1.3 Portfolio Protection

Exercise 1.13.

(a) (i) Since $R_f = 8\%$ we know that for $R_L = 6\%$ to minimize the probability of returns lower than 6% is equivalent to set that probability to zero, i.e. to deposit 100% of our wealth.

(ii) If we have $R_f = R_L = 8\%$ and Gaussian returns, all efficient portfolio have the same probability of returns lower than 8% .

(iii) If we have $R_L = 10\% > R_f = 8\%$ and Gaussian returns the optimal turns out to be to leverage up as much as possible (borrowing as much as possible) to invest more than our wealth in the tangent portfolio.

(b) Portfolios with the highest return-at-risk (RaR) are also the safest portfolio according to Kataoka. We, thus, are interested in portfolio returns with the probabilities lower or equal to 10%, i.e. in the worst 10% scenarios,

$$\begin{aligned}
&\Pr(R_p \le R_L) \le 10\% \\
&\Pr\left(\frac{R_p - \bar{R}_p}{\sigma_p} \le \frac{R_L - \bar{R}_p}{\sigma_p}\right) \le 10\% \\
&\Phi\left(\frac{R_L - \bar{R}_p}{\sigma_p}\right) \le 10\% \\
&\frac{R_L - \bar{R}_p}{\sigma_p} \le \Phi^{-1}(10\%) \\
&\bar{R}_p \ge R_L - \Phi^{-1}(10\%)\sigma_p \\
&\bar{R}_p \ge R_L + 1.2816\sigma_p
\end{aligned}$$

So the Kataoka lines are given by $\bar{R}_p = R_L + 1.2816\sigma_p$ and the goal is to maximize R_L along the EF.

From Exercise 1.11 recall that for $R_f = 8\%$, the EF is given by $\bar{R}_p = 0.08 + 2.081\sigma_p$. Since the slope of the EF is higher than the slope of the Kataoka lines, the maximum R_L will be the highest expected return portfolio. That is, the lowest RaR portfolio turns out to require extreme leverage (borrowing as much as possible) to invest more than our wealth in the tangent portfolio.

(c) Following the exact same steps as in (b) we get

$$\Pr(R_p \le 10\%) \le 10\% \qquad \Leftrightarrow \qquad \bar{R}_p \ge 0.1 + 1.2816\sigma_p \ .$$

Since the EF has a lower y-cross and a higher slope $\bar{R}_p = 0.08 + 2.081\sigma_p$, we need to find the crossing point.

$$0.1 + 1.2816\sigma_p = 0.08 + 2.081\sigma_p \quad \Leftrightarrow \quad \sigma_p = \frac{0.1 - 0.08}{2.081 - 1.2816} = 2.5\% \quad \Rightarrow \quad \bar{R}_p = 13.21\%.$$

(d) In (a) we deal with the Roy criterion, in (b) with the Kataoka criterion and in (c) with the Telser criterion. See Figure 1.13 for a graphical representation of the previous answers.

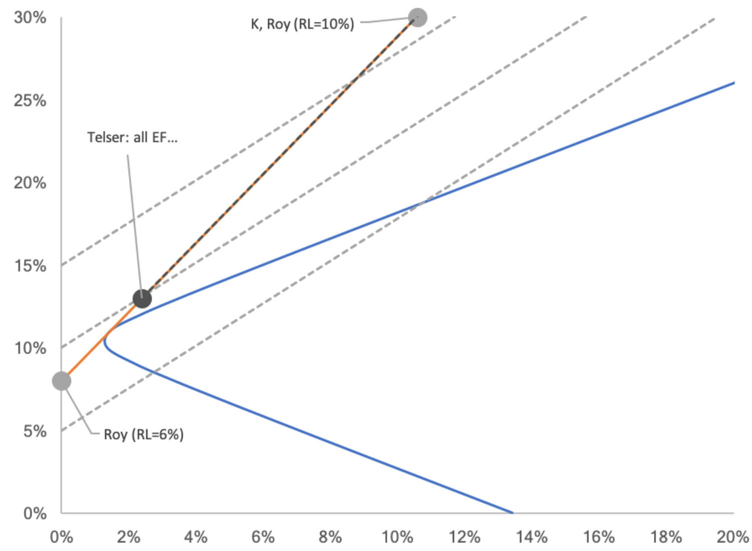

Figure 7: Exercise 1.13 – Efficient Frontier (red) and safety first portfolios determined in (a)–(c). Dashed grey lines are Kataoka lines for different R_L. All portfolios in the efficient frontier has the have probability of returns lower than $R_L = 8\%$. Dark grey dashed segment of line on the EF identify all portfolios that satisfy the Telser restriction.

Exercise 1.14.

(a) All combinations of A and B satisfy

$$\Pr\left(R_p \leq 5\%\right) = \Pr\left(\frac{R_p - \bar{R}_p}{\sigma_p} \leq \frac{5\% - \bar{R}_p}{\sigma_p}\right) = \Phi\left(\frac{5\% - \bar{R}_p}{\sigma_p}\right)$$

since, it must be less or equal than 15% we have

$$\Phi\left(\frac{5\% - \bar{R}_p}{\sigma_p}\right) \leq 15\% \quad \Leftrightarrow \quad \bar{R}_p \geq 5\% - \Phi^{-1}\left(15\%\right)\sigma_p \quad \Leftrightarrow \quad R_p \geq 5\% + 1.0364\ \sigma_p\ .$$

From Exercise 1.12 we also know all combinations of A and B are given by the hyperbola

$$\sigma_p^2 = 3\bar{R}_p^2 - 0.44\bar{R}_p + 0.0176.$$

From Figure 1.14 we clearly see that there is no combination of A and B that satisfies the safety condition.

(b) The combination that maximizes the likelihood of getting returns above 5% is the one that minimizes the probability of returns lower or equal to 5%, i.e. it is the Roy portfolio with $R_L = 5\%$. The Roy portfolio can be determine as a tangent portfolio, where R_L acts as a ficticious risk-free rate. In this case we get

$$Z = V^{-1}\left[\bar{R} - R_L\mathbb{1}\right] = \begin{pmatrix} 909.091 & -1136.364 \\ -1136.364 & 2045,454545 \end{pmatrix}\begin{pmatrix} 5\% \\ 3\% \end{pmatrix} = \begin{pmatrix} 11.3636 \\ 4.5455 \end{pmatrix}$$

$$\Downarrow$$

$$X^{\text{Roy}} = \begin{pmatrix} 71.43\% \\ 28.57\% \end{pmatrix}$$

The Roy portfolio is a concrete combination of A adnd B, so it belongs to the hyperbola. It has $\bar{R}^{\text{Roy}} = 9.43\%$ and $\sigma^{\text{Roy}} = 5.28\%$. Thus its probability of returns lower than 5% is

$$\Phi\left(\frac{5\% - 9.43\%}{5.28\%}\right) = \Phi\left(-0.894\right) = 20.1\% \ .$$

See the representation of the Roy portfolio in Figure 1.14.

(c) The combination with the highest 15% quantile is the Kataoka portfolio for an $\alpha = 15\%$. For a fixed α we have lines with a fixed slope equal to $-\Phi(\alpha)$. In this case we have $-\Phi(\alpha) = 1.0364$. For find the Kataoka portfolio we need to find the hyperbola point with the exact same slope.

From the Kataoka lines we get

$$\bar{R}_p = R_L + 1.0364\ \sigma_p \qquad \Rightarrow \qquad \left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{Kataoka} = 1.0364$$

From the hyperbola equation, $\sigma_p^2 = 3\bar{R}_p^2 - 0.44\bar{R}_p + 0.0176$, and considering only its upper part (the efficient part, we have

$$\bar{R}_p = \frac{+0.44 + \sqrt{0.44^2 - 4 \times 3 \times (0.0176 - \sigma_p^2)}}{6}$$

and differentiating w.r.t. σ_p

$$\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{hyperbola} = \frac{1}{6} \times \frac{1}{2}\left(0.44^2 - 4 \times 3 \times (0.0176 - \sigma_p^2)\right)^{-\frac{1}{2}} \times (-2\sigma_p)$$

Matching the slopes of the Kataoka lines with the hyperbola slope

$$\begin{aligned}
\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{Kataoka} &= \left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{hyperbola} \\
1.0364 &= -\frac{1}{6}\left(0.44^2 - 4 \times 3 \times (0.0176 - \sigma_p^2)\right)^{-\frac{1}{2}} \sigma_p \\
1.0364^2 &= \frac{1}{36}\left(0.44^2 - 4 \times 3 \times (0.0176 - \sigma_p^2)\right)^{-1} \sigma_p^2 \\
1.0741\left(0.44^2 - 12(0.0176 - \sigma_p^2)\right) &= \frac{1}{36}\sigma_p^2 \\
\sigma_p^2 &= 0.002203 \qquad \Rightarrow \qquad \sigma^{\text{Kataoka}} = 4.05\%
\end{aligned}$$

Which implies and expecte return of

$$\bar{R}_p = \frac{+0.44 + \sqrt{0.44^2 - 4 \times 3 \times (0.0176 - 0.002203)}}{6} = 8.9\%.$$

Finally, in terms of composition we have $X^{\text{Kataoka}} = \begin{pmatrix} 45\% \\ 55\% \end{pmatrix}$.

(d) In (a) we deal with the Telser criterion, but in this case there was no feasible portfolio satisfiying the safety condition. In (b) we address Roy's safety criterion and in (c) Kataoka's.

(e) If the returns were not Gaussian we could do the same type of computations but using the correct distribution function.

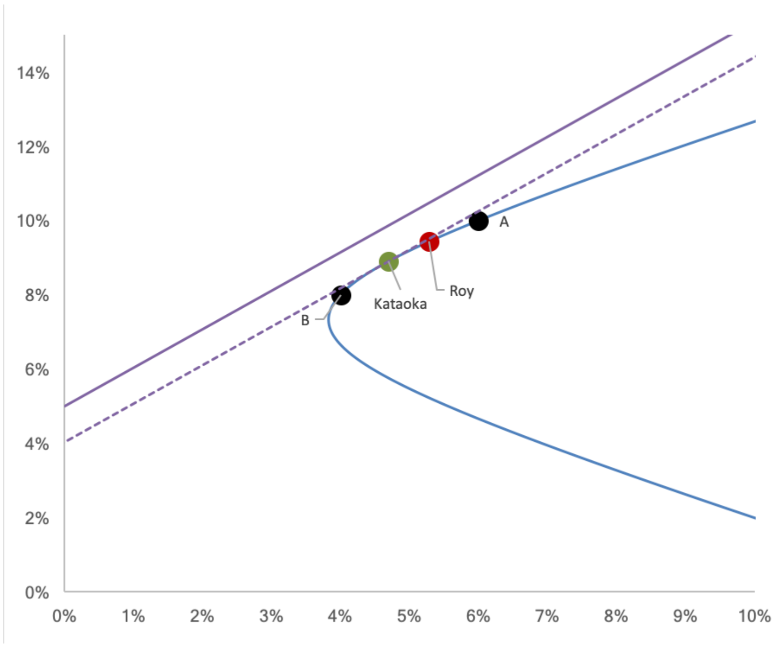

Figure 8: Exercise 1.14 – No Telser portfolio feasible. Representation of the two efficient portfolios A and B and the Roy and Kataoka portfolios.

Exercise 1.15. To be done!

1.4 International Diversification

Exercise 1.16. Diversification means combine different assets with different risk profiles such that we can manage to decrease our risk exposure while maintaining our return. Of course, diversification is only possible if the assets in the portfolio are not perfectly positively correlated ($\rho = 1$). Actually, the most idyllic scenario would perfectly negatively correlation ($\rho = -1$) among assets since it would allow us to cancel an important portion of portfolio's risk: the specific or idiosyncratic risk. Let,

$$\sigma_P^2 = \sum_{i=1}^{N} x_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} x_i x_j \sigma_{ij}$$

If $x_i = \frac{1}{N}$ then

$$\sigma_P^2 = \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{1}{N}\right) \left(\frac{1}{N}\right) \sigma_{ij}$$

Factoring out $1/N$ from the first summation and $(N-1)/N$ from the second and simplifying yields

$$\sigma_P^2 = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\sigma_i^2}{N}\right] + \frac{(N-1)}{N}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\frac{\sigma_{ij}}{N(N-1)}\right]$$
$$= \frac{1}{N}\bar{\sigma}_i^2 + \frac{N-1}{N}\bar{\sigma_{ij}}$$

This is a quite realistic representation of what occur when we invest in a portfolio of assets. The contribution to the portfolio variance of the variance of the individual securities goes to zero as N gets very large. However, the contribution of the covariance terms approaches the average covariance as N gets large. Actually, if we let $N \to \infty$, it cames

$$\sigma_P^2 = \frac{1}{N}\left(\bar{\sigma}_i^2 - \bar{\sigma}_{ij}\right) + \bar{\sigma}_{ij}$$

Thus, as said before, the individual risk of securities can be diversified ways. Of course the higher the number os securities in the portfolio, the better the diversification. If we only consider a domestic market, the available number of tradable securities is lower than when we also consider external markets. Therefore, the major effect of diversification is to allow for a better diversification. However, this is at a price, which is exchange rate risk.

Exercise 1.17.

(a) The return due to exchange-rate changes (R_X) is equal to $fx_t/fx_{t-1} - 1$, where fx_t is the foreign exchange rate at time t expressed in terms of the investor's home currency per unit of foreign currency. Let fx_t be the exchange rate expressed in terms of dollars and fx_t^* be the exchange rate expressed in terms of pounds. These two rates are simply reciprocals, i.e., $fx_t^* = 1/fx_t$. So from the table in the problem we have:

Period	$(1+R_X)$ (for US investor)	$(1+R_X^*)$ (for UK investor)
1	$2.5/3 = 0.833$	$3/2.5 = 1.200$
2	$2.5/2.5 = 1.000$	$2.5/2.5 = 1.000$
3	$2/2.5 = 0.800$	$2.5/2 = 1.250$
4	$1.5/2 = 0.750$	$2/1.5 = 1.333$
5	$2.5/1.5 = 1.667$	$1.5/2.5 = 0.600$

The total return to a U.S. investor from a U.K. investment is

$$(1+R_{US}) = (1+R_x)(1+R_{UK})$$

And the total return to a U.K. investor from a U.S. investment is

$$(1+R_{UK}) = (1+R_x)(1+R_{US})$$

So,

– Return to US investor

Period	From US investment	From UK investment
1	10%	$(0.833)(1.05) - 1 = 12.5\%$
2	15%	$(1)(0.95) - 1 = 5.0\%$
3	−5%	$(0.8)(1.15) - 1 = 8.0\%$
4	12%	$(0.75)(1.08) - 1 = 19.0\%$
5	6%	$(1.667)(1.1) - 1 = 83.3\%$
Average	7.6%	7.76%

– Return to UK investor

Period	From UK investment	From US investment
1	5%	$(1.2)(1.1) - 1 = 32.0\%$
2	−5%	$(1)(1.15) - 1 = 15.0\%$
3	15%	$(1.25)(0.95) - 1 = 18.75\%$
4	8%	$(1.333)(1.12) - 1 = 49.3\%$
5	10%	$(0.6)(1.06) - 1 = -36.4\%$
Average	6.6%	15.73%

(b) The standard deviation of return is given by

$$\sigma = \sqrt{\sum_{i=1}^{N} \frac{\left(R_i - \bar{R}_i\right)^2}{N}}$$

Thus,

– For US investor

$$\begin{aligned}\sigma_{US} &= \sqrt{\frac{(10-7.6)^2 + (15-7.6)^2 + (-5-7.6)^2 + (12-7.6)^2 + (6-7.6)^2}{5}} \\ &= 6.95\% \\ \sigma_{UK} &= \sqrt{\frac{(-12.5-7.76)^2 + (-5-7.76)^2 + (-8-7.76)^2 + (-19-7.76)^2 + (83.3-7.76)^2}{5}} \\ &= 38.06\%\end{aligned}$$

– For UK investor

$$\begin{aligned}\sigma_{UK} &= \sqrt{\frac{(5-6.6)^2 + (-5-6.6)^2 + (15-6.6)^2 + (8-6.6)^2 + (10-6.6)^2}{5}} \\ &= 6.65\% \\ \sigma_{US} &= \sqrt{\frac{(32-15.73)^2 + (15-15.73)^2 + (18.75-15.73)^2 + (49.3-15.73)^2 + (-36.4-15.73)^2}{5}} \\ &= 38.06\%\end{aligned}$$

Exercise 1.18. In general, we should hold non-domestic (N) securities instead of domestic securities(D) when foreign investment is more attractive than domestic investment. What happens when the following inequality holds

$$\frac{\bar{R}_N - R_F}{\sigma_N} > \frac{\bar{R}_D - R_F}{\sigma_D}\rho_{N,D}$$

Specifically, for an US investor

$$\frac{\bar{R}_N - R_F}{\sigma_N} > \frac{\bar{R}_{US} - R_F}{\sigma_{US}}\rho_{N,US}$$

$\bar{R}_{US}$ and $\bar{R}_N$, σ_N and $\sigma_{N,US}$ for the foreign countries are given in the problem and summarized below:

	$\bar{R}_N$ (%)	σ_N	$\sigma_{N,US}$
Austria	14	24.50	0.281
France	16	17.76	0.534
Japan	14	25.70	0.348
UK	15	15.59	0.646

We also know that $\bar{R}_{US} = 20\%$, $\sigma_{US} = 13.59$ and $R_F = 6\%$. Thus, we have

	$\frac{\bar{R}_N - R_F}{\sigma_N}$	$\frac{R_{US} - R_F}{\sigma_{US}} \rho_{N,US}$
Austria	0.327	0.289
France	0.563	0.550
Japan	0.311	0.358
UK	0.577	0.665

For Austria and France, the above inequality holds, so a US investor should consider those foreign markets as attractive investments; for Japan and the UK, the above inequality does not hold, so a US investor should not consider those foreign markets as attractive investments.

Exercise 1.19. The formula to find the minimum-risk portfolio of two assets is get by taking the first derivative of the portfolio variance to X_1 and equal 0, which gives

$$X_1^{mvp} = \frac{\sigma_2^2 - \sigma_1 \sigma_2 \rho_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho_{1,2}}$$

where X_1 is the investment weight for asset 1 and $X_2 = 1 - X_1$.

(a) For equities, $\sigma_{US} = 13.59$, $\sigma_N = 16.70$ and $\rho_{N,US} = 0.423$. So the minimum-variance portfolio is:

$$X_{US}^{mvp} = \frac{19.0^2 - 15.39 \times 19.0 \times 0.423}{15.39^2 + 19.0^2 - 2 \times 15.39 \times 19.0 \times 0.423} = 0.6771$$
$$X_N^{mvp} = 1 - X_{US}^{mvp} = 0.3266$$

(b) For bonds, $\sigma_{US} = 6.92$, $\sigma_N = 12.875$ and $\rho_{N,US} = 0.527$. So the minimum-variance portfolio is:

$$X_{US}^{mvp} = \frac{12.875^2 - 6.92 \times 12.875 \times 0.527}{6.92^2 + 12.875^2 - 2 \times 6.92 \times 12.875 \times 0.527} = 0.9924$$
$$X_N^{mvp} = 1 - X_{US}^{mvp} = 0.0076$$

(c) For T-bills, $\sigma_{US} = 1.068$, $\sigma_N = 10.057$ and $\rho_{N,US} = -0.220$. So the minimum-variance portfolio is:

$$X_{US}^{mvp} = \frac{10.057^2 + 1.068 \times 10.057 \times 0.220}{1.068^2 + 10.057^2 + 2 \times 1.068 \times 10.057 \times 0.220} = 0.9673$$
$$X_N^{mvp} = 1 - X_{US}^{mvp} = 0.0327$$

2 Portfolio Selection Models

2.1 Constant Correlation Model

Exercise 2.1.

(a)-(b) The only assumption of the Constant Correlation Model is that the correlation between any pair of securities is constant, such that $\rho_{ij} = \rho^* \, \forall i, j$. This is an unrealistic assumption that may lead to introduction of model risk. On the other hand, it allows us to decrease the number of parameters one needs to estimate to use MVT. So, the use of CCM may lead to a considerable reduction in estimation risk. It also allows us to use cut-off methods to find tangent portfolios.

Exercise 2.2.

(a) Yes, since all pairwise correlations are the same, this is the ideal scenario to use CCMs. In this case we have zero model risk.

(b) If short sales are allowed, all securities will be included in the optimal portfolio. Assuming constant correlation we can apply the cut-off method that consists in

1. Rank all securities accordingly to Sharpe's Ratio
2. Calculate the Cut-Off point
3. Compute Zs and the weights Xs.

In Table 1 below, given that the riskless rate equals 4%, the securities are ranked in descending order by their excess return over standard deviation. To calculate the cut-off point C^* we need a general expression that give us C_i. This expression is

$$C_i = \frac{\rho}{1 - \rho + i\rho} \sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$$

where ρ is the correlation coefficient - assumed constant for all securities. The subscript i indicates that C_i is calculated, using data on the first i securities. Each C_i is calculated as follows

$$C_1 = \frac{\rho}{1 - \rho + 1\rho} \sum_{i=1}^{1} \frac{\bar{R}_{10} - R_F}{\sigma_1 0} = \frac{0.5}{1 - 0.5 + 1 \times 0.5} \times \frac{12 - 4}{2} = 2$$

$$C_2 = \frac{\rho}{1 - \rho + 2\rho} \sum_{i=1}^{2} \frac{\bar{R}_3 - R_F}{\sigma_3} = \frac{0.5}{1 - 0.5 + 2 \times 0.5} \left(\frac{12 - 4}{2} + \frac{12 - 4}{4} \right) = 2$$

$$\vdots$$

Since short-sales are allowed, we include all securities, which implies that the cut-off rate is given by the C rate of the last security. In this exercise, $C^* = C^{10^{th}} = 1.41$.

The last step to find the optimal portfolio is to calculate Zs, which is given by

$$Z_i = \frac{1}{(1 - \rho)\,\sigma_i} \left(\frac{\bar{R}_i - R_F}{\sigma_i} - C^* \right)$$

Security	Rank i	$\bar{R}_i - R_F$	$\frac{\bar{R}_i - R_F}{\sigma_i}$	$\sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	C	Z_i	X_i
10	1	8	4.00	4.00	0.50	2.00	2.59	189.22%
3	2	8	2.00	6.00	0.33	2.00	0.30	21.68%
6	3	5	1.67	7.67	0.25	1.92	0.17	12.69%
9	4	6	1.50	9.17	0.20	1.83	0.05	3.44%
4	5	10	1.43	10.6	0.17	1.77	0.01	0.48%
1	6	6	1.20	11.8	0.14	1.69	-0.08	−6.00%
5	7	2	1.00	12.8	0.13	1.60	-0.41	−29.59%
7	8	1	1.00	13.8	0.11	1.53	-0.81	−59.17%
8	9	4	1.00	14.8	0.10	1.48	-0.20	−14.79%
2	10	4	0.67	15.47	0.09	1.41	-0.25	−17.97%

Table 1: Exercise 3.2 - Efficient Portfolio

Then,

$$Z_1 = \frac{1}{(1-\rho)\,\sigma_{10}}\left(\frac{\bar{R}_{10} - R_F}{\sigma_{10}} - C^*\right) = \frac{1}{(1-0.5)\,2}\left(\frac{12-4}{2} - 1.41\right) = 2.59$$
$$Z_2 = \frac{1}{(1-\rho)\,\sigma_3}\left(\frac{\bar{R}_3 - R_F}{\sigma_3} - C^*\right) = \frac{1}{(1-0.5)\,4}\left(\frac{12-4}{4} - 1.41\right) = 0.30$$
$$\vdots$$

Finally, to find the weights Xs and since $X_i = \frac{Z_i}{\sum_{i=1}^{N} Z_i}$, we have

$$X_1 = \frac{Z_1}{\sum_{i=1}^{10} Z_i} = \frac{2.59}{1.37} = 1.8922$$
$$X_2 = \frac{Z_2}{\sum_{i=1}^{10} Z_i} = \frac{0.3}{1.37} = 0.2168$$
$$\vdots$$

Table 1 presents all previous calculations and the efficient portfolio, T^A.

(e) The efficient portfolio, T^A found in part (b) is the unique efficient portfolio we have with a risk-free rate of 4%, being the tangent portfolio between the capital market line and the efficient frontier of risky assets. Applying the formulas for portfolio's return and risk, we have $R_T = 18.907\%$ and $\sigma_T = 3.297\%$. Now, we can draw the capital market line, which is the efficient frontier in this case (see Figure 9)

Exercise 2.3.

(a) The efficient frontier is the line from R_F and is tangent to the efficient frontier of risky assets. It is similar to Figure 9.

(b)

(i) In Table 2, given that the riskless rate equals 5%, the securities are ranked in descending order by their excess return over standard deviation. To calculate the

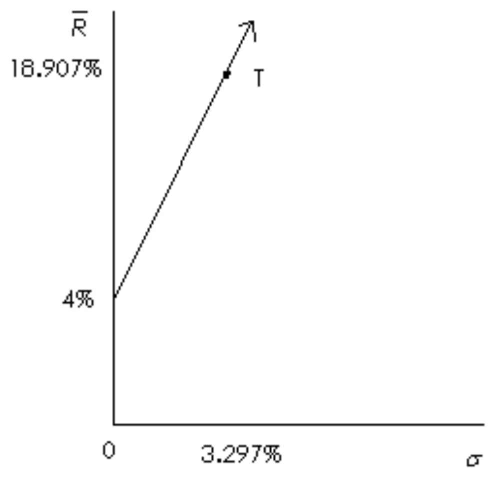

Figure 9: Exercise 2.2 - Efficient Frontier

cut-off point C^* we need a general expression that give us C_i. This expression is

$$C_i = \frac{\rho}{1-\rho+i\rho}\sum_{i=1}^{N}\frac{\bar{R}_i - R_F}{\sigma_i}$$

where ρ is the correlation coefficient - assumed constant for all securities. The subscript i indicates that C_i is calculated, using data on the first i securities. Each C_i is calculated as follows

$$C_1 = \frac{\rho}{1-\rho+1\rho}\sum_{i=1}^{1}\frac{\bar{R}_1 - R_F}{\sigma_1} = \frac{0.5}{1-0.5+1\times 0.5}\times\frac{15-5}{10} = 0.5$$

$$C_2 = \frac{\rho}{1-\rho+2\rho}\sum_{i=1}^{2}\frac{\bar{R}_2 - R_F}{\sigma_2} = \frac{0.5}{1-0.5+2\times 0.5}\left(\frac{15-5}{10}+\frac{20-5}{15}\right) = 0.6667$$

$$\vdots$$

With no short sales, we only include those securities for which $\frac{\bar{R}_i - R_F}{\sigma_i} > C_i$. Thus, only securities 1, 2, 5 and 6 (the four highest ranked securities in the above table) are in the optimal (tangent) portfolio. We could have stopped our calculations after the first time we found a ranked security for which $\frac{\bar{R}_i - R_F}{\sigma_i} < C_i$, (in this case the fifth highest ranked security, security 4), but we did not so that we could demonstrate that $\frac{\bar{R}_i - R_F}{\sigma_i} < C_i$ for all of the remaining lower ranked securities as well.
Since security 6 (the fourth highest ranked security, where i = 4) is the last ranked security in descending order for which $\frac{\bar{R}_i - R_F}{\sigma_i} > C_i$, we set $C^* = C_4 = 0.78$
The last step to find the optimal portfolio is to calculate Zs, which is given by

$$Z_i = \frac{1}{(1-\rho)\,\sigma_i}\left(\frac{\bar{R}_i - R_F}{\sigma_i} - C^*\right)$$

Security	Rank i	$\bar{R}_i - R_F$	$\frac{\bar{R}_i - R_F}{\sigma_i}$	$\sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	C	Z_i	X_i
1	1	10	1.00	1.00	0.5000	0.5000	0.0440	0.2375
2	2	15	1.00	2.00	0.3333	0.6667	0.0293	0.1581
5	3	5	1.00	3.00	0.2500	0.7500	0.0880	0.4749
6	4	9	0.90	3.90	0.2000	0.7800	0.0240	0.1295
4	5	7	0.70	4.60	0.1667	0.7668	-	-
3	6	13	0.65	5.25	0.1429	0.7502	-	-
7	7	11	0.55	5.80	0.1250	0.7250	-	-

Table 2: Exercise 3.3.b.i - Efficient Portfolio (short-selling not allowed)

Then,

$$Z_1 = \frac{1}{(1-\rho)\,\sigma_1}\left(\frac{\bar{R}_1 - R_F}{\sigma_1} - C^*\right) = \frac{1}{(1-0.5)\,10}\left(\frac{15-5}{10} - 0.78\right) = 0.0440$$
$$Z_2 = \frac{1}{(1-\rho)\,\sigma_2}\left(\frac{\bar{R}_2 - R_F}{\sigma_2} - C^*\right) = \frac{1}{(1-0.5)\,15}\left(\frac{20-5}{15} - 0.78\right) = 0.0293$$
$$\vdots$$

Finally, to find the weights Xs and since $X_i = \frac{Z_i}{\sum_{i=1}^{N} Z_i}$, we have

$$X_1 = \frac{Z_1}{\sum_{i=1}^{4} Z_i} = \frac{0.0440}{0.1853} = 0.2375$$
$$X_2 = \frac{Z_2}{\sum_{i=1}^{4} Z_i} = \frac{0.0293}{0.1853} = 0.1581$$
$$\vdots$$

Table 2 presents all previous calculations and the efficient portfolio without short-selling. Since $i = 1$ for security 1, $i = 2$ for security 2, $i = 3$ for security 5 and $i = 4$ for security 6, the tangent portfolio when short sales are not allowed consists of 23.75% invested in security 1, 15.81% invested in security 2, 47.49% invested in security 5 and 12.95% invested in security 6.

(ii) When short-selling is allowed, we set the cut-off rate to $C^* = 0.725$ ir order to include all securities in our efficient portfolio (see Table 3). The Zs and the weights Xs are calculated as before. However you should notice that Lintner Definition of short sales implies $\sum_{i=1}^{N} |Z_i|$. Thus,

$$Z_1 = \frac{1}{(1-\rho)\,\sigma_1}\left(\frac{\bar{R}_1 - R_F}{\sigma_1} - C^*\right) = \frac{1}{(1-0.5)\,10}\left(\frac{15-5}{10} - 0.725\right) = 0.0550$$
$$\vdots$$
$$Z_5 = \frac{1}{(1-\rho)\,\sigma_2}\left(\frac{\bar{R}_2 - R_F}{\sigma_2} - C^*\right) = \frac{1}{(1-0.5)\,10}\left(\frac{12-5}{10} - 0.725\right) = -0.0050$$
$$\vdots$$

Security	Rank i	$\bar{R}_i - R_F$	$\frac{\bar{R}_i - R_F}{\sigma_i}$	$\sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	C	Z_i	X_i
1	1	10	1.00	1.00	0.5000	0.5000	0.0550	0.2062
2	2	15	1.00	2.00	0.3333	0.6667	0.0367	0.1376
5	3	5	1.00	3.00	0.2500	0.7500	0.1100	0.4124
6	4	9	0.90	3.90	0.2000	0.7800	0.0350	0.1312
4	5	7	0.70	4.60	0.1667	0.7668	-0.0050	-0.0187
3	6	13	0.65	5.25	0.1429	0.7502	-0.0075	-0.0281
7	7	11	0.55	5.80	0.1250	0.7250	-0.0175	-0.0656

Table 3: Exercise 3.3.b.ii - Efficient Portfolio (short-selling allowed - Lintner Definition)

Security	Rank i	$\bar{R}_i - R_F$	$\frac{\bar{R}_i - R_F}{\sigma_i}$	$\sum_{i=1}^{N} \frac{\bar{R}_i - R_F}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	C	Z_i	X_i
1	1	10	1.00	1.00	0.5000	0.5000	0.0550	0.2661
2	2	15	1.00	2.00	0.3333	0.6667	0.0367	0.1776
5	3	5	1.00	3.00	0.2500	0.7500	0.1100	0.5322
6	4	9	0.90	3.90	0.2000	0.7800	0.0350	0.1703
4	5	7	0.70	4.60	0.1667	0.7668	-0.0050	-0.0242
3	6	13	0.65	5.25	0.1429	0.7502	-0.0075	-0.0363
7	7	11	0.55	5.80	0.1250	0.7250	-0.0175	-0.0847

Table 4: Exercise 3.3.b.iii - Efficient Portfolio (short-selling allowed - Standard Definition)

And

$$X_1 = \frac{Z_1}{\sum_{i=1}^{7} |Z_i|} = \frac{0.05500}{0.2667} = 0.2062$$

$$\vdots$$

$$X_5 = \frac{Z_2}{\sum_{i=1}^{7} |Z_i|} = \frac{-0.0050}{0.2667} = -0.0187$$

$$\vdots$$

Table 3 presents all previous calculations and the efficient portfolio with short-selling (Lintner definition).

(iii) Again, we want to calculate the efficient portfolio allowing short-sales, but this time using the standard definition that states $\sum_{i=1}^{N} Z_i$. In this exercise $\sum_{i=1}^{N} Z_i = 0.2061$. Therefore, we proceed just as before arriving to Table 4.

(c) If the risk-free asset does not exist, their are an infinite number of efficient portfolios of risky assets. Determine all these portfolios imply the calculation of the efficient frontier, which can be done using pretty sophisticated matricial equations, which are outside the scope of this course. Nevertheless, we have a different and easier way to do this calculation. We just need to assume the existence of a fictitious risk-free rate of return to find an efficient portfolio. Then we assume a second fictitious frontier to have a second efficient portfolio. Now, with these two portfolios we can find any other portfolio applying the Efficient Portfolios Theorem and we can, also, derive the representative equation of the hyperbole that corresponds to the efficient frontier.

2.2 Single-Index Model

Exercise 2.4.

(a) The β of Security A is lower than 1 therefore it is considered a defensive stock. On the other side, security B has a β higher than 1, so that it is an aggressive stock.

(b) (i) To compute the portfolio's β we proceed as follows

$$\beta_p = \sum x_i \beta_i \Leftrightarrow \beta_p = x_A\beta_A + x_B\beta_B = 0.25 \times 0.75 + 0.75 \times 2 = 1.6875$$

(ii) Using the single-index model (SIM), the portfolio's risk is

$$\sigma_p^2 = \beta_p^2\sigma_m^2 + \sum x_i^2\sigma_{ei}^2 = 1.6875^2 \times 0.25^2 + \left[(0.25)^2 \times 0.02 + (0.75)^2 \times 0.03\right]$$

$$\sigma_p^2 = 0.177978 + 0.018125 = 0.1961 \qquad \sigma_p = \sqrt{0.1961} = 44.28\%.$$

(c) A portfolio with A and B, which risk equals the market risk is a portfolio, which risk equals the market risk, thus $\sigma_p^2 = \beta_p^2\sigma_m^2 = 0.25^2 = 0.0625$. To calculate the weight of stock A (X_A)we need to solve the portfolio variance equation in order to X_A. To do so we us first need to compute the return's variance for stock A and B and the covariance between this returns using the single-index model:

$$\sigma_A^2 = \beta_A^2\sigma_m^2 + \sigma_{eA}^2 = 0.75^2 \times 0.25^2 + 0.02 = 0.0552$$

$$\sigma_B^2 = \beta_B^2\sigma_m^2 + \sigma_{eB}^2 = 2^2 \times 0.25^2 + 0.03 = 0.28$$

$$\sigma_{AB} = \beta_A \times \beta_B \times \sigma_M^2 = 0.75 \times 2 \times 0.25^2 = 0.09375$$

Then,

$$\begin{aligned}
\sigma_p^2 &= X_A^2\sigma_A^2 + (1-X_A)^2\sigma_B^2 + 2X_A(1-X_A) \times \sigma_{AB}^2 \\
0.25^2 &= 0.0552X_A^2 + 0.28(1-X_A)^2 + 2 \times 0.75 \times 2 \times 0.25^2 X_A(1-X_A) \\
0.0625 - 0.028 &= (0.0552 + 0.28 - 2 \times 0.09375)X_A^2 + 2 \times (0.09375 - 0.28)X_A \\
0 &= 0.1477X_A^2 - 0.3725X_A + 0.2175 \\
X_A &= \frac{0.3725 \pm \sqrt{0.3724^2 - 4 \times 0.1477 \times 0.2175}}{2 \times 0.1477} \Leftrightarrow X_A = 160.39\% \vee X_A = 91.85\%
\end{aligned}$$

Their are two possible solutions to X_A, nevertheless just one makes sense, since just one is efficient. Such solution is $X_A = 91.85\%$. The β of this portfolio is $\beta_P = X_A\beta_A + (1 - X_A)\beta_B = 0.9185 \times 0.75 + 0.0815 \times 2 \approx 0.85$.

(d) In part (c) we calculated the stocks variance using SIM. When we compare these results with the new data we realize that $\sigma_A^{2SIM} = 0.0552 \neq 0.1$ and $\sigma_B^{2SIM} = 0.28 \approx 0.3$. Thus, the SIM does not seems to hold when we use it with stock A, despite it seems to be a good approximation when applied to stock B.

Exercise 2.5.

(a) The covariance between stock B and the market portfolio is $\sigma_{BM} = \beta_B\beta_M\sigma_M^2 = 1.125 \times 1 \times 0.4^2 = 0.18$

(b) If the Single Index Model (SIM) holds, the portfolio variance is as follows

$$\sigma_p^2 = \underbrace{\beta_p^2\sigma_m^2}_{\text{systematic variance}} + \underbrace{\sum_{i=1}^{n} x_i^2\sigma_{\varepsilon i}^2}_{\text{residual variance}}$$

Thus, the residual variance in this homogenous portfolio (in a homogenous portfolio each security weight is given by $1/N$, where N is the number of securities, in this case $X_i = 1/2 = 0.5$) is

$$\sigma_{ep}^2 = \sum_{i=1}^{n} x_i^2\sigma_{\varepsilon i}^2 = 0.5^2 \times 0.1 + 0.5^2 \times 0.15 = 0.0625$$

(c) Since the covariance between the residual variances of security A and B are not zero, the single-index model does not apply. Therefore, the residual variance calculated in part b is not the effective residual variance of a homogeneous portfolio, which is given by the modern portfolio's theory. Thus, for two securities, the variance is

$$\begin{aligned}\sigma_{e_P}^2 &= x_A^2\sigma_{e_B}^2 + x_B^2\sigma_{e_B}^2 + 2x_Ax_B\sigma_{e_Ae_B} \\ &= 0.5^2 \times 0.1 + 0.5^2 \times 0.15 + 2 \times 0.5 \times 0.5 \times 0.1 \\ &= 0.1125\end{aligned}$$

(d) As seen in part b, the systematic risk, under SIM, is $\sigma_{e_{Syst}}^2 = \beta_p^2\sigma_m^2$ and $\beta_p = \sum_{i=1}^{2} x_i\beta_i = 0.5 \times 0.875 + 0.5 \times 1.125 = 1$. Thus,

$$\sigma_{e_{Syst}}^2 = \beta_p^2\sigma_m^2 = 1^2 \times 0.4^2 = 0.16$$

(e) (i) Total risk for each individual security calculated with SIM or with Portfolio Theory is the same as long as SIM's assumptions hold, namely that $\sigma_{e_iM} = 0$. In this case nothing is said about this, therefore anything definitive can be said.

(ii) In the general case, total risk for a portfolio computed under SIM or Markowitz assumptions is the same, as long as SIM's assumptions hold, namely that $\sigma_{e_ie_j} = 0$. However, this is not the case when we use securities A and B to construct a portfolio, since $\sigma_{e_Ae_B} = 0.1$. Actually, under Markowitz total variance is $\sigma_p^2 = \beta_p^2\sigma_m^2 + x_A^2\sigma_{e_B}^2 + x_B^2\sigma_{e_B}^2 + 2x_Ax_B\sigma_{e_Ae_B} = 0.16 + 0.1125 = 0.2725$ and under SIM total variance is $\sigma_p^2 = \beta_p^2\sigma_m^2 + \sum_{i=1}^{n} x_i^2\sigma_{\varepsilon i}^2 = 0.16 + 0.0625 = 0.2225$. Thus, their total risk is also different.

Exercise 2.6.

(a) This exercise is based on the single-index model, more precisely in the market model, which a positive correlation between any given stock returns and the market returns, such that the return on a stock can be written as

$$R_i = a_i + \beta_i R_m$$

The term a_i represents that component of return insensitive to the return on the market, i.e. it represents specific risk. The term a_i can be broken into two components: $alpha_i$ that denotes the expected value for a_i; and ε_i representing the random element of a_i, which expected value is zero ($\mathbb{E}\varepsilon_i = 0$). Then $a_i = \alpha_i + \varepsilon_i$ and

$$R_i = \alpha_i + \beta_i R_m + \varepsilon_i$$

Note that both ε_i and R_m are random variables with standard deviations denoted by σ_{ε_i} and σ_m, respectively. The term $\beta_i R_m$ represent the systematic risk and measure how sensitivity the stock's return is to the market's return.

The model's main assumptions are:

- ε_i is uncorrelated with R_m, such that the model ability to explain stock returns is independent of what the return on the market happens to be. More formally $cov(\varepsilon_i R_m) = \mathbb{E}\left[(\varepsilon_i - 0)\left(R_m - \bar{R}_m\right)\right] = 0$
- ve_i is independent of e_j for all values of i and j. which implies that the only reason stocks vary together, systematically, is because of a common co-movement with the market. More formally $\mathbb{E}(\varepsilon_i \varepsilon_j) = 0$

(b) (i) The expected return is given by $\bar{R}_i = a_i + \beta_i \bar{R}_m$. Thus,

$$\begin{cases} \bar{R}_A = a_A + \beta_A \bar{R}_m \\ \bar{R}_B = a_B + \beta_B \bar{R}_m \\ \bar{R}_C = a_C + \beta_C \bar{R}_m \\ \bar{R}_D = a_D + \beta_D \bar{R}_m \end{cases} \Leftrightarrow \begin{cases} \bar{R}_A = 2 + 1.5 \times 8 \\ \bar{R}_B = 3 + 1.3 \times 8 \\ \bar{R}_C = 1 + 0.8 \times 8 \\ \bar{R}_D = 4 + 0.9 \times 8 \end{cases} \Leftrightarrow \begin{cases} \bar{R}_A = 14 \\ \bar{R}_B = 13.4 \\ \bar{R}_C = 7.4 \\ \bar{R}_D = 11.2 \end{cases}$$

(ii) The security variance is given by $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2$. Therefore,

$$\begin{cases} \sigma_A^2 = \beta_A^2 \sigma_m^2 + \sigma_{\varepsilon_A}^2 \\ \sigma_B^2 = \beta_B^2 \sigma_m^2 + \sigma_{\varepsilon_B}^2 \\ \sigma_C^2 = \beta_C^2 \sigma_m^2 + \sigma_{\varepsilon_C}^2 \\ \sigma_D^2 = \beta_D^2 \sigma_m^2 + \sigma_{\varepsilon_D}^2 \end{cases} \Leftrightarrow \begin{cases} \sigma_A^2 = 1.5^2 \times 25 + 9 \\ \sigma_B^2 = 1.3^2 \times 25 + 1 \\ \sigma_C^2 = 0.8^2 \times 25 + 4 \\ \sigma_D^2 = 0.9^2 \times 25 + 16 \end{cases} \Leftrightarrow \begin{cases} \sigma_A^2 = 65.25 \\ \sigma_B^2 = 43.25 \\ \sigma_C^2 = 20 \\ \sigma_D^2 = 36.25 \end{cases}$$

(iii) The covariance is given by $\sigma_{ij} = \beta_i \beta_j \sigma_m^2$. Therefore,

$$\begin{cases} \sigma_{AB} = \beta_A \beta_B \sigma_m^2 \\ \sigma_{AC} = \beta_A \beta_C \sigma_m^2 \\ \sigma_{AD} = \beta_A \beta_D \sigma_m^2 \\ \sigma_{BC} = \beta_B \beta_C \sigma_m^2 \\ \sigma_{BD} = \beta_B \beta_D \sigma_m^2 \\ \sigma_{CD} = \beta_C \beta_D \sigma_m^2 \end{cases} \Leftrightarrow \begin{cases} \sigma_{AB} = 1.5 \times 1.3 \times 25 \\ \sigma_{AC} = 1.5 \times 0.8 \times 25 \\ \sigma_{AD} = 1.5 \times 0.9 \times 25 \\ \sigma_{BC} = 1.3 \times 0.8 \times 25 \\ \sigma_{BD} = 1.3 \times 0.9 \times 25 \\ \sigma_{CD} = 0.8 \times 0.9 \times 25 \end{cases} \Leftrightarrow \begin{cases} \sigma_{AB} = 48.75 \\ \sigma_{AC} = 30 \\ \sigma_{AD} = 33.75 \\ \sigma_{BC} = 26 \\ \sigma_{BD} = 29.25 \\ \sigma_{CD} = 18 \end{cases}$$

The covariance matrix Σ is

$$\begin{pmatrix} 65.25 & 48.75 & 30 & 33.75 \\ 48.75 & 43.25 & 26 & 29.25 \\ 30 & 26 & 20 & 18 \\ 33.75 & 29.25 & 36.25 & 16 \end{pmatrix}$$

(c) A homogenous portfolio is a portfolio where each security weight is given by $1/n$, where n denotes the number of security. Now, $n = 4$, thus each security weight is $1/4 = 0.25$.

(i) The portfolio's β is the weighted average β of all securities

$$\beta_P = \sum_{i=1}^{4} x_i \beta_i = 1.5 \times 0.25 + 1.3 \times 0.25 + 0.8 \times 0.25 + 0.9 \times 0.25 = 1.125$$

(ii) Like β_P, α_P is given by the weighted average α of all securities

$$\alpha_P = \sum_{i=1}^{4} x_i \alpha_i = 2 \times 0.25 + 3 \times 0.25 + 1 \times 0.25 + 4 \times 0.25 = 2.5$$

(iii) The portfolio's variance is $\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{4} x_i^2 \sigma_{\varepsilon_i}^2$. Thus,

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{4} x_i^2 \sigma_{\varepsilon_i}^2 = 1.125^2 \times 25 + \left(9 \times 0.25^2 + 1 \times 0.25^2 + 4 \times 0.25^2 + 16 \times 0.25^2\right) = 33.52$$

(iv) To find the portfolio's expected return we apply the market model using the portfolio's α and β. Therefore,

$$\bar{R}_P = \alpha_P + \beta_P \bar{R}_m = 2.5 + 1.125 \times 8 = 11.5$$

(d) Using the suggested adjustment to find the β of the following period, we have

$$\beta_{2A} = 0.343 + 0.677\beta_{1A} = 0.343 + 0.677 \times 1.5 = 1.3585$$

$$\beta_{2B} = 0.343 + 0.677\beta_{1B} = 0.343 + 0.677 \times 1.3 = 1.2231$$

$$\beta_{2C} = 0.343 + 0.677\beta_{1C} = 0.343 + 0.677 \times 0.8 = 0.8846$$

$$\beta_{2D} = 0.343 + 0.677\beta_{1D} = 0.343 + 0.677 \times 0.9 = 0.9523$$

(e) Applying the Vasiček technique with the provided data and knowing the Vasiček β is given by

$$\beta_{2i} = \frac{\sigma_{\beta_{1i}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1i}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1i}}^2} \beta_{1i}$$

we have

$$\beta_{2A} = \frac{\sigma_{\beta_{1A}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1A}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1A}}^2} \beta_{1A} = \frac{0.0441}{0.0441 + 0.00625} \cdot 1 + \frac{0.0625}{0.0441 + 0.0625} \cdot 1.5 = 1.2932$$

$$\beta_{2B} = \frac{\sigma_{\beta_{1B}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1B}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1B}}^2} \beta_{1B} = \frac{0.1024}{0.1024 + 0.0625} \cdot 1 + \frac{0.0625}{0.1024 + 0.0625} \cdot 1.3 = 1.1137$$

$$\beta_{2C} = \frac{\sigma_{\beta_{1C}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1C}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1C}}^2} \beta_{1C} = \frac{0.0324}{0.0324 + 0.0625} \cdot 1 + \frac{0.0625}{0.0324 + 0.0625} \cdot 0.8 = 0.8683$$

$$\beta_{2D} = \frac{\sigma_{\beta_{1D}}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1D}}^2} \bar{\beta}_1 + \frac{\sigma_{\bar{\beta}_1}^2}{\sigma_{\bar{\beta}_1}^2 + \sigma_{\beta_{1D}}^2} \beta_{1D} = \frac{0.04}{0.04 + 0.0625} \cdot 1 + \frac{0.0625}{0.04 + 0.0625} \cdot 0.9 = 0.9390$$

Exercise 2.7.

(a) The covariance between any two securities can be written as

$$\sigma_{ij} = \mathbb{E}\left[\left(R_i - \bar{R}_i\right)\left(R_j - \bar{R}_j\right)\right]$$

Substituting for R_i, $\bar{R}_i$, R_j and $\bar{R}_j$ yields

$$\sigma_{ij} = \mathbb{E}\left\{\left[(\alpha_i + \beta_i R_m + \varepsilon_i) - \left(\alpha_i + \beta_i \bar{R}_m + \varepsilon_i\right)\right]\left[(\alpha_j + \beta_j R_m + \varepsilon_j) - \left(\alpha_j + \beta_j \bar{R}_m + \varepsilon_j\right)\right]\right\}$$

Simplifying by canceling the α's and combining the terms involving β's yields

$$\sigma_{ij} = \mathbb{E}\left\{\left[\beta_i\left(R_m - \bar{R}_m\right) + \varepsilon_i\right]\left[\beta_j\left(R_m - \bar{R}_m\right) + \varepsilon_j\right]\right\}$$

Carrying out the multiplication

$$\sigma_{ij} = \beta_i\beta_j\mathbb{E}\left(R_m - \bar{R}_m\right)^2 + \beta_j\mathbb{E}\left[\varepsilon_i\left(R_m - \bar{R}_m\right)\right] + \beta_i\mathbb{E}\left[\varepsilon_j\left(R_m - \bar{R}_m\right)\right] + \mathbb{E}\left(\varepsilon_i\varepsilon_j\right)$$

From the single-index model assumptions we know

$$\mathbb{E}\left(R_m - \bar{R}_m\right)^2 = \sigma_m^2$$

$$\mathbb{E}\left[\varepsilon_i\left(R_m - \bar{R}_m\right)\right] = 0$$

$$\mathbb{E}\left[\varepsilon_j\left(R_m - \bar{R}_m\right)\right] = 0$$

And from the data in the problem

$$k = cov\left(\varepsilon_i\varepsilon_j\right) = \mathbb{E}\left[\varepsilon_i\varepsilon_j\right] - \underbrace{\mathbb{E}\left[\varepsilon_i\right]}_{0}\underbrace{\mathbb{E}\left[\varepsilon_j\right]}_{0} = \mathbb{E}\left[\varepsilon_i\varepsilon_j\right]$$

Thus,

$$\sigma_{ij} = \beta_i\beta_j\sigma_m^2 + k$$

(b) The general equation for the portfolio variance is

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} X_iX_j\sigma_{ij} \tag{2}$$

From the Single-Index Model we know that

$$\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_{\varepsilon_i}^2 \tag{3}$$

and

$$\sigma_{ij} = \beta_i\beta_j\sigma_m^2$$

However, in this case, the covariance among the returns residuals is K and, therefore,

$$\sigma_{ij} = \beta_i\beta_j\sigma_m^2 + k \tag{4}$$

as calculates in part b. Applying (3) and (4) in (2) we get

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2\left(\beta_i^2\sigma_m^2 + \sigma_{\varepsilon_i}^2\right) + \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} X_iX_j\left(\beta_i\beta_j\sigma_m^2 + k\right)$$

Doing some transformations we finally have

$$\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{N} X_i^2\left(\beta_i^2\sigma_m^2 + \sigma_{\varepsilon_i}^2\right) + \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} X_iX_j\left(\beta_i\beta_j\sigma_m^2 + k\right)\\
&= \sum_{i=1}^{N} X_i^2\beta_i^2\sigma_m^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} X_iX_j\beta_i\beta_j\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{\varepsilon_i}^2 + + \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} X_iX_jk\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} X_iX_j\beta_i\beta_j\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{\varepsilon_i}^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} X_iX_jk\\
&= \underbrace{\left(\sum_{i=1}^{N} X_i\beta_i\right)}_{\beta_P}\underbrace{\left(\sum_{i=1}^{N} X_i\beta_i\right)}_{\beta_P}\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{\varepsilon_i}^2 + k\left(\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} X_iX_j\right)\\
&= \beta_P^2 + \sum_{i=1}^{N} X_i^2\sigma_{\varepsilon_i}^2 + k\left(\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} X_iX_j\right)
\end{aligned}$$

Exercise 2.8.

(a) This is a standard portfolio selection exercise, in which we have to choose the tangent portfolio between the capital market line and the efficient frontier of risky assets. The solution for this problem involves solving the following system of simultaneous equations in order to Z_i, $\forall i \geqslant 0$

$$\begin{cases} \bar{R}_1 - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13} + \cdots + Z_N\sigma_{1N} \\ \bar{R}_2 - R_F = Z_1\sigma_{21} + Z_2\sigma_2^2 + Z_3\sigma_{23} + \cdots + Z_N\sigma_{2N} \\ \bar{R}_3 - R_F = Z_1\sigma_{31} + Z_2\sigma_{32} + Z_3\sigma_3^2 + \cdots + Z_N\sigma_{3N} \\ \vdots \\ \bar{R}_N - R_F = Z_1\sigma_{NN}^2 + Z_2\sigma_{N2} + Z_3\sigma_{N3} + \cdots + Z_N\sigma_N^2 \end{cases}$$

which can be written using matricial notation

$$Z = V^{-1}\left(R - R_F 1\right)$$

where Σ^{-1} is the inverse covariance matrix, R is a column vector with the securities returns, R_F is a scalar and 1 is a column vector of 1s. The Zs are proportional to the optimum amount to invest in each security. Then the optimum proportions to invest in stock k is X_k, where

$$X_k = \frac{Z_k}{\sum_{i=1}^{N} Z_i}$$

Thus, we need to calculate the covariance matrix and then invert it. To find each pair of covariances we can use the variance and covariance definitions used in the Single-Index Model $\left(\sigma_i^2 = \beta^2\sigma_m^2 + \sigma_{\varepsilon_i}^2 \text{ and } \sigma_{ij} = \beta_i\beta_j\sigma_m^2\right)$. Thus, for security 1 and for the pair 1, 2 it comes

$$\sigma_i^2 = \beta^2\sigma_m^2 + \sigma_{\varepsilon_i}^2 = 1^2 \times 10 + 30 = 40$$

$$\sigma_{ij} = \beta_i\beta_j\sigma_m^2 = 1 \times 1.5 \times 10 = 15$$

Proceeding similarly for the other securities we arrive to the covariance matrix

$$V = \begin{pmatrix} 40 & 15 & 20 & 8 & 10 & 15 \\ 15 & 42.5 & 30 & 12 & 15 & 22.5 \\ 20 & 30 & 80 & 16 & 20 & 30 \\ 8 & 12 & 16 & 16.4 & 8 & 12 \\ 10 & 15 & 20 & 8 & 30 & 15 \\ 15 & 22.5 & 30 & 12 & 15 & 32.5 \end{pmatrix}$$

Then the inverse matrix is

$$V^{-1} = \begin{pmatrix} 0.0317 & -0.0037 & -0.0024 & -0.0039 & -0.0024 & -0.0073 \\ -0.0037 & 0.0418 & -0.0055 & -0.0089 & -0.0055 & -0.0164 \\ -0.0024 & -0.0055 & 0.0213 & -0.0058 & -0.0037 & -0.0110 \\ -0.0039 & -0.0088 & -0.0058 & 0.0907 & -0.0058 & -0.0175 \\ -0.0024 & -0.0055 & -0.0037 & -0.0058 & 0.0463 & -0.0110 \\ -0.0073 & -0.0164 & -0.0110 & -0.0175 & -0.0110 & 0.0671 \end{pmatrix}$$

And $\bar{R} - R_F 1$ is

$$\bar{R} - R_F 1 = \begin{pmatrix} 15 \\ 12 \\ 11 \\ 8 \\ 9 \\ 14 \end{pmatrix} - 5 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \\ 6 \\ 3 \\ 4 \\ 9 \end{pmatrix}$$

Finally,

$$
\begin{aligned}
Z &= \Sigma^{-1}(R - R_F 1) \\
&= \begin{pmatrix} 0.0317 & -0.0037 & -0.0024 & -0.0039 & -0.0024 & -0.0073 \\ -0.0037 & 0.0418 & -0.0055 & -0.0089 & -0.0055 & -0.0164 \\ -0.0024 & -0.0055 & 0.0213 & -0.0058 & -0.0037 & -0.0110 \\ -0.0039 & -0.0088 & -0.0058 & 0.0907 & -0.0058 & -0.0175 \\ -0.0024 & -0.0055 & -0.0037 & -0.0058 & 0.0463 & -0.0110 \\ -0.0073 & -0.0164 & -0.0110 & -0.0175 & -0.0110 & 0.0671 \end{pmatrix} \begin{pmatrix} 10 \\ 7 \\ 6 \\ 3 \\ 4 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} 0.18983 \\ 0.02711 \\ -0.06526 \\ -0.04410 \\ -0.01526 \\ 0.25422 \end{pmatrix}
\end{aligned}
$$

Since short sales are not allowed, we need to use the cut-off method to know how many securities will show up in the tangent portfolio.
We known we will not invest in any security with a negative "Z" and we may even not invest in some of the securities with positive Z.
In this case it turns out the optimal portfolio will have 3 securities, i.e., securities 1, 2 and 6 (which are the first three in the ranking and for which we have $Z_i > 0$). Thus, summing over the Zs from these three securities we get $\sum_{i=1}^{3} Z_i = 0.47116$ and the weights to invest in each security are

$$X_1 = \frac{0.18983}{0.47116} = 0.4029 \ , \ X_2 = \frac{0.02711}{0.47116} = 0.0575 \ , \ X_6 = \frac{0.25422}{0.47116} = 0.5396$$

(b) If short sales are allowed, using the standard definition, $\sum_{i=1}^{6} Z_i = 0.34623$ and the weights to invest in each security are

$$
\begin{aligned}
X_1 &= \frac{0.18983}{0.34623} = 0.5483 & X_2 &= \frac{0.02711}{0.34623} = 0.0783 \\
X_3 &= -\frac{0.06526}{0.34623} = -0.1885 & X_4 &= -\frac{0.0441}{0.34623} = -0.1283 \\
X_5 &= -\frac{0.01526}{0.34623} = -0.0441 & X_6 &= \frac{0.25422}{0.34623} = 0.7343
\end{aligned}
$$

Using Lintner definition, $\sum_{i=1}^{6} |Z_i| = 0.59609$ and the weights to invest in each security are

$$
\begin{aligned}
X_1 &= \frac{0.18983}{0.59609} = 0.3185 & X_2 &= \frac{0.02711}{0.59609} = 0.0485 \\
X_3 &= -\frac{0.06526}{0.59609} = -0.1095 & X_4 &= -\frac{0.0441}{0.59609} = -0.0745 \\
X_5 &= -\frac{0.01526}{0.59609} = -0.0256 & X_6 &= \frac{0.25422}{0.59609} = 0.4265
\end{aligned}
$$

(c) If the risk-free asset does not exist, their are an infinite number of efficient portfolios of risky assets. Determine all these portfolios imply the calculation of the efficient frontier, which can be done using pretty sophisticated matricial equations, which are outside the scope of this course. Nevertheless, we have a different and easier way to do this calculation. We just need to assume the existence of a fictitious risk-free rate of return to find an efficient portfolio. Then we assume a second fictitious frontier to have a second efficient portfolio. Now, with these two portfolios we can find any other portfolio applying the Efficient Portfolios Theorem and we can, also, derive the representative equation of the hyperbole that corresponds to the efficient frontier.

Exercise 2.9. We know β_i can be written as σ_{im}/σ_m^2. We also know that $\sigma_{im} = \rho_{im}\sigma_i\sigma_m$. Then,

$$\beta_i = \frac{\rho_{im}\sigma_i\sigma_m}{\sigma_m^2} = \frac{\rho_{im}\sigma_i}{\sigma_m} \tag{5}$$

Since we have constant correlation ρ^* between each pair of securities we should be to express ρ_{im} as a function of ρ^*. If the Single-Index Model holds, then $\sigma_{ij} = \beta_i\beta_j\sigma_m^2$ that can be rewritten as follows

$$\sigma_{ij} = \beta_i\beta_j\sigma_j^2 = \frac{\rho_{im}\sigma_i\sigma_m}{\sigma_m^2} \times \frac{\rho_{jm}\sigma_j\sigma_m}{\sigma_m^2} \times \sigma_m^2 = \rho_{im}\rho_{jm}\sigma_i\sigma_j$$

From statistics we have $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$. If we let correlations to be constant, then $\sigma_{ij} = \rho^*\sigma_i\sigma_j$. If correlations are constant and the Single-Index Model holds, we have

$$\rho^*\sigma_i\sigma_j = \rho_{im}\rho_{jm}\sigma_i\sigma_j$$

$$\rho^* = \rho_{im}\rho_{jm}$$

As the correlation is constant between each pair of securities we must have $\rho_{im} = \rho_{jm}$. Then,

$$\rho^* = \rho_{im}\rho_{im} = \rho_{im}^2$$

and,

$$\rho_{im} = \sqrt{|\rho^*|} \tag{6}$$

Finally, using (6) into (5), we have

$$\beta_i = \frac{\sqrt{|\rho_*|}}{\sigma_m}\sigma_i$$

So, if correlations are constant and equal to ρ^*, then, under the Single-Index Model assumptions, each security β is a proportion of its volatility. This proportion is constant and equal to all securities and defined as $\frac{\sqrt{|\rho_*|}}{\sigma_m}$.

Exercise 2.10. Accordingly to the Single-Index Model, the expected return and risk are given by

$$\bar{R}_i = \alpha_i + \beta_i\bar{R}_m$$

$$\sigma_i^2 = \underbrace{\beta_A^2\sigma_M^2}_{\text{Systematic Variance}} + \underbrace{\sigma_{\varepsilon_A}^2}_{\text{Specific Variance}}$$

$$\sigma_i = \sqrt{\beta_A^2\sigma_M^2 + \sigma_{\varepsilon_A}^2}$$

Therefore, the table can be filled using the equations. Notice that to calculate systematic risk we assume specific risk to be zero. On the other hand, when we calculate the specific risk we assume that systematic risk is zero.

Securities	Expected Return	Systematic Risk	Specific Risk	Total Risk
A	$\alpha_A + \beta_A\bar{R}_m$	$\sqrt{\beta_A^2\sigma_M^2}$	$\sqrt{\sigma_{\varepsilon_A}^2}$	$\sqrt{\beta_A^2\sigma_m^2 + \sigma_{\varepsilon_A}^2}$
B	$\alpha_B + \beta_B\bar{R}_m$	$\sqrt{\beta_B^2\sigma_M^2}$	$\sqrt{\sigma_{\varepsilon_B}^2}$	$\sqrt{\beta_B^2\sigma_m^2 + \sigma_{\varepsilon_B}^2}$
C	$\alpha_C + \beta_C\bar{R}_m$	$\sqrt{\beta_C^2\sigma_M^2}$	$\sqrt{\sigma_{\varepsilon_C}^2}$	$\sqrt{\beta_C^2\sigma_m^2 + \sigma_{\varepsilon_C}^2}$
Portfolio K	$\alpha_K + \beta_K\bar{R}_m$	$\sqrt{\beta_K^2\sigma_M^2}$	σ_{ε_K}	$\sqrt{\beta_K^2\sigma_m^2 + \sigma_{\varepsilon_K}^2}$

which give us

Securities	Expected Return	Systematic Risk	Specific Risk	Total Risk
A	$2 + 1.5 \times 20 = 32$	$\sqrt{1.5^2 \times 10^2} = 15$	4	$\sqrt{15^2 + 4^2} = 15.52$
B	$4 + 0.8 \times 20 = 20$	$\sqrt{0.8^2 \times 10^2} = 8$	3	$\sqrt{8^2 + 3^2} = 8.54$
C	$6 + 0.4 \times 20 = 14$	$\sqrt{0.4^2 \times 10^2} = 4$	2	$\sqrt{4^2 + 2^2} = 4.47$
Portfolio K	$4.6 + 0.74 \times 20 = 19.4$	$\sqrt{0.74^2 \times 10^2} = 7.4$	1.56	$\sqrt{7.4^2 + 1.56^2} = 7.56$

For the portfolio K, α_K, β_K and σ_{ε_K} are as follows

$$\alpha_K = \sum_{i=1}^{3} X_i \alpha_i = 2 \times 0.2 + 4 \times 0.3 + 6 \times 0.5 = 4.6$$

$$\beta_K = \sum_{i=1}^{3} X_i \beta_i = 1.5 \times 0.2 + 0.8 \times 0.3 + 0 - 4 \times 0.5 = 0.74$$

$$\sigma_{\varepsilon_K} = \sqrt{\sum_{i=1}^{3} X_i \sigma_{\varepsilon_i}} = \sqrt{4^2 \times 0.2^2 + 3^2 \times 0.3^2 + 2^2 \times 0.5^2} = 1.56$$

2.3 Multi-Index Model

Exercise 2.11. Let us start with multi-index model with 3 correlated indexes I_1^*, I_2^* and I_3^*:

$$R_i = a_i^* + b_{i1}^* \times I_1^* + b_{i2}^* \times I_2^* + b_{i3}^* \times I_3^* + c_i \tag{7}$$

To reduce a general three-index model to a three-index model with orthogonal indexes we need first to set $I_1^* = I_1$. Then, since I_1^* and I_2^* are correlated, we can express I_2^* in terms of I_1, defining an index I_2 which is orthogonal to I_1 as follows

$$I_2^* = \gamma_0 + \gamma_1 \times I_1 + d_t$$

The part from I_2^* that is independent of I_1^* and adds new information to it is given by the residuals in the linear regression, such that $I_2 = d_t$. Thus

$$I_2 = d_t = I_2^* - (\gamma_0 + \gamma_1 \times I_1)$$

$$I_2^* = \gamma_0 + \gamma_1 \times I_1 + I_2$$

Substituting the above expression into equation 7 and rearranging we get:

$$R_i = (a_i^* + b_{i2}^* \times \gamma_0) + (b_{i1}^* + b_{i2}^* \times \gamma_1) \times I_1 + b_{i2}^* \times I_2 + b_{i3}^* \times I_3^* + c_i$$

The first term in the above equation is a constant, which we can define as a_i' . The coefficient in the second term of the above equation is also a constant, which we can define as b_{i1}'. We can then rewrite the above equation as:

$$R_i = a_i' + b_{i1}' \times I_1 + b_{i2}^* \times I_2 + b_{i3}^* \times I_3^* + c_i \tag{8}$$

This model is equivalent to equation 7, but with two orthogonal indexes, I_1 and I_2, and a third index I_3^* that can be explained by I_1 and I_2, through a linear regression

$$I_3^* = \theta_0 + \theta_1 \times I_1 + \theta_2 \times I_2 + e_t$$

As before, all new information due to I_3^* is captured by the residuals e_t. Therefore,

$$I_3 = e_t = I_3^* - (\theta_0 + \theta_1 \times I_1 + \theta_2 \times I_2)$$

$$I_3^* = \theta_0 + \theta_1 \times I_1 + \theta_2 \times I_2 + I_3$$

Substituting the above expression into equation 8 and rearranging we get:

$$R_i = (a_i' + b_{i3} \times \theta_0) + (b_{i1}' + b_{i3} \times \theta_2) + (b_{i2}^* + b_{i3} * \theta_2) \times I_2 + b_{i3}^* \times I_3 + c_i$$

In the above equation, the first term and all the coefficients of the new orthogonal indices are constants, so we can rewrite the equation as follows, getting a three-index model with orthogonal indexes:

$$R_i = a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_i$$

Where $a_i = a_i^* + b_{i2}^* \times \gamma_0 + b_{i3} \times \theta_0$, $b_{i1} = b_{i1}^* + b_{i2}^* \times \gamma_1 + b_{i3}^* \times \theta_1$, $b_{i2} = b_{i2}^* + b_{i3}^* \times \theta_2$ and $b_{i3} = b_{i3}^*$.

Exercise 2.12.

(a) In a three-index model we have:

$$R_i = a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_i$$

Since $\mathbb{E}[C_i] = 0$, we

$$\mathbb{E}[R_i] = a_i + b_{i1} \times \mathbb{E}[I_1] + b_{i2} \times \mathbb{E}[I_2] + b_{i3} \times \mathbb{E}[I_3]$$

(b) To derive the variance we need to recall three assumptions of a multi-index model

1. the indexes are uncorrelated: $\mathbb{E}[I_i I_j] = \mathbb{E}[I_i]\,\mathbb{E}[I_j]$
2. the specific factors of each security are independent: $\mathbb{E}[c_i c_j] = 0$
3. For any security, each index factors are independent of the specific factors of that same security: $\mathbb{E}[I_i c_i] = 0$
4. $\mathbb{E}[c_i]^2 = \sigma_{ci}^2$

Now we can apply the variance formula $\sigma_i^2 = \mathbb{E}\left[\left(R_i - \bar{R}_i\right)^2\right]$, such that

$$\begin{aligned}\sigma_i^2 &= \mathbb{E}\left[\left(a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_i - \left(a_i + b_{i1} \times \bar{I}_1 + b_{i2} \times \bar{I}_2 + b_{i3} \times \bar{I}_3\right)\right)^2\right] \\ &= \mathbb{E}\left[\left(b_{1i}\left(I_1 - \bar{I}_1\right) + b_{2i}\left(I_2 - \bar{I}_2\right) + b_{3i}\left(I_3 - \bar{I}_3\right)\right)^2\right]\end{aligned}$$

Carrying out the squaring, noting that the indices are all orthogonal with each other and using the stated assumptions gives us

$$\sigma_i^2 = b_{i1}^2 \sigma_{I1}^2 + b_{i2}^2 \sigma_{I2}^2 + b_{i3}^2 \sigma_{I3}^2 + \sigma_{c_i}^2$$

(c) Here we apply exactly the same reasoning that we used in part b. Covariance is given by $\sigma_{ij} = \mathbb{E}\left[\left(R_i - \bar{R}_i\right)\left(R_j - \bar{R}_j\right)\right]$.Thus,

$$\begin{aligned}\sigma_{ij} &= \mathbb{E}\left[\begin{array}{c}\left(a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_i - \left(a_i + b_{i1} \times \bar{I}_1 + b_{i2} \times \bar{I}_2 + b_{i3} \times \bar{I}_3\right)\right) \times \\ \times \left(a_i + b_{i1} \times I_1 + b_{i2} \times I_2 + b_{i3} \times I_3 + c_j - \left(a_j + b_{j1} \times \bar{I}_1 + b_{j2} \times \bar{I}_2 + b_{j3} \times \bar{I}_3\right)\right)\end{array}\right] \\ &= \mathbb{E}\left[\begin{array}{c}\left(b_{1i}\left(I_1 - \bar{I}_1\right) + b_{2i}\left(I_2 - \bar{I}_2\right) + b_{3i}\left(I_3 - \bar{I}_3\right)\right) \times \\ \times \left(b_{1j}\left(I_1 - \bar{I}_1\right) + b_{2j}\left(I_2 - \bar{I}_2\right) + b_{3j}\left(I_3 - \bar{I}_3\right)\right)\end{array}\right]\end{aligned}$$

Carrying out the squaring, noting that the indices are all orthogonal with each other and using the stated assumptions gives us

$$\sigma_{ij} = b_{i1} b_{j1} \sigma_{I1}^2 + b_{i2} b_{j2} \sigma_{I2}^2 + b_{i3} b_{j3} \sigma_{I3}^2$$

Exercise 2.14. To build such model, we can use all kind of economic explanatory factors, such as, GDP growth rate, inflation rate, interest rate, or firms characteristics that proxies risk factors as size, book to market ratio, sales/equity ratio, price/earnings or a market factor. For example, Fama and French (1992 and 2003) developed in the early 90s a three factor model, whose factors were variables built to capture size, the relation between book-value and market-value and the market return. Earlier, late 80s, Burmeister, McElroy (1987 and 1988) and other found that five variables are sufficient to describe security returns: two variables were related to the discount rate used to find the present value of cash flows; one related to both size of the cash flows and discount rates; one related only to cash flows; and a remaining variable that captures the impact of the market not incorporated in the first four variables.

Exercise 2.15.

(a) By definition the risk-free asset does not have any risk, so that the sensitivity to risk factors must be zero. Thus, $b_{F1} = 0 \wedge b_{F2} = 0$

(b) From the presented two-index model we know the expected return of any security is

$$\bar{R}_i = a_i + b_{i1}\bar{R}_{I_1} + b_{i2}\bar{R}_{I_2}$$

The above model is valid for any security including security B that is explained by factor 2, since $b_{i1} = 0$. Thus, we have

$$\bar{R}_B = a_B + b_{B2}\bar{R}_{I_2}$$

$$9.5 = -0.1 + 1.2\bar{R}_{I_2}$$

$$\bar{R}_{I_2} = \frac{9.6}{1.2} = 8$$

(c) The expected return of security A is

$$\begin{aligned}\bar{R}_A &= a_A + b_{A1}\bar{R}_{I_1} + b_{A2}\bar{R}_{I_2}\\ &= 0.2 + 1.2 \times 15 - 0.15 \times 8\\ &= 17\end{aligned}$$

(d) Total risk as measured by standard deviation is

$$\sigma_i = \sqrt{b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \sigma_{c_i}^2}$$

And the systematic risk is measure by

$$\sigma_i = \sqrt{b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2}$$

Thus, the risk of A, B and C is

$$\sigma_A = \sqrt{b_{A1}^2\sigma_{I1}^2 + b_{A2}^2\sigma_{I2}^2} = \sqrt{1.2^2 \times 25^2 - 0.15^2 \times 5^2} = 30$$

$$\sigma_B = \sqrt{b_{B1}^2\sigma_{I1}^2 + b_{B2}^2\sigma_{I2}^2} = \sqrt{0.8^2 \times 25^2 + 0^2 \times 5^2} = 20$$

$$\sigma_C = \sqrt{b_{C1}^2\sigma_{I1}^2 + b_{C2}^2\sigma_{I2}^2} = \sqrt{0^2 \times 25^2 + 1.2^2 \times 5^2} = 6$$

(e) Variance and covariance are measured, respectively, by

$$\sigma_i^2 = b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \sigma_{c_i}^2$$

$$\sigma_i^2 = b_{i1}b_{j1}\sigma_{I1}^2 + b_{i2}b_{j2}\sigma_{I2}^2$$

Applying the data in the exercise,

$$\sigma_A^2 = b_{A1}^2\sigma_{I1}^2 + b_{A2}^2\sigma_{I2}^2 + \sigma_{c_A}^2 = 1.2^2 \times 25^2 - 0.15^2 \times 5^2 + 5^2 = 925.56$$

$$\sigma_B^2 = b_{B1}^2\sigma_{I1}^2 + b_{B2}^2\sigma_{I2}^2 + \sigma_{c_B}^2 = 0.8^2 \times 25^2 + 0^2 \times 5^2 + 2^2 = 404$$

$$\sigma_B^2 = b_{C1}^2\sigma_{I1}^2 + b_{C2}^2\sigma_{I2}^2 + \sigma_{c_C}^2 = 0^2 \times 25^2 + 1.2^2 \times 5^2 + 1^2 = 37$$

$$\sigma_{AB} = \sigma_{BA} = b_{A1}b_{B1}\sigma_{I1}^2 + b_{A2}b_{B2}\sigma_{I2}^2 = 1.2 \times 0.8 \times 25^2 - 0.15 \times 0 \times 5^2 = 600$$

$$\sigma_{AC} = \sigma_{CA} = b_{A1}b_{C1}\sigma_{I1}^2 + b_{A2}b_{C2}\sigma_{I2}^2 = 1.2 \times 0 \times 25^2 - 0.15 \times 1.2 \times 5^2 = -4.5$$

$$\sigma_{BC} = \sigma_{BC} = b_{B1}b_{C1}\sigma_{I1}^2 + b_{B2}b_{C2}\sigma_{I2}^2 = 0.8 \times 0 \times 25^2 + 0 \times 1.2 \times 5^2 = 0$$

So that, the covariance matrix is

$$\begin{pmatrix} 925.56 & 600 & -4.5 \\ 600 & 404 & 0 \\ -4.5 & 0 & 37 \end{pmatrix}$$

(f) (i) To find the minimum variance portfolio (mvp) we need to take the derivative and equal to 0 of the portfolio variance in order to X_B, which is the weight of security B in the mvp. Since securities B and C are not correlated and, therefore, $\rho_{BC} = 0$, we have

$$\sigma_V^2 = X_B^2\sigma_B^2 + (1 - X_B)^2\,\sigma_C^2$$

Taking the derivative, equaling 0 and solving for X_B

$$\frac{\partial\sigma_V^2}{\partial X_B} = 2X_B\sigma_B^2 + 2\,(1 - X_B)\,(-1)^2\,\sigma_C^2 = 0$$

$$X_B = \frac{\sigma_C^2}{\sigma_B^2 + \sigma_C^2}$$

Consequently,

$$X_B = \frac{\sigma_C^2}{\sigma_B^2 + \sigma_C^2} = \frac{37}{404 + 37} = 0.084$$
$$X_C = 1 - X_B = 1 - 0.084 = 0.916$$

Finally the portfolio's risk is

$$\sigma_V = \sqrt{X_B^2\sigma_B^2 + X_C^2\sigma_C^2} = \sqrt{0.084^2 \times 404 + 0.916^2 \times 37} = 0.0582$$

(ii) If we could invest in a risk-free security, the mvp would be 100% composed with the risk-free security, since, of course, it is impossible to build a portfolio with less risk then the risk-free security.

(g) (i) This is a standard portfolio selection exercise, in which we have to choose the tangent portfolio between the capital market line and the efficient frontier of risky assets. The solution for this problem involves solving the following system of simultaneous equations in order to Z_i, $\forall i = A,\ B,\ C$

$$\begin{cases} \bar{R}_A - R_F = Z_A\sigma_A^2 + Z_B\sigma_{AB} + Z_C\sigma_{AC} \\ \bar{R}_B - R_F = Z_A\sigma_{BA} + Z_B\sigma_B^2 + Z_C\sigma_{BC} \\ \bar{R}_C - R_F = Z_A\sigma_{CA} + Z_B\sigma_{CB} + Z_C\sigma_C^2 \end{cases}$$

Applying the data in the problem,

$$\begin{cases} 17-5 = 925.56Z_A + 600Z_B - 4.5Z_C \\ 12.5-5 = 600Z_A + 404Z_B \\ 9.5-5 = -4.5Z_A + 37Z_C \end{cases} \Leftrightarrow \begin{cases} Z_A = 0.041525 \\ Z_B = -0.04311 \\ Z_C = 0.12667 \end{cases}$$

Then, $\sum_{i=A}^{C} Z_i = 0.12509$. Therefore, the weights of the tangent portfolio are

$$\begin{aligned} X_A &= \frac{Z_A}{\sum_{i=A}^{C} Z_i} = \frac{0.041525}{0.12509} = 0.332 \\ X_B &= \frac{Z_B}{\sum_{i=A}^{C} Z_i} = \frac{-0.04311}{0.12509} = -0.3446 \\ X_C &= \frac{Z_C}{\sum_{i=A}^{C} Z_i} = \frac{0.12793}{0.12509} = 1.0126 \end{aligned}$$

Finally, the portfolio's expected return is

$$\bar{R}_T = \sum_{i=A}^{C} X_i \bar{R}_i = 0.332 \times 17 - 0.3446 \times 12.5 + 1,0126 \times 9.5 = 10.96$$

The portfolio's variance is

$$\begin{aligned} \sigma_T^2 &= X'\Sigma X \\ &= \begin{pmatrix} 0.332 & -0.3446 & 1.0126 \end{pmatrix} \begin{pmatrix} 925.56 & 600 & -4.5 \\ 600 & 404 & 0 \\ -4.5 & 0 & 37 \end{pmatrix} \begin{pmatrix} 0.332 \\ -0.3446 \\ 1.0126 \end{pmatrix} \\ &= 47.61 \end{aligned}$$

And portfolio's risk is

$$\sigma_T = 6.9$$

(ii) The capital market line is

$$\begin{aligned} \bar{R}_i &= R_F + \frac{R_T - R_F}{\sigma_T}\sigma_i \\ &= 5 + \frac{10.96-5}{6.9}\sigma_i \\ &= 5 + 0.86\sigma_i \end{aligned}$$