given by the spread of the defaultable over the default-free continuously compounded forward rates:

$$\boxed{h(t,T) = \overline{f}(t,T) - f(t,T).} \tag{7.118}$$

REDUCED FORM MODELS

1. *The dynamics of the default free forward rates are given by:*

$$df(t,T) = \alpha(t,T)\,dt + \sum_{i=1}^{n} \sigma_i(t,T)\,dW^i(t). \qquad (7.119)$$

2. *The dynamics of the defaultable forward rates are given by:*

$$d\overline{f}(t,T) = \overline{\alpha}(t,T)\,dt + \sum_{i=1}^{n} \overline{\sigma}_i(t,T)\,dW^i(t). \qquad (7.120)$$

prices of default-free and defaultable zero-coupon bonds are given by:

$$B(t,T) = \exp\left\{-\int_t^T f(t,s)\,ds\right\}.$$

$$\overline{B}(t,T) = (1 - N(t))\exp\left\{-\int_t^T \overline{f}(t,s)\,ds\right\}$$

REDUCED FORM MODELS

1. *Given the dynamics of the risk-free forward rates* (7.119):

   (i) *The dynamics of the risk-free bond prices are given by*

$$\frac{dB(t,T)}{B(t-,T)} = \left[-\gamma(t,T) + r(t) + \frac{1}{2}\sum_{i=1}^{n} a_i^2(t,T) \right] dt + \sum_{i=1}^{n} a_i(t,T)\, dW^i(t) \qquad (7.123)$$

where $a_i(t,T)$ *and* $\gamma(t,T)$ *are*

$$a_i(t,T) = -\int_t^T \sigma_i(t,v)\, dv, \qquad (7.124)$$

$$\gamma(t,T) = \int_t^T \alpha(t,v)\, dv. \qquad (7.125)$$

(ii) *The dynamics of the risk-free short rate are given by*

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t) ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t) dW^i(s). \qquad (7.126)$$

REDUCED FORM MODELS

Gaussian HJM model – vols and bond prices are deterministic functions of time t and maturity T, with forward rates normally distributed.

2. *Given the dynamics of the defaultable forward rates (7.120):*
   (i) *The dynamics of the defaultable bond prices are given by*

$$\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} = \left[-\overline{\gamma}(t,T) + \overline{r}(t) + \frac{1}{2}\sum_{i=1}^{n}\overline{a}_i^2(t,T)\right]dt + \sum_{i=1}^{n}\overline{a}_i(t,T)\,dW^i(t) - dN(t) \qquad (7.127)$$

where $\overline{a}_i(t,T)$ *and* $\overline{\gamma}(t,T)$ *are defined by*

$$\overline{a}_i(t,T) = -\int_t^T \overline{\sigma}_i(t,v)\,dv, \qquad (7.128)$$

$$\overline{\gamma}(t,T) = \int_t^T \overline{\alpha}(t,v)\,dv. \qquad (7.129)$$

(ii) *The dynamics of the defaultable short rate are given by*

$$\overline{r}(t) = \overline{f}(t,t) = \overline{f}(0,t) + \int_0^t \overline{\alpha}(s,t)ds + \sum_{i=1}^{n}\int_0^t \overline{\sigma}_i(s,t)dW^i(s). \qquad (7.130)$$

4 – CREDIT RATING MODELS

- We have focused so far on the modeling the stochastic structure of the default event by an intensity. Other types of credit events, such as rating transitions, can be modeled in terms of intensities as well.

- Example:

Firm ABC is currently rated A, but obviously this rating can change. We assume that there are only three possible ratings, A, B and D, the rating for defaulted debt. For simplicity we set the recovery rate of defaulted debt to zero.

The rating agency publishes the following rating migration data:

	A	B	D
A	$p_{AA} = 0.80$	$p_{AB} = 0.15$	$p_{AD} = 0.05$
B	$p_{BA} = 0.10$	$p_{BB} = 0.80$	$p_{BD} = 0.10$
D	$p_{DA} = 0.00$	$p_{DB} = 0.00$	$p_{DD} = 1.00$

Only 2 performing ratings, as D is an absorbing state

For example, of the companies rated A at the beginning of a year:

- $p_{AA} = 80\%$ were still rated A after one year,
- $p_{AB} = 15\%$ were rated B after that year, and
- $p_{AD} = 5\%$ were rated D, they had defaulted within the year.

We assumed that no firm can recover from default, the state D is called *absorbing*.

What is the probability of a default of the A-rated ABC debt within the next two years? There is an obvious, but wrong, answer.

One could say, we have two events of default each with a probability of $p_{AD} = 0.05$ or a survival probability of $(1 - p_{AD}) = 0.95$, giving a total survival probability of $(1 - p_{AD})^2 = 0.9025$ and a default probability of 0.0975. But this answer does not take *rating migration* into account. In the next two years a default can also occur via a transition to the B rating. Default can be reached via the following transitions:

$$
\begin{array}{lll}
A \to A \to D & \text{with probability} & p_{AA}p_{AD} = 0{,}80 \times 0.05 = 0.04, \\
A \to B \to D & \text{with probability} & p_{AB}p_{BD} = 0.15 \times 0.10 = 0.015, \\
A \to D(\to D) & \text{with probability} & p_{AD}p_{DD} = 0.05 \times 1 = 0.05.
\end{array}
$$

This gives a total default probability of 0.1075, which is a full percentage point larger than before. This effect is even stronger with real-world ratings. Here the credit risk for investment-grade bonds lies mainly in the risk of downgrading (with subsequently very much higher risk of default), and not in the risk of direct default.

4 – CREDIT RATING MODELS

To sum up, the two-period probability of default given initial rating A, i.e. the *two-period transition probability from A to D* is:

$$p_{AD}^{(2)} = p_{AA}p_{AD} + p_{AB}p_{BD} + p_{AD}p_{DD} = (p_{AA} \quad p_{AB} \quad p_{AD}) \begin{pmatrix} p_{AD} \\ p_{BD} \\ p_{DD} \end{pmatrix}.$$

If one takes the transition matrix

$$A = \begin{pmatrix} p_{AA} & p_{AB} & p_{AD} \\ p_{BA} & p_{BB} & p_{BD} \\ p_{DA} & p_{DB} & p_{DD} \end{pmatrix}$$

then it is easily seen that the two-period transition probability $p_{AD}^{(2)}$ is exactly the (A, D) component of the square A^2 of A. This also holds for the other two-period transition probabilities, and we reach the two-period transition probability matrix as:

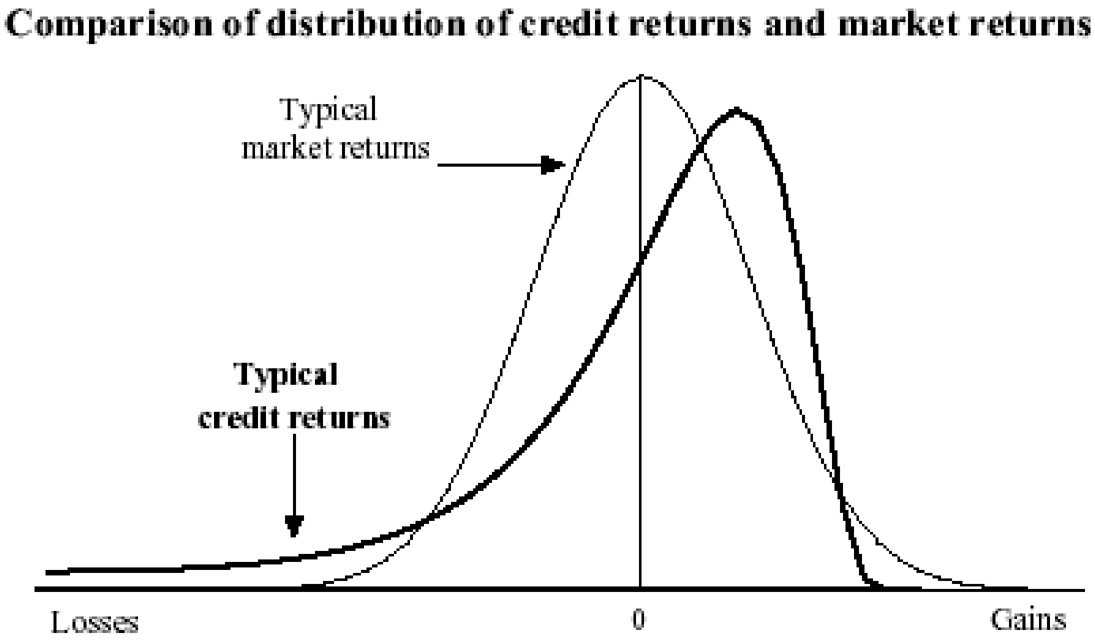
$$A^{(2)} = A \cdot A = A^2.$$

TYPES OF MODELS

- Default mode (DM) – take into consideration only the changes in the value of bonds due to defaults.
- Marked-to-market (MTM) – allows to assess the impact on the credit value of any change in its risk.
- Individual models – focus on the changes of a credit value, regardless the correlations with other credits in the portfolio.
- Portfolio models – incorporate the correlations between the several assets of a credit portfolio.

Challenges in Estimating Portfolio Credit Risk

- Non-normal returns - credit returns are highly skewed and fat-tailed.
- Difficulty in modeling correlations - lack of data, contrary to equities.



Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

Example: Portfolio with a single BBB 5y-bond

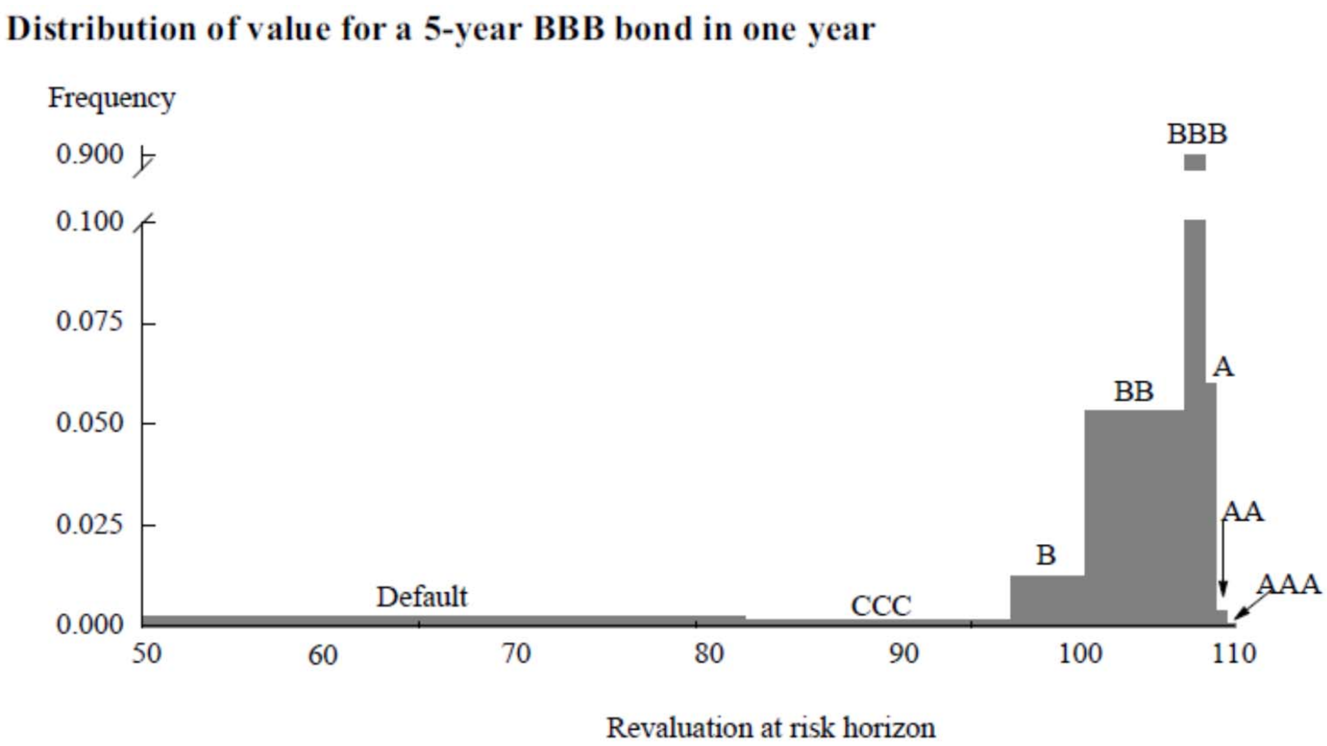
Ratings	Probability of Transition (%) (1)	Loan Value at year-end (2)	Difference to the mean (3)=(2)-μ	Contribution to the variance (4)=(1)x(3)^2
AAA	0.02	109.37	2.27	0.00
AA	0.33	109.19	2.09	0.01
A	5.96	108.66	1.56	0.15
BBB	86.93	107.55	0.45	0.18
BB	5.3	102.02	-5.08	1.37
B	1.17	98.10	-9.00	0.95
CCC	0.12	83.64	-23.46	0.66
Default	0.18	51.13	-55.97	5.64
Mean (μ)		107.10		
Variance (Σ(4))		8.95		
Standard-dev.		2.99		

Source: JPMorgan (1997), “CreditMetrics - Technical document”

Note: The Loan value is calculated using forward rates obtained from the term structure of interest rates for each rating level, to discount the remaining cash-flows (from year 1 to 5). The default price is the expected recovery rate.

Credit-VaR

Histogram for the credit value (from (1) and (2)):



Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

1-year 99% Credit-Var = Mean-$P_{1,B}$ (as the probability of having 1 year after a rating not above B = P(B)+P(CCC)+P(D)) = 1,17+0,12+0,18 ≈ 1% = 107,1-98,1 = 9.

Credit-VaR

Portfolio with BBB 5y-bond + single-A 3y bond, with annual coupon rate of 5%.

- Year-end price of the single-A 3y bond, after the several potential rating migrations:

	Year-end Bond Price	Probability of Transition (%)
AAA	106.59	0.09
AA	106.49	2.27
A	106.30	91.05
BBB	105.64	5.52
BB	103.15	0.74
B	101.39	0.60
CCC	88.71	0.01
Default	51.13	0.06

Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

All potential values of the portfolio will result from the combination of the 8 potential values for each bond (8x8):

All possible 64 year-end values for a two-bond portfolio ($)

Obligor #1 (BBB)		Obligor #2 (single-A)							
		AAA	AA	A	BBB	BB	B	CCC	Default
		106.59	106.49	106.30	105.64	103.15	101.39	88.71	51.13
AAA	109.37	215.96	215.86	215.67	215.01	212.52	210.76	198.08	160.50
AA	109.19	215.78	215.68	215.49	214.83	212.34	210.58	197.90	160.32
A	108.66	215.25	215.15	214.96	214.30	211.81	210.05	197.37	159.79
BBB	107.55	214.14	214.04	213.85	213.19	210.70	208.94	196.26	158.68
BB	102.02	208.61	208.51	208.33	207.66	205.17	203.41	190.73	153.15
B	98.10	204.69	204.59	204.40	203.74	201.25	199.49	186.81	149.23
CCC	83.64	190.23	190.13	189.94	189.28	186.79	185.03	172.35	134.77
Default	51.13	157.72	157.62	157.43	156.77	154.28	152.52	139.84	102.26

Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

- The joint probabilities would just be product of the rating migration probability for each bond, if these ratings were independent.

Joint migration probabilities with zero correlation (%)

Obligor #1 (BBB)		Obligor #2 (single-A) AAA	AA	A	BBB	BB	B	CCC	Default
		0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
AAA	0.02	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.00
AA	0.33	0.00	0.01	0.30	0.02	0.00	0.00	0.00	0.00
A	5.95	0.01	0.14	5.42	0.33	0.04	0.02	0.00	0.00
BBB	86.93	0.08	1.98	79.15	4.80	0.64	0.23	0.01	0.05
BB	5.30	0.00	0.12	4.83	0.29	0.04	0.01	0.00	0.00
B	1.17	0.00	0.03	1.06	0.06	0.01	0.00	0.00	0.00
CCC	0.12	0.00	0.00	0.11	0.01	0.00	0.00	0.00	0.00
Default	0.18	0.00	0.00	0.16	0.01	0.00	0.00	0.00	0.00

Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

- However, ratings do not tend to be independent, as they may be moved by the same macroeconomic factors.

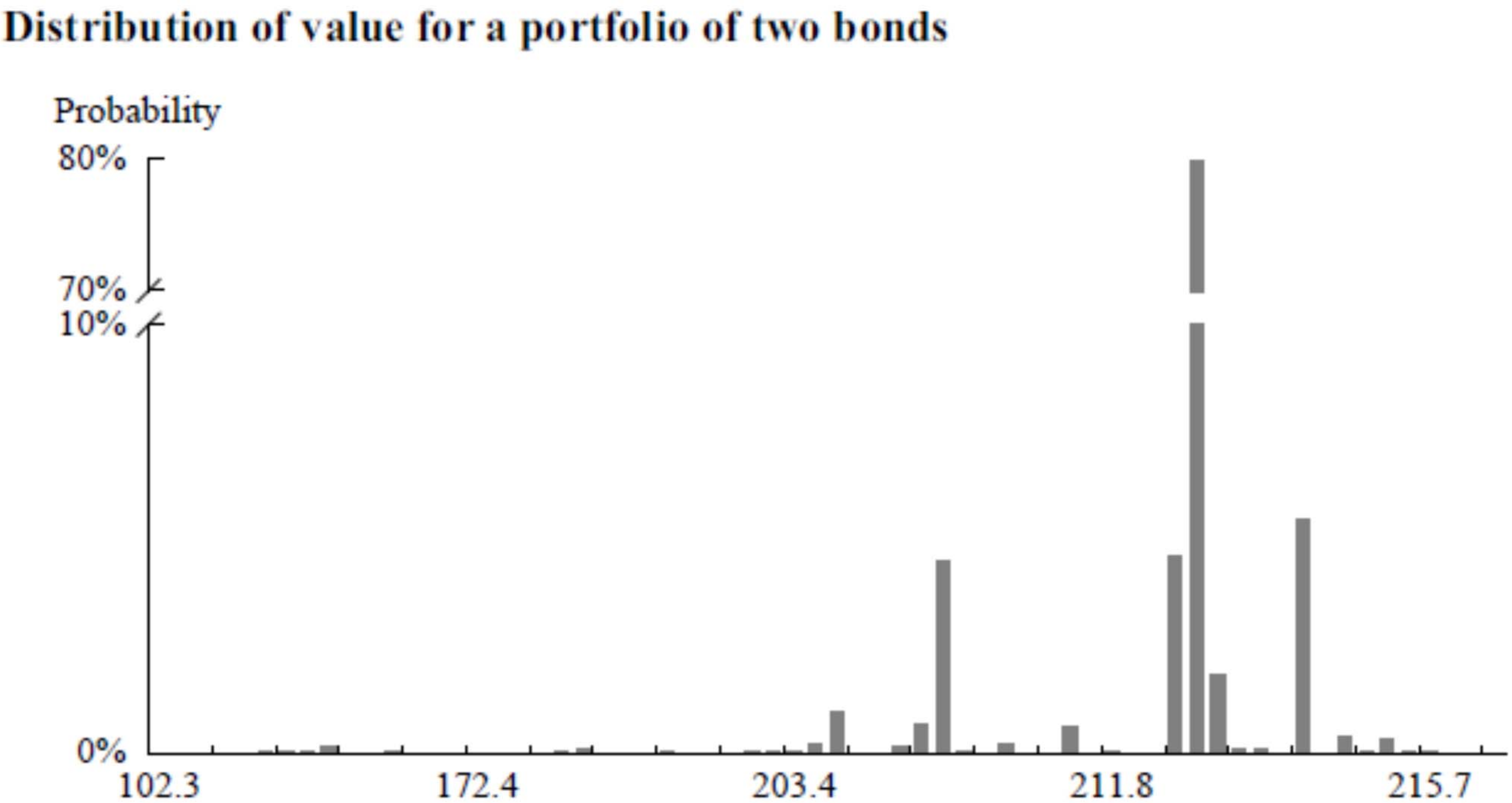
Joint rating migration probabilities with correlated bonds:

Joint migration probabilities with 0.30 asset correlation (%)

Obligor #1 (BBB)		Obligor #2 (single-A)							
		AAA	AA	A	BBB	BB	B	CCC	Default
		0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
AAA	0.02	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.00
AA	0.33	0.00	0.04	0.29	0.00	0.00	0.00	0.00	0.00
A	5.95	0.02	0.39	5.44	0.08	0.01	0.00	0.00	0.00
BBB	86.93	0.07	1.81	79.69	4.55	0.57	0.19	0.01	0.04
BB	5.30	0.00	0.02	4.47	0.64	0.11	0.04	0.00	0.01
B	1.17	0.00	0.00	0.92	0.18	0.04	0.02	0.00	0.00
CCC	0.12	0.00	0.00	0.09	0.02	0.00	0.00	0.00	0.00
Default	0.18	0.00	0.00	0.13	0.04	0.01	0.00	0.00	0.00

Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR



Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

Credit-VaR

There are at least four interesting features in the joint likelihood table above:

1. The probabilities across the table necessarily sum to 100%.
2. The most likely outcome is that both obligors simply remain at their current credit ratings. In fact, the likelihoods of joint migration become rapidly smaller as the migration distance grows.
3. The effect of correlation is generally to increase the joint probabilities along the diagonal drawn through their current joint standing (in this case, through BBB-A).
4. The sum of each column or each row must equal the chance of migration for that obligor standing alone. For instance, the sum of the last row must be 0.18%, which is the default likelihood for Obligor #1 (BBB) in isolation.

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

Credit-VaR

The credit risk in a portfolio arises because there is variability in the value of the portfolio due to credit quality changes. Therefore, we expect any credit risk measure to reflect this variability. Loosely speaking, the greater the dispersion in the range of possible values, the greater the absolute amount at credit risk. With this background, we next provide two alternative measures of credit risk that we use in CreditMetrics.

(i) standard-deviation

$$(Standard\ Deviation)^2 = p_1 \cdot (V_1 - Mean)^2 + p_2 \cdot (V_2 - Mean)^2 + \ldots + p_{64} \cdot (V_{64} - Mean)^2$$

where p_1 refers to the probability or likelihood of being in State 1 at the end of the risk horizon, and V_1 refers to the value in State 1. $Mean = p_1 \cdot V_1 + p_2 \cdot V_2 + \ldots + p_{64} \cdot V_{64}$

(ii) Percentile level:

We define this second measure of risk as a specified *percentile level* of the portfolio value distribution. The interpretation of the percentile level is much simpler than the standard deviation: the lowest value that the portfolio will achieve 1% of the time is the 1^{st} percentile.

Source: Riskmetrics Group (2007), “CreditMetrics – Technical Document”.

Credit-VaR

$$\text{Mean: } \mu_{Total} = \sum_{i=1}^{S=64} p_i \mu_i = 213.63$$

$$\text{Variance: } \sigma^2_{Total} = 11.22$$

	BBB Bond	A Bond	Portfolio
Mean	107,09	106,55	213,63
St.-Dev.	2,99	1,49	3,35

Conclusion: The means of the BBB and the A bonds sum directly, but the risk (standard deviations) is much less than the summed individuals due to diversification.

1-year 99% Credit-Var = Mean-$P^{P}_{1,(B,A)}$ (as the probability of having 1 year after a rating not above B in the 1st bond and A in the 2nd bond = P(B,A)+P(B,BBB)+…+P(D,D) = 0,92+0,18+…+0 =1,45. ≈ **1**%) = 213,63-204,4 = 9,23.

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

Credit-VaR

- Assuming a normal distribution, the Var would be:

$$VaR(\alpha\%) = N(1-\alpha\%)*\sigma \Rightarrow$$

$$VaR(5\%) = 1.65*\sigma = 1.65*3.35=5.53$$
$$VaR(1\%) = 2.33*\sigma = 2.33*3.35=7.81$$

(lower than the observed value, due to fat tails)

Credit-VaR

- The decision to hold a bond or not is likely to be made within the context of some existing portfolio.
- Thus, the more relevant calculation is the marginal increase to the portfolio risk that would be created by adding a new bond to it = 0,36 (much smaller than the A-Bond st-dev = 1,49) in standard-deviation and 0,23 in Credit-Var.

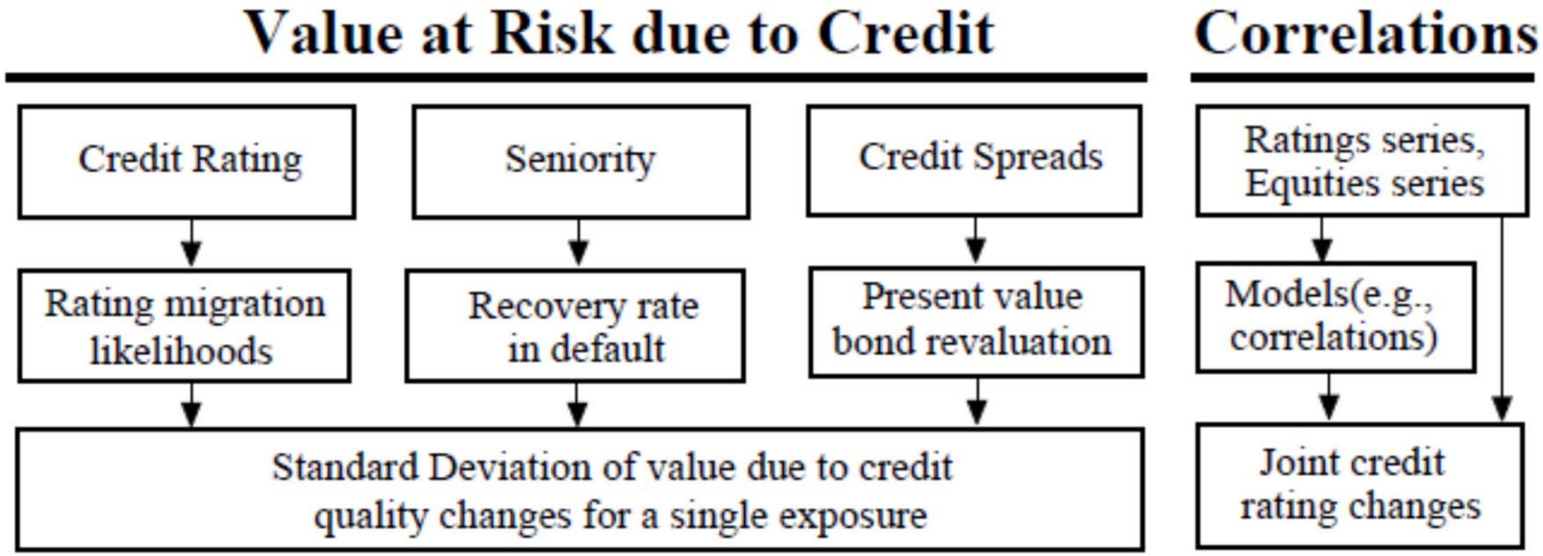
	BBB-Bond (1)	Portfolio (2)	A-Bond Marginal Risk (3) = (2)-(1)
Standard-deviation	2,99	3,35	0,36
99% Credit-Var	9,00	9,23	0,23

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

Creditmetrics

In our examples of one and two bond portfolios, we have been able to specify the entire distribution of values for the portfolio. We remark that this becomes inconvenient, and finally impossible, to do this in practice as the size of the portfolio grows. Noting that for a three asset portfolio, there are 512 (that is, 8 times 8 times 8) possible joint rating states. For a five asset portfolio, this number jumps to 32,768, and in general, for a portfolio with N assets, there are 8^N possible joint rating states.

Information required:



Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

5 – DEFAULT CORRELATION MODELS

- Credit spreads of different issuers are correlated through time.
- However, a good model for the default correlations across firms is still an open challenge for credit risk researchers.
- Correlations across equities are considerably higher than observed default correlations.

↓

- **Two patterns** are found in time series of spreads:

1st) Spreads vary smoothly with general macro-economic factors in a correlated fashion.

↓

Cyclical correlation between defaults

2nd) Jumps are common on several firm credit spreads. This suggest that the sudden variation in the credit risk of one issuer, which causes the jump in first place, can propagate to other issuers as well.

5 – Default Correlation Models

Historically, defaults tended to cluster as the following examples from the USA show.

- Oil industry: 22 companies defaulted in 1982–1986.
- Railroad conglomerates: 1 default each year 1970–1977.
- Airlines: 3 defaults in 1970–1971, 5 defaults in 1989–1990.
- Thrifts (savings and loan crisis): 19 defaults in 1989–1990.
- Casinos/hotel chains: 10 defaults in 1990.
- Retailers: >20 defaults in 1990–1992.
- Construction/real estate: 4 defaults in 1992.

If defaults were indeed independent, such clusters of defaults should not occur.

5 – DEFAULT CORRELATION MODELS

- Conditional probabilities:

$$p_{A|B} = \frac{p_{AB}}{p_B}, \qquad p_{B|A} = \frac{p_{AB}}{p_A}$$

- Correlation coefficient:

$$\varrho_{AB} = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}}.$$

The joint default probability is given by:

$$p_{AB} = p_A p_B + \varrho_{AB}\sqrt{p_A(1-p_A)p_B(1-p_B)}$$

and the conditional default probabilities are:

Dividing $p_{A|B}$ by p_B

$$p_{A|B} = p_A + \varrho_{AB}\sqrt{\frac{p_A}{p_B}(1-p_A)(1-p_B)}.$$

5 – DEFAULT CORRELATION MODELS

- Calculation of default correlation:

  - Historically observed joint rating and default events: The obvious source of information on default correlation is the historical incidence of joint defaults of similar firms in a similar time frame. We used such data in Section 10.1.1 when we analysed the evidence for default dependency in aggregated historical US default rate data. Such data is objective and directly addresses the modelling problem. Unfortunately, because joint defaults are rare events, historical data on joint defaults is very sparse. To gain a statistically useful number of observations, long time ranges (several decades) have to be considered and the data must be aggregated across industries and countries. In the majority of cases direct data will therefore not be available.

  - Credit spreads: Credit spreads contain much information about the default risk of traded bonds, and changes in credit spreads reflect changes in the markets' assessment of the riskiness of these investments. If the credit spreads of two obligors are strongly correlated it is likely that the defaults of these obligors are also correlated. Credit spreads have the further advantage that they reflect market information (therefore they already contain risk premia) and that they can be observed far more frequently than defaults. Disadvantages are problems with data availability, data quality (liquidity), and the fact that there is no theoretical justification for the size and strength of the link between credit spread correlation and default correlation.[2]

5 – DEFAULT CORRELATION MODELS

- Calculation of default correlation:

  - Equity correlations: Equity price data is much more readily available and typically of better quality than credit spread data. Unfortunately, the connection between equity prices and credit risk is not obvious. This link can only be established by using a theoretical model, and we saw that these models have difficulties in explaining the credit spreads observed in the market. Consequently, a lot of pre-processing of the data is necessary until a statement about default correlations can be made.

Independent Defaults

If defaults are independent and happen with probability p over the time horizon T, then the loss distribution of a portfolio of N loans is described by the binomial distribution function.

Definition 10.1 (binomial distribution) *Consider a random experiment with success probability p which is repeated N times and let X be the number of successes observed. All repetitions are independent from each other. The binomial frequency function $b(n; N, p)$ gives the probability of observing $n \leq N$ successes. The binomial distribution function $B(n; N, p)$ gives the probability of observing less than or equal to n successes:*

$$b(n; N, p) := \mathbf{P}[X = n] = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$$B(n; N, p) := \mathbf{P}[X \leq n] = \sum_{m=0}^{n} \binom{N}{m} p^m (1-p)^{N-m}.$$

In our credit setting, the probability of exactly $X = n$ (with $n \leq N$) defaults until time T is $b(n; N, p)$ and the probability of up to n defaults is $B(n; N, p)$.

Independent Defaults

Distribution of default losses under independence
(number of obligors =100 and p =0,5)

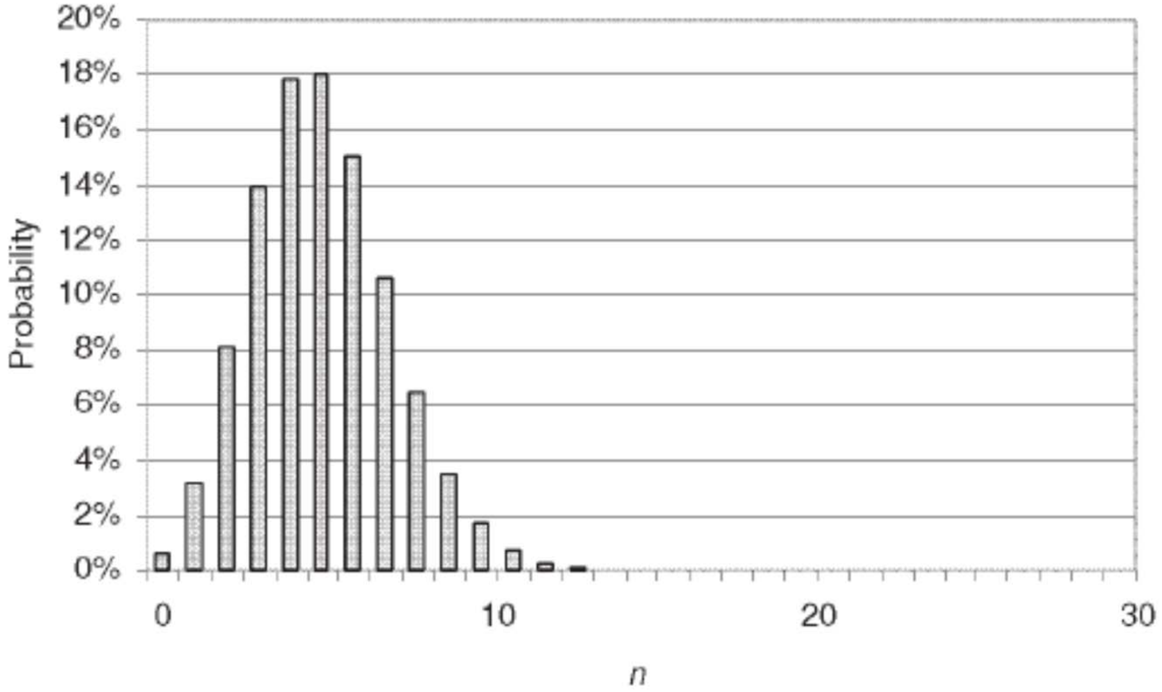


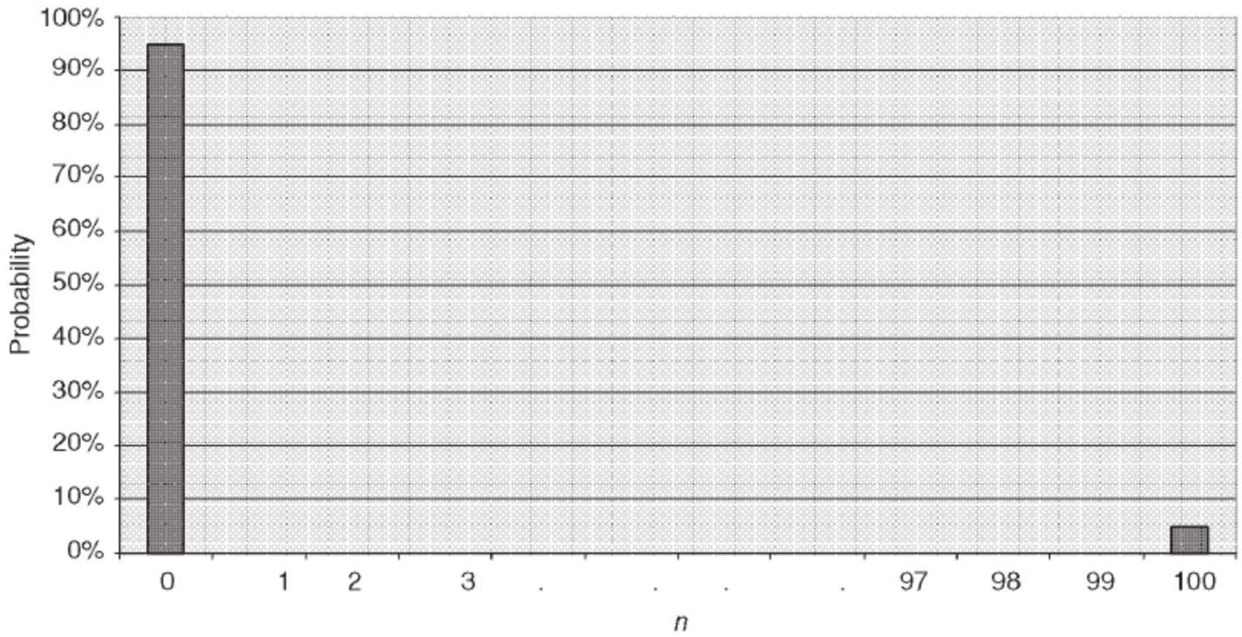
Figure 10.5 Distribution of default losses under independence. Parameters: number of obligors $N = 100$, individual default probability $p = 5\%$

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

Perfectly Correlated Defaults

- *Either all* obligors default (with 5% probability),
- *Or none* of the obligors defaults (with 95% probability).

Perfectly dependent defaults



Source: Schonbucher, Philipp J. (2003), “Credit Derivatives Pricing Models – Models, Pricing and Implementation”, Wiley.

BINOMIAL EXPANSION MODEL

The binomial expansion technique (BET) is a method used by the ratings agency Moody's to assess the default risk in bond and loan portfolios. It was one of the first attempts to quantify the risk of a portfolio of defaultable bonds. The method is not based upon a formal portfolio default risk model, it can be inaccurate and it is generally unsuitable for pricing, yet it has become something of a market standard in risk assessment and portfolio credit risk concentration terminology.[5]

The BET is based upon the following observation. Assume we analyse a loan portfolio of $N = 100$ loans of the same size, with the same loss L in default and the same default probability $p = 5\%$. If the defaults of these obligors are independent, we know from the previous section that the loss distribution function is given by the binomial distribution function. The probability of a loss of exactly $X = nL$ (with $n \le N$) until time T is (10.8):

$$\mathbf{P}[X = nL] = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} =: b(n; N, p).$$

BINOMIAL EXPANSION MODEL

Let us now consider the other extreme. If all defaults are perfectly dependent (i.e. either *all* or *none* of the obligors default), we have:

$$\mathbf{P}[X > 0] = p = 5\% = \mathbf{P}[X = NL],$$
$$\mathbf{P}[X = 0] = 1 - p = 95\% = \mathbf{P}[X = 0].$$

The key point to note here is that this can also be represented as a binomial distribution function with probability $p = 5\%$, but this time only *one* binomial draw is taken and the stakes are much higher: a loss of NL if the 5% event occurs.

BINOMIAL EXPANSION MODEL

Thus we have the following results.

- Perfect independence is $N = 100$ obligors with loss L and loss probability $p = 5\%$ each. The probability of a loss X of less than x is

$$\mathbf{P}[X \leq x] = B(n; N, p),$$

where the parameters are:
– $N = 100$;
– $n = \lfloor x/L \rfloor$ ("rounding down", the largest integer less than or equal to x/L);
– $p = 5\%$.

BINOMIAL EXPANSION MODEL

- Perfect dependence is equivalent to $N' = 1$ obligors with loss $L' = NL$ and loss probability $p = 5\%$. The probability of a loss X of less than x is

$$\mathbf{P}[X \leq x] = B(n'; N', p),$$

where:
– $N' = 1$, an adjusted number of obligors;
– $n' = \lfloor x/L' \rfloor$ ($n' = 0$ here);
– $p = 5\%$.

FACTOR MODELS

Assumption 10.3 (one-factor model) *The values of the assets of the obligors are driven by a common, standard normally distributed factor Y component and an idiosyncratic standard normal noise component ϵ_n:*

$$V_n(T) = \sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, \epsilon_n \quad \forall\, n \leq N,$$

where Y and ϵ_n, $n \leq N$ are independent normally distributed random variables with mean 0 and variance 1 and $\varrho \in [0, 1]$.

Using this approach the values of the assets of two obligors n and $m \neq n$ are correlated with linear correlation coefficient ϱ. The important point is that *conditional on the realisation of the systematic factor Y, the firm's values and the defaults are* ***independent***.

FACTOR MODELS

The systematic risk factor Y can be viewed as an indicator of the state of the business cycle, and the idiosyncratic factor ϵ_n as a firm-specific effects factor such as the quality of the management or the innovations of the firm. The default threshold K of the firm is mainly determined by the firm's reserves and balance-sheet structure. The relative sizes of the idiosyncratic and systematic components are controlled by the correlation coefficient ϱ. If $\varrho = 0$, then the business cycle has no influence on the fates of the firms, if $\varrho = 1$, then it is the only driver of defaults, and the individual firm has no control whatsoever. Empirically calibrated values of ϱ are around 10%.

In the following we assume that all obligors have the same default barrier $K_n = K$ and the same exposure $L_n = 1$. Following the intuition above, the distribution of defaults in the portfolio can be derived. First, the business cycle variable Y materialises, and conditional on the general state of the economy, the individual defaults occur independently from each other, but with a default probability $p(y)$ which depends on the state of the economy. This default probability is

$$p(y) = \Phi\left(\frac{K - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\right)$$

FACTOR MODELS

where $\Phi(\cdot)$ is the cumulative normal distribution function. This can be seen as follows. The individual conditional default probability $p(y)$ is the probability that the firm's value $V_n(T)$ is below the barrier K, given that the systematic factor Y takes the value y:

$$\begin{aligned} p(y) &= \mathbf{P}[V_n(T) < K \mid Y = y] \\ &= \mathbf{P}\left[\sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, \epsilon_n < K \,\middle|\, Y = y\right] \\ &= \mathbf{P}\left[\epsilon_n < \frac{K - \sqrt{\varrho}\, Y}{\sqrt{1-\varrho}} \,\middle|\, Y = y\right] \\ &= \Phi\left(\frac{K - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\right). \end{aligned}$$

The probability of having exactly n defaults is the average of the conditional probabilities of n defaults, averaged over the possible realisations of Y and weighted with the probability density function $\phi(y)$:

$$\mathbf{P}[X = n] = \int_{-\infty}^{\infty} \mathbf{P}[X = n \mid Y = y]\, \phi(y) dy$$

(from the 3 previous equations)

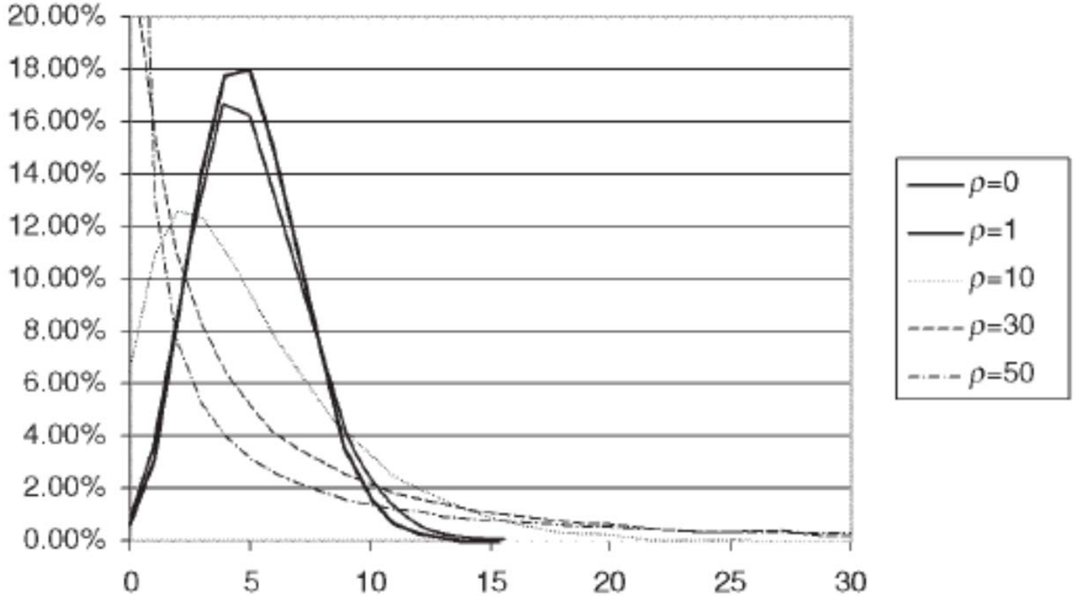
$$\mathbf{P}[X = n] = \int_{-\infty}^{\infty} \binom{N}{n} \left(\Phi\left(\frac{K - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\right)\right)^n \left(1 - \Phi\left(\frac{K - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\right)\right)^{N-n} \phi(y) dy$$

FACTOR MODELS

Thus, the resulting distribution function of the defaults is:

$$\mathbf{P}[X \leq m] = \sum_{n=0}^{m} \binom{N}{n} \int_{-\infty}^{\infty} \left(\Phi \left(\frac{K - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}} \right) \right)^{n} \left(1 - \Phi \left(\frac{K - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}} \right) \right)^{N-n} \phi(y) dy.$$

Figure 10.10 Default losses under correlation (one-factor model). Parameters: number of obligors $N = 100$, individual default probability $p = 5\%$, asset correlation ρ in percentage points: 0, 1, 10, 30, 50



FACTOR MODELS

Figure 10.10 shows the distribution of the default losses for our benchmark portfolio (100 obligors, 5% individual default probability) under different asset correlations. Increasing asset correlation (and thus default correlation) leads to a shift of the probability weight to the left ("good" events) and to the tail on the right. Very good events (no or very few defaults) become equally more likely as very bad events (many defaults). It should be noted that the deviation of the loss distribution function from the distribution under independence (i.e. zero correlation $\varrho = 0$) is already significant for low values for the asset correlation (e.g. 10%).

Table 10.3 99.9% and 99% VaR levels as a function of the asset correlation in the one-factor model. Parameters: 100 obligors, 5% individual default probability

Asset correlation (%)	99.9% VaR level	99% VaR level
0	13	11
1	14	12
10	27	19
20	41	27
30	55	35
40	68	44
50	80	53

The most significant effect for risk management is the increased mass of loss distribution in the tails => VaR increases with asset correlation

6 – Recovery Issues

Different recovery specifications

The recovery payment at default can be measured in different units.

- In the **recovery of par (or recovery of face value) scheme** it is given as a fraction of the security's face value.
- In the **recovery of treasury (or equivalent recovery) scheme** it is given as a fraction of an equivalent but default-free version of the security.
- In the **recovery of market value (or fractional recovery) scheme** investors receive a fraction of the asset market value just before default.

All these different specifications of recovery rates lead to different prices.

EVOLUTION OF RECOVERY MODELS

- Most papers have focused on modeling the default intensity process.
- Recovery issues are often ignored.
- When treated it is common to make unrealistic assumptions about the recovery

  - Constant recovery
  - Stochastic recovery

BUT independent of the default arrival

⬆

unrealistic

Empirical Facts:

- Recovery rates change over time, probably in a stochastic way
- Probability of Default (PD) and Loss given default (LGD) are correlated

RECOVERY MODELS

	MAIN MODELS & RELATED EMPIRICAL STUDIES	TREATMENT OF LGD	RELATIONSHIP BETWEEN RR AND PD
	Credit Pricing Models		
First generation structural-form models	Merton (1974), Black and Cox (1976), Geske (1977), Vasicek (1984), Crouhy and Galai (1994), Mason and Rosenfeld (1984).	PD and RR are a function of the structural characteristics of the firm. RR is therefore an endogenous variable.	PD and RR are inversely related (see Appendix A).
Second generation structural-form models	Kim, Ramaswamy e Sundaresan (1993), Nielsen, Saà-Requejo, Santa Clara (1993), Hull and White (1995), Longstaff and Schwartz (1995).	RR is exogenous and independent from the firm's asset value.	RR is generally defined as a fixed ratio of the outstanding debt value and is therefore independent from PD.
Reduced-form models	Litterman and Iben (1991), Madan and Unal (1995), Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Lando (1998), Duffie and Singleton (1999), Duffie (1998) and Duffee (1999).	Reduced-form models assume an exogenous RR that is either a constant or a stochastic variable independent from PD.	Reduced-form models introduce separate assumptions on the dynamic of PD and RR, which are modeled independently from the structural features of the firm.
Latest contributions on the PD-RR relationship	Frye (2000a and 2000b), Jarrow (2001), Carey and Gordy (2003), Altman, Brady, Resti and Sironi (2001 and 2004).	Both PD and RR are stochastic variables which depend on a common systematic risk factor (the state of the economy).	PD and RR are negatively correlated. In the "macroeconomic approach" this derives from the common dependence on one single systematic factor. In the "microeconomic approach" it derives from the supply and demand of defaulted securities.
	Credit Value at Risk Models		
CreditMetrics®	Gupton, Finger and Bhatia (1997).	Stochastic variable (beta distr.)	RR independent from PD
CreditPortfolioView®	Wilson (1998).	Stochastic variable	RR independent from PD
CreditRisk+®	Credit Suisse Financial Products (1997).	Constant	RR independent from PD
KMV CreditManager®	McQuown (1997), Crosbie (1999).	Stochastic variable	RR independent from PD

Loss determinants

- Collateral
- Debt seniority
- Loan type (namely for individuals)
- Region
- Business cycle
- Economic sector
- PD

LGD features

1. Most of the time recovery as a percentage of exposure is either relatively high (around 70-80%) or low (around 20-30%). The recovery (or loss) distribution is said to be "bimodal" (two-humped). Hence thinking about an "average" recovery or loss given default can be very misleading.
2. The most important determinants of which mode a defaulted claim is likely to fall into is whether or not it is secured and its place in the capital structure of the obligor (the degree to which the claim is subordinated). Thus bank loans, being at the top of the capital structure, typically have higher recovery than bonds.
3. Recoveries are systematically lower in recessions, and the difference can be dramatic: about one-third lower. That is, losses are higher in recessions, lower otherwise.
4. Industry of the obligor seems to matter: tangible asset-intensive industries, especially utilities, have higher recovery rates than service sector firms, with some exceptions such as high tech and telecom.
5. Size of exposure seems to have no strong effect on losses.

Estimation Methods

- NPV of recoveries
- Recovery distributions
- Bond prices after default
- LGD implied in bond prices
- LGD implied in observed losses and in PD estimates.
- Econometric adjustment of the LGD as a function of several variables (LossCalc, Moody's (2002)).

Table 9

Classification of the objective methods to obtain LGDs

Source	Measure	Type of facilities in the RDS: Defaulted facilities	Type of facilities in the RDS: Non-defaulted facilities	Most applicable to
Market values	Price differences	Market LGD		Large corporate, sovereigns, banks
	Credit spreads		Implied market LGD	Large corporate, sovereigns, banks
Recovery and cost experience	Discounted cash flows	Workout LGD		Retail, SMEs, large corporate
	Historical total losses and estimated PD	Implied historical LGD		Retail

Source: Basel Committee on Banking Supervision (2005)

Statistics

- Recoveries exhibit a bimodal distribution:

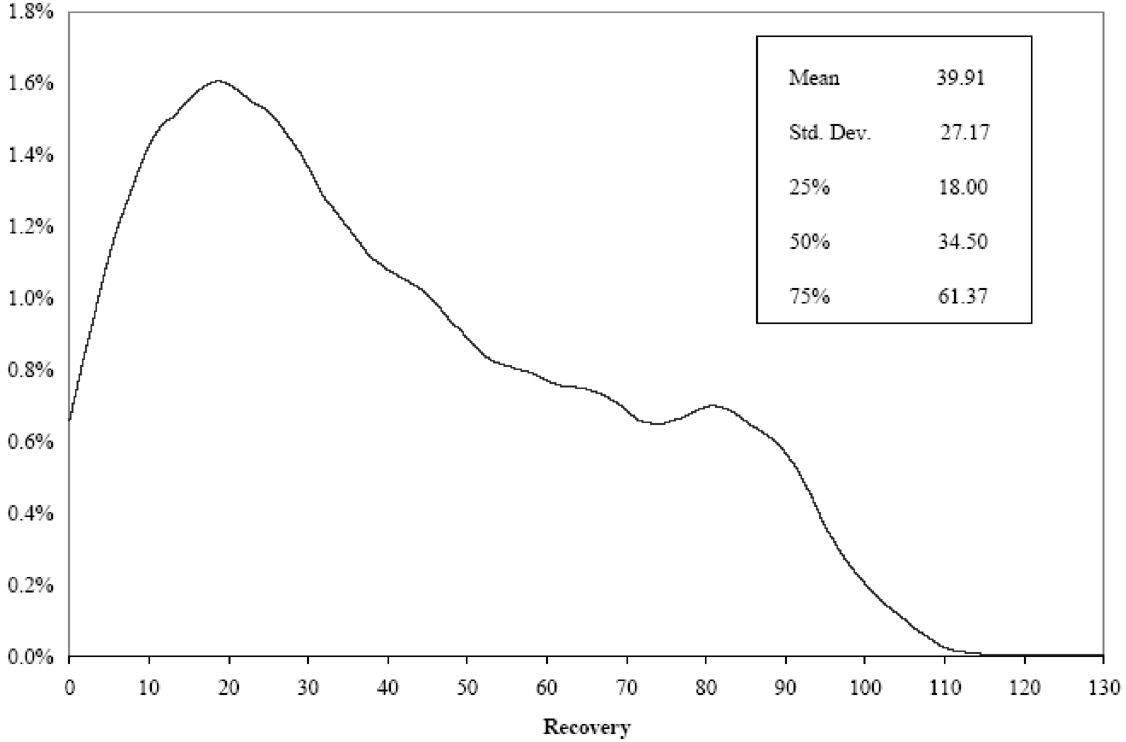


Figure 1: Probability Distribution of Recoveries, 1970-2003: All Bonds & Loans (Moody's)

Source: Schuermann (2004)

Seniority

- Higher recoveries in senior debt:

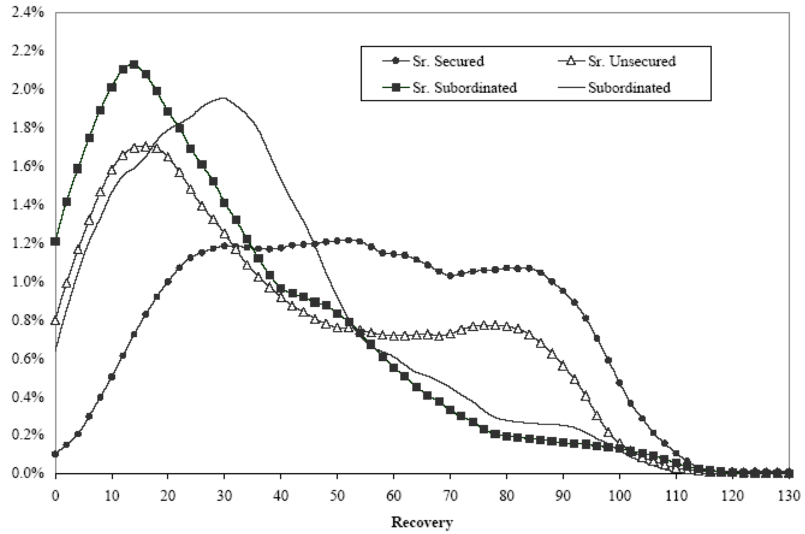


Figure 2: Probability Densities of Recovery by Seniority (Moody's, 1970-2003)

Source: Schuermann (2004) and Moody's (2009)

Average Annual Bond and Loan Recovery Rates[1]

	Loan	Bond					
Year	Sr. Sec.[2]	Sr. Sec.	Sr. Unsec.	Sr. Sub.	Sub.	Jr. Sub.	All Bonds
1982	n.a.	72.50%	35.79%	48.09%	29.99%	n.a.	35.57%
1983	n.a.	40.00%	52.72%	43.50%	40.54%	n.a.	43.64%
1984	n.a.	n.a.	49.41%	67.88%	44.26%	n.a.	45.49%
1985	n.a.	83.63%	60.16%	30.88%	39.42%	48.50%	43.66%
1986	n.a.	59.22%	52.60%	50.16%	42.58%	n.a.	48.38%
1987	n.a.	71.00%	62.73%	44.81%	46.89%	n.a.	50.48%
1988	n.a.	55.40%	45.24%	33.41%	33.77%	36.50%	38.98%
1989	n.a.	46.54%	43.81%	34.57%	26.36%	16.85%	32.31%
1990	75.25%	33.81%	37.01%	25.64%	19.09%	10.70%	25.50%
1991	74.67%	48.39%	36.66%	41.82%	24.42%	7.79%	35.53%
1992	61.13%	62.05%	49.19%	49.40%	38.04%	13.50%	45.89%
1993	53.40%	n.a.	37.13%	51.91%	44.15%	n.a.	43.08%
1994	67.59%	69.25%	53.73%	29.61%	38.23%	n.a.	45.57%
1995	75.44%	62.02%	47.60%	34.30%	41.54%	n.a.	43.28%
1996	88.23%	47.58%	62.75%	43.75%	22.60%	n.a.	41.54%
1997	78.75%	75.50%	56.10%	44.73%	35.96%	30.58%	49.39%
1998	51.40%	46.82%	41.63%	44.99%	18.19%	62.00%	39.25%
1999	75.82%	43.00%	38.04%	28.01%	35.64%	n.a.	34.33%
2000	68.32%	39.23%	23.81%	20.75%	31.86%	15.50%	25.18%
2001	64.87%	37.98%	21.45%	19.82%	15.94%	47.00%	22.21%
2002	58.80%	48.37%	29.69%	21.36%	24.51%	n.a.	29.95%
2003	73.43%	63.46%	41.87%	37.18%	12.31%	n.a.	40.72%
2004	87.74%	73.25%	52.09%	42.33%	94.00%	n.a.	58.50%
2005	83.78%	71.93%	54.88%	26.06%	51.25%	n.a.	55.97%
2006	83.60%	74.63%	55.02%	41.41%	56.11%	n.a.	55.02%
2007	68.63%	80.54%	53.25%	54.47%	n.a.	n.a.	54.69%
2008	63.38%	57.98%	33.80%	23.02%	23.56%	n.a.	34.83%

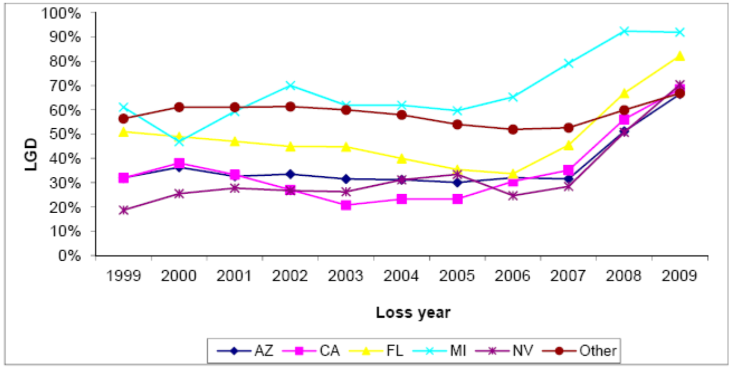
1. Issuer-weighted, based on 30-day post-default market prices.

2. Second-lien loans excluded.

Region

- Often regions where customers are based exhibit different recovery perspectives:

Figure 4: LGD over Loss Years by State



Source: Zhang, Yanan Lu Ji and Fei Liu (2010), "Local Housing Market Cycle and Loss Given Default: Evidence from Sub-Prime Residential Mortgages", IMF WP WP/10/167.

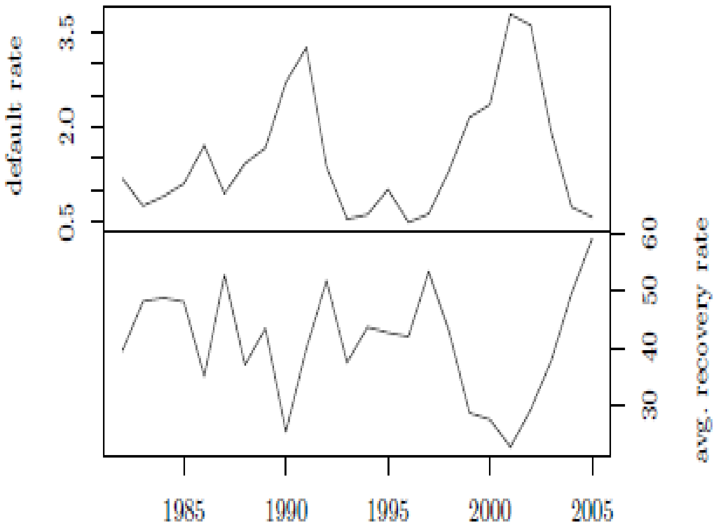
Table 5.5: Discounted recovery rates by country (12%)

	Mean	Median	Std. dev.	No. in sample
U.K.	65.8%	82.8%	36.4%	92
France	38.0%	31.9%	33.6%	336
Germany	54.9%	56.7%	24.0%	35
Total				463

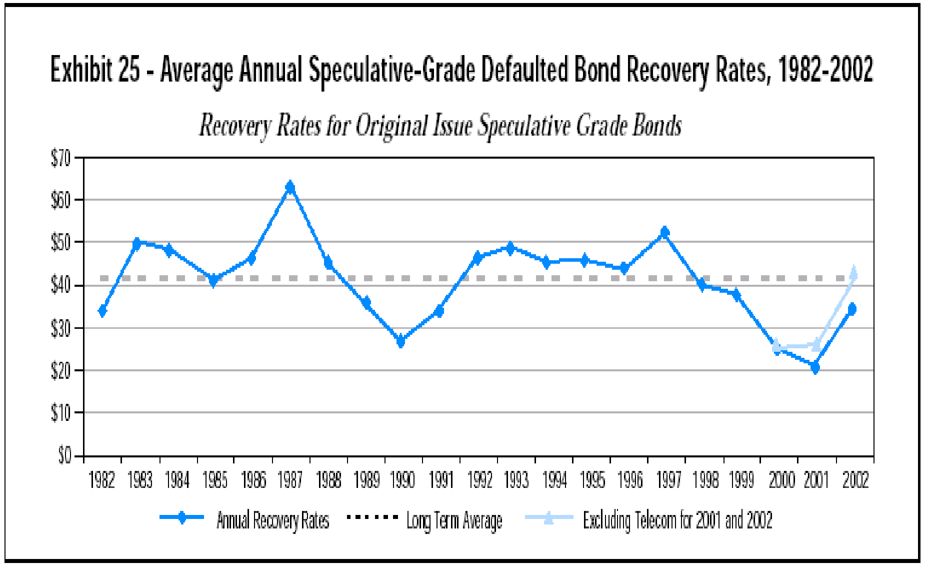
Source: Franks *et al* (2004).

Business Cycle

- LGD is typically higher during the lower stages of the business cycle.



Source: Bruche, Max and Carlos Gonzalez-Aguado (2007), “Recovery Rates, Default Probabilities and the Credit Cycle”.



Source: Moody’s (2003).

Business Cycle

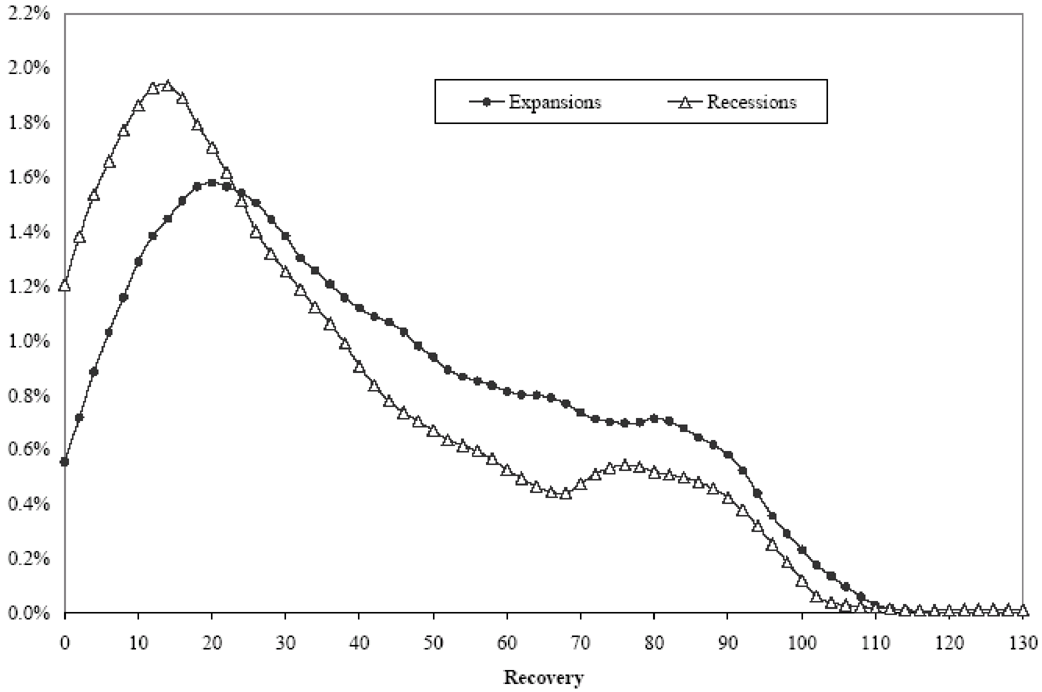


Figure 4: Probability Densities of Recoveries across the Business Cycle (Moody's, 1970-2003)

Source: Schuermann (2004)

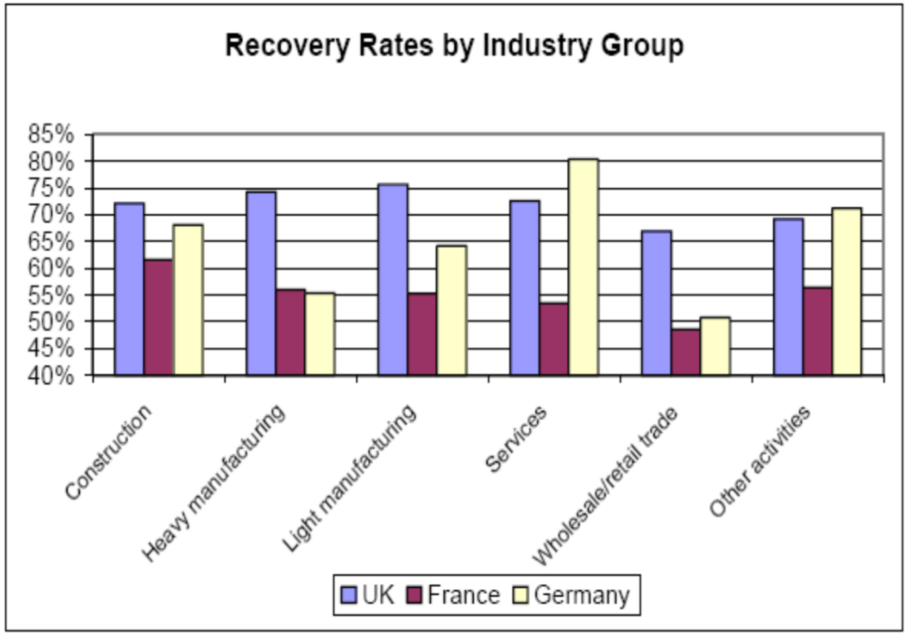
Economic Sectors

- In Altman and Kishore (1996), differences between sectors are identified.
- The LGD is usually higher for sectors with higher PD.

Exhibit 16 - Average Recovery Rates by Industry Category

Industry	Issuer Weighted Mean Recovery Rate		
	2003	2002	1982-2003
Utility-Gas	48.0	54.6	51.5
Oil and Oil Services	NA	44.1	44.5
Hospitality	64.5	60.0	42.5
Utility-Electric	5.3	39.8	41.4
Transport-Ocean	76.8	31.0	38.8
Media, Broadcasting and Cable	57.5	39.5	38.2
Transport-Surface	NA	37.9	36.6
Finance and Banking	18.8	25.6	36.3
Industrial	33.4	34.3	35.4
Retail	57.9	58.2	34.4
Transport - Air	22.6	24.9	34.3
Automotive	39.0	39.5	33.4
Healthcare	52.2	47.0	32.7
Consumer Goods	54.0	22.8	32.5
Construction	22.5	23.0	31.9
Technology	9.4	36.7	29.5
Real Estate	NA	5.0	28.8
Steel	31.8	28.5	27.4
Telecommunications	45.9	21.4	23.2
Miscellaneous	69.5	46.5	39.5

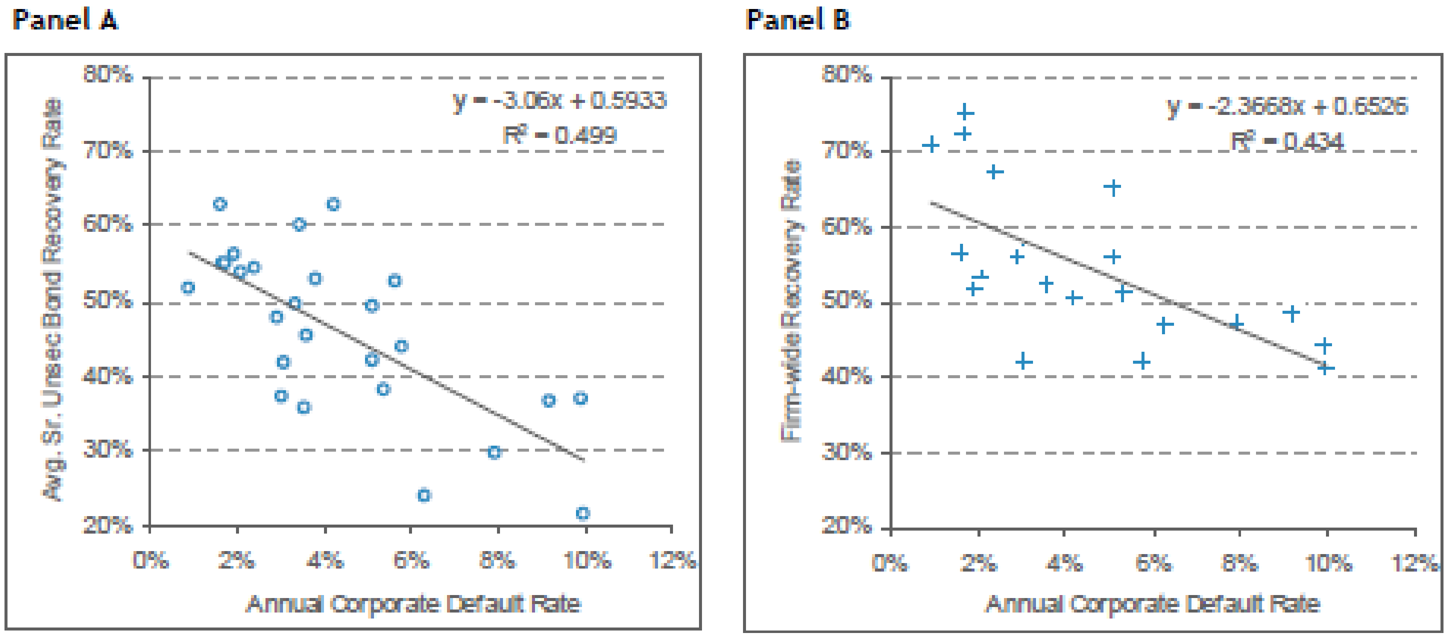
Source: Moody's (2004).



Source: Franks *et al* (2004).

PD

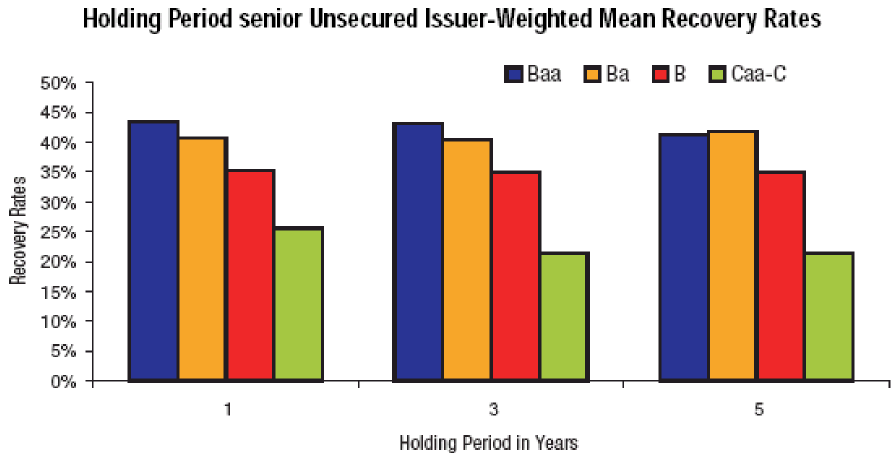
- The correlation between LGD and PD along time is high (0.66 according to S&P (2007)).



Source: Moody's (2008).

PD

- Higher ratings typically exhibit lower LGDs:



Average Sr. Unsecured Bond Recovery Rates by Year Prior to Default, 1982-2008[1]

	Year 1	Year 2	Year 3	Year 4	Year 5
Aaa	n.a.	3.33%[2]	n.a.	97.00%	85.55%
Aa	43.60%	40.15%	43.45%	57.61%	43.40%
A	42.48%	45.45%	44.50%	38.28%	40.95%
Baa	41.85%	44.56%	44.09%	45.44%	42.68%
Ba	48.00%	42.68%	41.58%	41.15%	41.12%
B	36.98%	35.41%	35.88%	36.91%	40.68%
Caa-C	33.96%	33.25%	33.11%	39.59%	41.94%
Investment-Grade	42.05%	44.23%	44.24%	44.57%	43.37%
Speculative-Grade	36.26%	35.71%	36.30%	38.26%	40.90%
All Rated	36.56%	36.65%	37.50%	39.52%	41.51%

1. *Issuer-weighted, based on 30-day post default market prices.*
2. *Based on three Icelandic bank defaults.*

Source: Moody's (2003; 2008).

Listed bonds

- Usually, in these exposures the LGD is measured as 1-Price (as a % of EAD) in a given period (usually 1 month after the default).
- Empirical evidence points to LGDs between 30% and 40% in non-colateralized exposures (around 60% for collateralized loans).

Average Corporate Debt Recovery Rates Measured by Post-Default Trading Prices

	ISSUER-WEIGHTED			VALUE-WEIGHTED		
LIEN POSITION	2009	2008	1982-2009	2009	2008	1982-2009
1st Lien Bank Loan	54.0%	61.7%	65.6%	56.6%	46.9%	59.1%
2nd Lien Bank Loan	16.0%	40.4%	32.8%	20.5%	36.6%	31.9%
Sr. Unsecured Bank Loan	34.5%	31.6%	48.7%	38.1%	22.8%	40.0%
Sr. Secured Bond	37.5%	54.9%	49.8%	29.5%	40.3%	48.5%
Sr. Unsecured Bond	37.7%	33.8%	36.6%	35.5%	26.2%	32.6%
Sr. Subordinated Bond	22.4%	23.7%	30.7%	17.9%	10.4%	25.0%
Subordinated Bond	46.8%	23.6%	31.3%	24.7%	7.3%	23.5%
Jr. Subordinated Bond	n.a.	n.a.	24.7%	n.a.	n.a.	17.1%

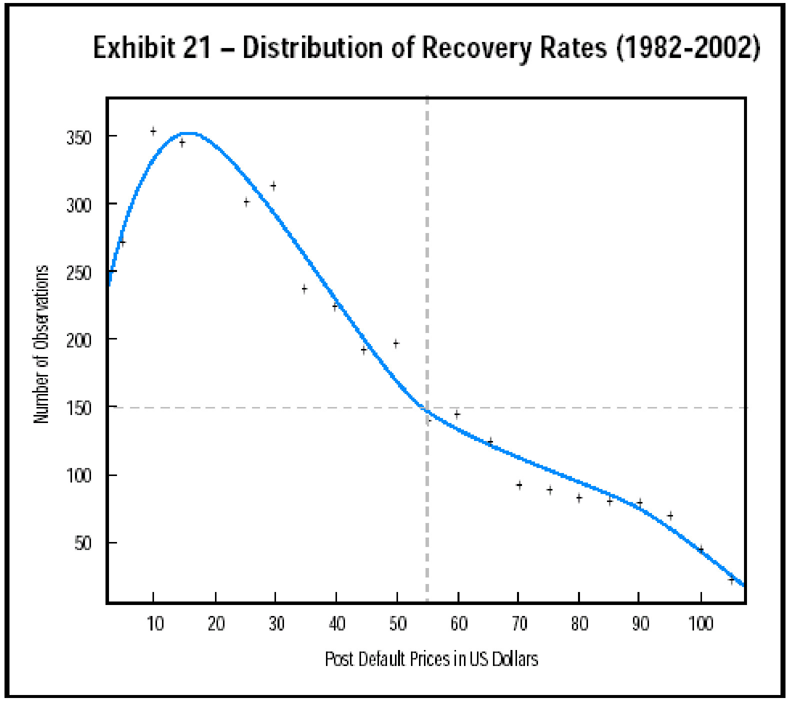
Source: Moody's (2010).

Listed bonds

Senior Unsecured Bond Recovery Rates for Financial Institution Defaults in 2008[1]

Company	Domain	Default Volume ($Mil)	Sr. Unsecured Bond Recovery
Lehman Brothers Holdings, Inc.	United States	120,164	9.3%
Kaupthing Bank hf	Iceland	20,063	4.0%
Glitnir banki hf	Iceland	18,773	3.0%
GMAC LLC	United States	17,190	69.9%
Washington Mutual Bank	United States	13,600	26.5%
Residential Capital, LLC	United States	12,315	51.7%
Landsbanki Islands hf	Iceland	12,161	3.0%
Washington Mutual, Inc.	United States	5,746	57.0%
GMAC of Canada Ltd	Canada	265	70.7%
Downey Financial Corp.	United States	200	0.5%
Fremont General Corporation	United States	166	46.0%
Luminent Mortgage Capital, Inc.	United States	131	27.3%
Triad Financial Corporation	United States	89	76.5%
Franklin Bank Corp.	United States	80	0.0%
GMAC International Finance B.V.	Netherlands	51	85.5%
Average	35.4%	Median	27.3%

Source: Moody's (2009).



Source: Moody's (2003).