3 Selecting the Optimal Portfolio

3.1 Expected Utility Theory

Exercise 3.1. A *fair game* is a game where the initial investment equals the expected value of the payoff, i.e., where we have $\mathbb{E}(W) = W_0$.

We also know the utility functions of risk neutral investors are linear, while utility functions of risk averse are concave and of risk lovers are convex functions. See general shapes of utility function in Figure 3.1

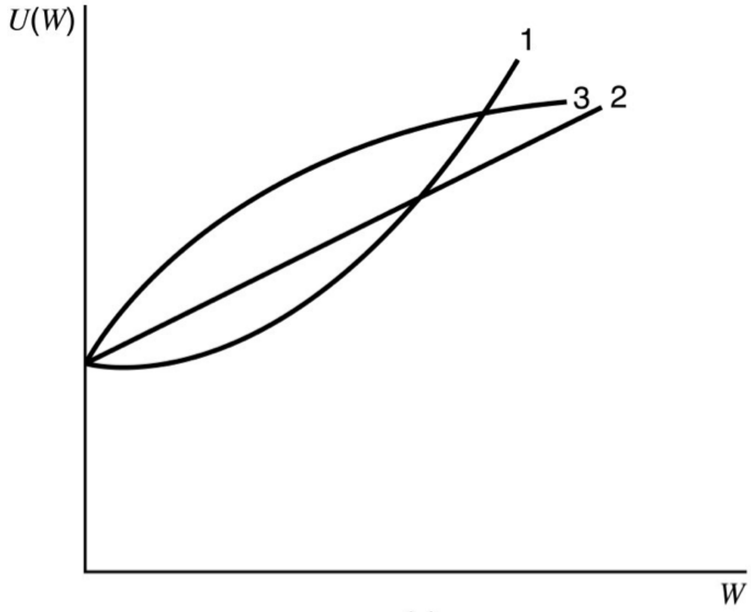


Figure 10: Exercise 3.1 – Shape of utility functions for risk (1) lovers, (2) neutral and (3) averse.

By definition of linear, concave and convex functions we have.

(a) For any a and b and $p \in [0,1]$ if the utility function U is linear we have

$$U(pa + (1-p)b) = pU(a) + (1-p)U(b) \qquad \Leftrightarrow \qquad U\underbrace{(\mathbb{E}(W))}_{W_0} = \mathbb{E}(U(W)) \ ,$$

thus, we conclude that any risk neutral investor would be indifferent between entering or not a fair game.

(b) For any a and b and $p \in [0,1]$ if the utility function U is concave we have

$$U(pa + (1-p)b)p \geq U(a) + (1-p)U(b) \qquad \Leftrightarrow \qquad U\underbrace{(\mathbb{E}(W))}_{W_0} \geq \mathbb{E}(U(W)) \ .$$

So, investors with concave utilities do not enter fair games.

(c) For any a and b and $p \in [0,1]$ if the utility function U is convex we have

$$U(pa + (1-p)b)p \leq U(a) + (1-p)U(b) \qquad \Leftrightarrow \qquad U\underbrace{(\mathbb{E}(W))}_{W_0} \leq \mathbb{E}(U(W)) \ .$$

So, investors with convex utilities enter fair games.

Exercise 3.2.

(a) For the investor with utility $U(W) = -W^{-1/3}$ we compute the expected utility of both investments,

$$\mathbb{E}\left[U(W_A)\right] = 0.25U(4) + 0.5U(6) + 0.25U(8) = -0.5576$$
$$\mathbb{E}\left[U(W_B)\right] = \frac{1}{3}U(4) + \frac{1}{3}U(6.2) + \frac{1}{3}U(8) = -0.5581$$

and conclude that investor 1, $A \succ B$.

(b) For $U(W) = -W^{-0.1}$ we get,

$$\mathbb{E}\left[U(W_A)\right] = 0.25U(4) + 0.5U(6) + 0.25U(8) = -0.8386$$
$$\mathbb{E}\left[U(W_B)\right] = \frac{1}{3}U(4) + \frac{1}{3}U(6.2) + \frac{1}{3}U(8) = -0.8387$$

and conclude that also investor 2, $A \succ B$.

(c) Both investors have power utility, thus

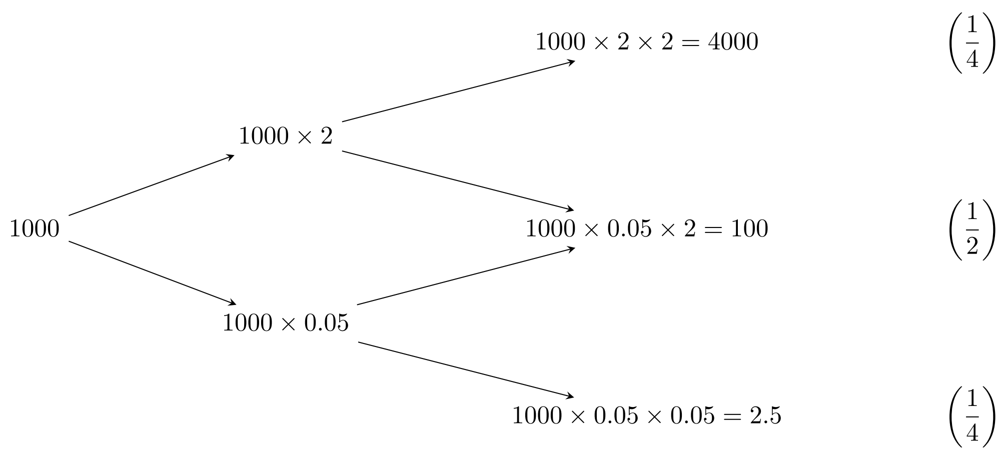
$$U(W) = -W^{-\alpha} \qquad \text{for} \alpha > 0$$
$$U'(W) = \alpha W^{-\alpha-1} > 0$$
$$U''(W) = -\alpha(\alpha+1)W^{-\alpha-2} < 0$$
$$ARA(W) = -\frac{U''(W)}{U'(W)} = \frac{\alpha(\alpha+1)W^{-\alpha-2}}{\alpha W^{-\alpha-1}} = \frac{1+\alpha}{W} \qquad \Rightarrow \qquad ARA'(W) = -\frac{1+\alpha}{W^2} < 0$$
$$RRA(W) = \frac{1+\alpha}{W}W = 1+\alpha \qquad \Rightarrow \qquad RRA'(W) = 0$$

they prefer more to less and they are risk averse with decreasing absolute risk aversion and constant relative risk aversion.

So, they always keep the same proportion of wealth invested in risky assets. Despite their similarities in terms of profiles, investor 1 has $\alpha = 1/3 = 0.3(3)$ while investor 2 has $\alpha = 0.1$, so their coefficients of RRAs are of 1.3(3) and 1.1, respectively, and we can conclude investor 1 has a higher degree of risk aversion than investir 2.

Exercise 3.3.

Since the coin is tossed twice the game can be summarised by the scheme below.

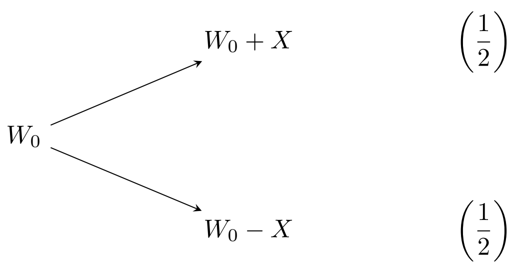


For log utility we have $U(W) = \log(W)$, and we have

$$\mathbb{E}\left[U\left(Game\right)\right] = \frac{1}{4}\underbrace{U(4000)}_{8.295} + \frac{1}{2}\underbrace{U(100)}_{4.6051} + \frac{1}{4}\underbrace{U(2.5)}_{0.916} = 4.6051$$

Since $U(100) = 4.60651$, we know the certainty equivalent of the game is $C = 100$ and, thus, the investor would be willing to pay up to 900 to avoid the situation.

Exercise 3.4.



(a) (i) For $W_0 = 1000$ and $X = 250$, the expected utility of the game, and the associated certainty equivalent, for each of the investor are:

$$\mathbb{E}\left[U(Game)\right] = \frac{1}{2}U(W_0 + X) + \frac{1}{2}U(W_0 - X)$$

$$U(W)) = \log(W) \qquad \mathbb{E}\left[U(Game)\right] = \frac{1}{2}\log(1250) + \frac{1}{2}\log(750)$$

$$= \frac{1}{2}(7.13) + \frac{1}{2}(6.62) = 6.875$$

$$\ln(C) = 6.875 \quad \Rightarrow \quad C = 967.78$$

$$V(W)) = 1 - e^{-0.001W} \qquad \mathbb{E}\left[U(Game)\right] = \frac{1}{2}\left(1 - e^{-0.001\times 1250}\right) + \frac{1}{2}\left(1 - e^{-0.001\times 750}\right)$$

$$= \frac{1}{2}(0.7135) + \frac{1}{2}(0.5276) = 0.62055$$

$$1 - e^{-0.001C} = 0.62055 \quad \Rightarrow \quad C = 969.03$$

Investor 1 is willing to pay $1000 - 967.78 = 32.22$ and investor 2 is willing to pay $1000 - 969.03 = 30.97$.

(ii) The expected utility of the game is

$$\max_X \qquad \mathbb{E}\left[U(Game)\right] = \frac{1}{2}U(W_0 + X) + \frac{1}{2}U(W_0 - X)$$

the value X that maximizes expected utility is given by the first-order-condition (F.O.C)

$$\frac{1}{2}U'(W_0 + X) - \frac{1}{2}U'(W_0 - X) = 0$$

For both investors we get

$$U(W) = \log(W)$$

$$U'(W) = \frac{1}{W}: \quad \frac{1}{2}\frac{1}{W_0 + X} - \frac{1}{2}\frac{1}{W_0 - X} = 0 \quad \Leftrightarrow \quad X = 0$$

$$V(W) = 1 - e^{-0.001W}$$

$$U'(W) = 0.001e^{-0.001W}: \quad \frac{0.001}{2}e^{-0.001(W_0+X)} - \frac{0.001}{2}e^{-0.001(W_0-X)} = 0 \quad \Leftrightarrow \quad X = 0$$

Which is not surprising as risk averse investors would rather not enter fair games (no matter the W_0 or X).

(b) The optimal $X = 0$ does not change. The amount investors are willing to pay to avoid the game, however, does depend on the initial wealth

$$
\begin{aligned}
U(W)) = \log(W) \qquad \mathbb{E}\left[U(Game)\right] &= \frac{1}{2}\log(100250) + \frac{1}{2}\log(99750) \\
&= \frac{1}{2}(11.5154) + \frac{1}{2}(11.5104) = 11.5129 \\
\ln(C) &= 11.5129 \quad \Rightarrow \quad C = 99997.45 \\
V(W)) = 1 - e^{-0.001W} \qquad \mathbb{E}\left[U(Game)\right] &= \frac{1}{2}\left(1 - e^{-0.001\times 100250}\right) + \frac{1}{2}\left(1 - e^{-0.001\times 99750}\right) \\
&= \frac{1}{2}(0.9999) + \frac{1}{2}(0.9999) = 0.9999 \\
1 - e^{-0.001C} &= 0.9999 \quad \Rightarrow \quad C = 99999.99
\end{aligned}
$$

As the wealth increases the curvature of both utility functions decrease and so they are willing to pay less to avoid the game.

Exercise 3.5.

(a) Starting from an initial wealth of $W_0 = 50$, the final outcome may be $W = 25$ or $W = 75$, with equal probability.

Given the utility function, we have

$$
\begin{aligned}
\text{If he enters the game :} \qquad \mathbb{E}\left[U(Game)\right] &= \frac{1}{2}U(25) + \frac{1}{2}U(75) \\
&= \frac{1}{2}\left[25 - 0.005(25)^2 + 75 - 0.005(75)^2\right] \\
&= 34.375 \\
\text{If he does not enter the game :} \qquad U(50) &= 50 - 0.005(50)^2 = 37.5
\end{aligned}
$$

So, he chooses not to play the game.

(b) To be indifferent betweem palying the same or not we need the expected utility of the game to be the same as the utility of not playing the game. Let us assign a probability p to the outcome 75 and $(1 - p)$ to 25. We, thus have

$$
\begin{aligned}
p\left[25 - 0.005(25)^2\right] + (1 - p)\left[75 - 0.005(75)^2\right] &= 37.5 \\
46.875\,p + 21.875(1 - p) &= 37.5 \\
p &= 62.5\%
\end{aligned}
$$

(c) The certainty equivalent of the game is the fixed amount that would make the investor indifferent between playing the game or nor.

In this case we have

$$
\begin{aligned}
U(C) &= \mathbb{E}\left[U(Game)\right] \\
C - 0.005C^2 &= 34.375 \\
C &= \frac{-1 \pm \sqrt{1 - 4 \times (-0.005) \times (-34.375)}}{2 \times (-0.005)} = \frac{1 \pm 0.5590}{0.01} \\
&\Rightarrow C = 44.1
\end{aligned}
$$

Exercise 3.6.

From the ranking of the projects, $X \succ Y \succ Z$, we know $\mathbb{E}(U_X) > \mathbb{E}(U_Y)$ and $\mathbb{E}(U_Y) > \mathbb{E}(U_Z)$.

Using a second order Taylor approximation of the RTF we also have

$$\mathbb{E}(U) = f(\sigma, \bar{R}) \approx \bar{R} - \frac{1}{2}RRA(W_0)(\bar{R}^2 + \sigma^2) \ .$$

For each project we get

$$\begin{aligned} f_X(30\%, 20\%) &\approx 0.2 - \frac{1}{2}RRA(W_0)(0.3^2 + 0.2^2) = 0.2 - 0.065RRA(W_0) \\ f_Y(35\%, 15\%) &\approx 0.15 - \frac{1}{2}RRA(W_0)(0.15^2 + 0.35^2) = 0.15 - 0.0725RRA(W_0) \\ f_Z(5\%, 8\%) &\approx 0.08 - halfRRA(W_0)(0.08^2 + 0.05^2) = 0.08 - 0.00445RRA(W_0) \ . \end{aligned}$$

and it musty hold

$$\begin{cases} f_X(30\%, 20\%) > f_Y(35\%, 15\%) \\ f_Y(35\%, 15\% > f_Z(5\%, 8\%) \end{cases} \Leftrightarrow \begin{cases} 0.2 - 0.065\ RRA(W_0) > 0.15 - 0.0725\ RRA(W_0) \\ 0.15 - 0.0725\ RRA(W_0) > 0.08 - 0.00445\ RRA(W_0) \end{cases}$$

Solving the system we get $1.06 > RRA(W_0) > -6.67$, so any investor with $RRA(W_0)$ within that range would have the suggested ranking of projects. In particular for risk neutral investors, with $RRA(W_0) = 0$, we also get $X \succ Y \succ Z$.

Exercise 3.7.

(a) The preferred investment will be the one with the highest level of expected utility. Thus, we have to calculate the utility in each state of economy for the three investments. Given the utility function $U(W) = 20W - 0.5 * W^2$ we get,

For investment A:

$$\begin{aligned} U(5) &= 20 * 5 - 0.5 \times 5^2 = 87.5 \\ U(6) &= 20 * 6 - 0.5 \times 6^2 = 102 \\ U(9) &= 20 * 9 - 0.5 \times 9^2 = 139.5 \end{aligned}$$

For investment B:

$$\begin{aligned} U(4) &= 20 * 4 - 0.5 \times 4^2 = 72 \\ U(7) &= 20 * 7 - 0.5 \times 7^2 = 115.5 \\ U(10) &= 20 * 10 - 0.5 \times 10^2 = 150 \end{aligned}$$

For investment C:

$$\begin{aligned} U(1) &= 20 * 1 - 0.5 \times 1^2 = 19.5 \\ U(9) &= 20 * 9 - 0.5 \times 9^2 = 139.5 \\ U(18) &= 20 * 18 - 0.5 \times 18^2 = 198 \end{aligned}$$

Therefore, the expected utility for each investment is

$$\begin{aligned} \mathbb{E}\left[U(W_A)\right] &= 87.5 \times 1/3 + 102 \times 1/3 + 139.5 \times 1/3 = 109.67 \\ \mathbb{E}\left[U(W_B)\right] &= 72 \times 1/4 + 115.5 \times 1/2 + 150 \times 1/4 = 113.25 \\ \mathbb{E}\left[U(W_C)\right] &= 19.5 \times 1/5 + 139.5 \times 3/5 + 198 \times 1/5 = 127.20 \end{aligned}$$

So, Investment C is preferred because it has the highest level of expected utility.

(b) As before, the preferred investment will be the one with the highest level of expected utility, so that we have to calculate the utility in each state of economy for the three investments, now considering the new utility function $U(W) = -\frac{1}{\sqrt{W}}$.

For investment A:

$$U(5) = -\frac{1}{\sqrt{5}} = -0.4472$$
$$U(6) = -\frac{1}{\sqrt{6}} = -0.4082$$
$$U(9) = -\frac{1}{\sqrt{9}} = -0.3333$$

For investment B:

$$U(4) = -\frac{1}{\sqrt{4}} = -0.5$$
$$U(7) = -\frac{1}{\sqrt{7}} = -0.3750$$
$$U(10) = -\frac{1}{\sqrt{10}} = -0.3162$$

For investment C:

$$U(1) = -\frac{1}{\sqrt{1}} = -1$$
$$U(9) = -\frac{1}{\sqrt{9}} = -0.3333$$
$$U(18) = -\frac{1}{\sqrt{18}} = -0.2351$$

Therefore, the expected utility for each investment is

$$\mathbb{E}\left[U(W_A)\right] = -0.4472 \times 1/3 - 0.4082 \times 1/3 - 0.3333 \times 1/3 = -0.3963$$
$$\mathbb{E}\left[U(W_B)\right] = -0.5 \times 1/4 - 0.3780 \times 1/2 - 0.3162 \times 1/4 = -0.3930$$
$$\mathbb{E}\left[U(W_C)\right] = -1 \times 1/5 - 0.3333 \times 3/5 - 0.2357 \times 1/5 = -0.4471$$

With this new utility function, Investment B is preferred because it has the highest level of expected utility.

(c) For investments A and B be indifferent, using the first utility function, their expected utility must equal. Therefore, what must be the probability π associated to payoffs 4 and 10 of investment B to have such equality?

$$A \sim B \Longleftrightarrow \mathbb{E}\left[U(W_A)\right] = \mathbb{E}\left[U(W_B)\right]$$

Thus

$$\mathbb{E}\left[U(W_A)\right] = \mathbb{E}\left[U(W_B)\right]$$
$$109.67 = 72 \times \pi + 115.5 \times (1 - 2\pi) + 150 \times \pi$$
$$\pi = 0.648$$

Since we must have $0 \leq \pi \leq 0.5$, otherwise the new probabilities would not be between 0 and 1, this means investor 1 will never be indifferent between investments A and B. He always prefer B to A .

(d) For investments B and C be indifferent, using the second utility function, their expected utility must be the same. In part c we vary the probability associated to certain payoffs, now we allow for a change in the lowest payoff of these two investments, which is 1 for Investment C. So,

$$B \sim C \iff \mathbb{E}\left[U(W_B)\right] = \mathbb{E}\left[U(W_C)\right]$$

Thus

$$\mathbb{E}\left[U(W_B)\right] = \mathbb{E}\left[U(W_C)\right]$$

$$-0.3963 = U(x) \times 1/5 - 0.3333 \times 3/5 - 0.2357 \times 1/5$$

$$U(x) = -0.7456$$

Since $U(x) = -\frac{1}{\sqrt{x}}$ we finally have

$$U(x) = -\frac{1}{\sqrt{x}}$$

$$-0.7456 = -\frac{1}{\sqrt{x}}$$

$$x = 1.7987$$

Exercise 3.8. (a) To analise the investor behaviour towards risk we need to study its utility function and its economics proprieties, which is done taking the first and the second derivative. With the utility function $U(W) = -w^{-1/2}$ and assuming $W > 0$, we have

$$U'(W) = \frac{1}{2}W^{-3/2}$$

Since $W > 0$ it comes $U'(W) > 0$, which means the investor prefers more to less. This attribute is known as nonsatiation. The second derivative is

$$U''(W) = -\frac{3}{4}W^{-5/2}$$

Which smaller than 0, so that the investor shows risk aversion.

(b) Absolute aversion is calculated by taking the first derivative of a measure of absolute aversion that is

$$ARA(W) = -\frac{U''(W)}{U'(W)}$$

Therefore,

$$ARA(W) = -\frac{U''(W)}{U'(W)} = \frac{\frac{3}{4}W^{-5/2}}{\frac{1}{2}W^{-3/2}} = \frac{3}{2}W^{-1}$$

And,

$$ARA'(W) = -\frac{3}{2}W^{-2}$$

Since $ARA'(W) < 0$, the investor exhibits decreasing absolute risk aversion. In practical terms, this means the investor increases the amount of money invested in risky assets when her wealth increases.

Relative aversion is a similar to absolute aversion, but its calculated in proportional terms. So, we need to take the first derivative of a measure of relative risk aversion that is

$$RRA(W) = -\frac{WU''(W)}{U'(W)}$$

Therefore,

$$RRA(W) = -\frac{WU''(W)}{U'(W)} = \frac{\frac{3}{4}W^{-5/2}W}{\frac{1}{2}W^{-3/2}} = \frac{3}{2}$$

And,

$$RRA'(W) = 0$$

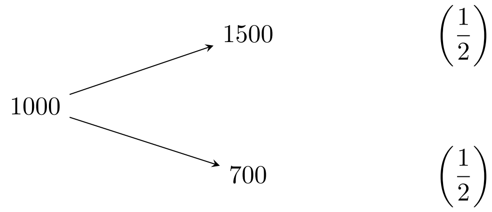
Since $RRA'(W) = 0$, the investor exhibits constant relative risk aversion. In practical terms, this means the percentage invested in risky assets remains constant when her wealth increases.

Exercise 3.9.

(a) Since $U(W) = ae^{-bW}$, we have $U'(W) = -abe^{-bW}$ and $U''(W) = ab^2e^{-bW}$. To have a risk averse investor we need $U''(W) = ab^2e^{-bW} < 0$. Since, $e^{-bW} > 0$ and b^2 is positive, then a must be negative ($a < 0$). On the other hand, to respect the nonsatiation assumption we need $U'(W) = -abe^{-bW} > 0$. Again $e^{-bW} > 0$. Because $a < 0$ we have $-a > 0$, which implies a positive b.

(b) (i) If the investor decides not to do the risky investment, he keep the 1000 and has an utility of $\mathbb{E}\left[U(Invest)\right] = ae^{-b1000}$.

If he decides do do the risky investment he faces



and his expected utility from the investment is

$$\mathbb{E}\left[U(Invest)\right] = \frac{1}{2}ae^{-b1500} + \frac{1}{2}ae^{-b700} = ae^{-b1000}\frac{e^{-b500} + e^{+b300}}{2}$$

To compare the utility of not investing with the expected utility of the investment we need to compare 1 with $\frac{e^{-b500}+e^{+b300}}{2}$, which does not depend on a, but only on b. The investor chooses the risky investment when

$$\begin{aligned} \mathbb{E}\left[U(Invest)\right] &> U(1000) \\ ae^{-b1000}\frac{e^{-b500} + e^{+b300}}{2} &> ae^{-b1000} \\ e^{-b500} + e^{+b300} &> 2\ . \end{aligned}$$

(ii)

$$\begin{aligned} U(C) &= \mathbb{E}\left[U(Invest)\right] \\ e^{-Cb} &= \frac{1}{2}e^{-b1500} + \frac{1}{2}e^{-b700} \\ -Cb &= \ln\left(\frac{1}{2}e^{-b1500} + \frac{1}{2}e^{-b700}\right) \\ C &= -\frac{\ln\left(\frac{1}{2}e^{-b1500} + \frac{1}{2}e^{-b700}\right)}{b} \end{aligned}$$

The certainty equivalent of a risky investment is the certain (fixed) amount that makes the investor indifferent between keeping that fixed amount or entering the risky investment. It can also be interpreted as the maximum amount the investor would be willing to "pay" to enter the risky investment.

(iii) For $b = 0.01$ we have $C = -\frac{\ln\left(\frac{1}{2}e^{-b1500}+\frac{1}{2}e^{-b700}\right)}{0.01} = 769.28$. Since it is less than 1000 we can conclude that in this case the investor will not do the risky investment.

Exercise 3.10.

(a) See Figure 11.

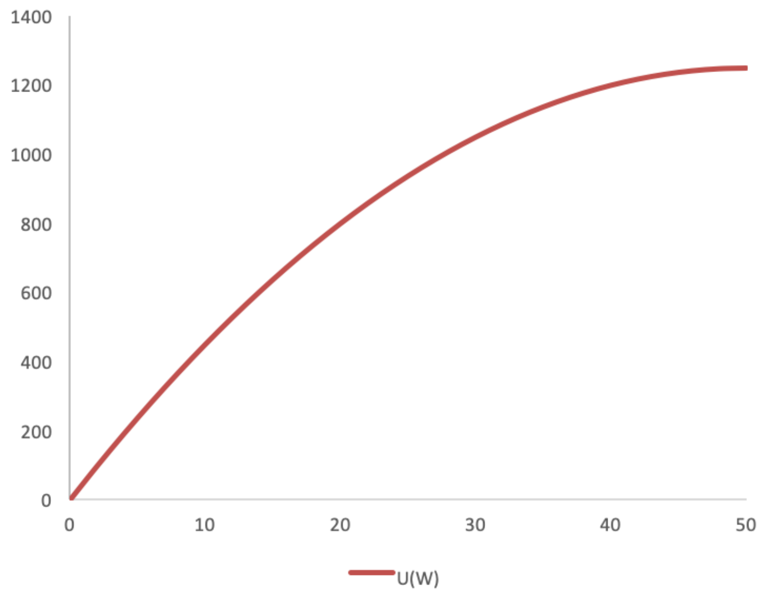


Figure 11: Exercise 3.10 – Utility function for relevant wealth levels ($W < 50$).

(b) To describe this investor behaviour towards risk we need to study the following properties

- Nonsaciation
- Risk attitude (risk aversion)
- Absolute risk aversion
- Relative risk aversion

The investor respects the nonsaciation assumption if $U'(W) > 0$. Since

$$U'(W) = 50 - W$$

This propriety is respected if and only if $W < 50$.

To study the second property we take te second derivative

$$U''(W) = -1 < 0$$

Consequently, the investor shows risk aversion for the feasible values for wealth ($W \in]0, 50[$). Geometrically, in the allowed domain, the function is increasing and concave, being a parable turned down (see Figure 11).

About absolute risk aversion we known

$$ARA(W) = -\frac{U''(W)}{U'(W)} = (50 - W)^{-1} \qquad ARA'(W) = (50 - W)^{-2} > 0$$

Thus, this investor exhibits an increasing absolute risk aversion, i.e. when her wealth increases she will invest a small amount of money in risky assets.

About relative risk aversion we have

$$RRA(W) = -\frac{WU''(W)}{U'(W)} = W\,(50 - W)^{-1} \qquad RRA'(W) = \frac{50}{(50 - W)^2} > 0$$

Thus, this investor exhibits an increasing relative risk aversion, i.e. when her wealth increases she will invest a small percentage of her wealth in risky assets.

(c) This investor will chose the project with higher expected utility. Thus for investment X, we have for each state of economy

$$U(10) = 50W - \frac{1}{2}W^2 = 50 \times 10 - \frac{1}{2} \times 10^2 = 450$$

$$U(40) = 50W - \frac{1}{2}W^2 = 50 \times 40 - \frac{1}{2} \times 40^2 = 1,200$$

$$U(25) = 50W - \frac{1}{2}W^2 = 50 \times 25 - \frac{1}{2} \times 25^2 = 937.5$$

For investment Y,

$$U(20) = 50W - \frac{1}{2}W^2 = 50 \times 20 - \frac{1}{2} \times 20^2 = 800$$

$$U(40) = 50W - \frac{1}{2}W^2 = 50 \times 40 - \frac{1}{2} \times 40^2 = 1,200$$

$$U(45) = 50W - \frac{1}{2}W^2 = 50 \times 45 - \frac{1}{2} \times 45^2 = 1,237.5$$

Thus, expected utilities are

$$\mathbb{E}\left[U\left(W_X\right)\right] = \sum_{i=1}^{3} P_i U(W_{X_i}) = 0.1 \times 450 + 0.2 \times 1,200 + 0.7 \times 937,5 = 941.25$$

$$\mathbb{E}\left[U\left(W_Y\right)\right] = \sum_{i=1}^{3} P_i U(W_{y_i}) = 0.05 \times 800 + 0.9 \times 1,237.5 + 0.05 \times 1200 = 1,181.88$$

As $\mathbb{E}\left[U\left(W_Y\right)\right] > \mathbb{E}\left[U\left(W_X\right)\right]$, we have $Y \succ X$, i.e. investor's choice should be project Y.

(d) The risk premium π is the amount the investor is willing to pay to insure against risk, such that this is a measure of absolute risk aversion. The risk premium is calculated as $\pi = \mathbb{E}[W] - c$ where c is the certain equivalent. The certain equivalent is the amount received with certainty that has the same utility than a lottery

$$U(c) = \mathbb{E}\left[U(W)\right] \tag{9}$$

Thus, for Investment X, we have $\pi_X = \mathbb{E}[W_X] - c_X$, where

$$\mathbb{E}[W_X] = \sum_{i=1}^{3} P_i W_{X_i} = 0.1 \times 10 + 0.2 \times 40 + 0.7 \times 25 = 26.5$$

To find c_X we need to use (9)

$$U(c_X) = \mathbb{E}[U(W_X)]$$

$$50c_X - \frac{1}{2}c_X^2 = 941.25$$

$$c_X = 74.85 \vee c_X = 25.15$$

Since c_X must be in the range of possible values for W_X we have $c_X = 25.15$. Finally, the risk premium is

$$\pi_X = \mathbb{E}[W_X] - c_X = 26.5 - 25.15 = 1.35$$

Similarly for Investment Y, we have $\pi_Y = \mathbb{E}[W_Y] - c_Y$, where

$$\mathbb{E}[W_Y] = \sum_{i=1}^{3} P_i W_{Y_i} = 0.05 \times 20 + 0.9 \times 40 + 0.05 \times 45 = 39.25$$

To find c_Y we use again (9)

$$U(c_Y) = \mathbb{E}[U(W_Y)]$$

$$50c_Y - \frac{1}{2}c_Y^2 = 1181.88$$

$$c_Y = 38.33 \vee c_Y = 61.67$$

Since c_Y must be in the range of possible values for W_Y we have $c_Y = 38.33$. Finally, the risk premium is

$$\pi_Y = \mathbb{E}[W_Y] - c_Y = 39.25 - 38.33 = 0.92$$

As expected the risk premium for investment X is higher due its higher risk level.

Exercise 3.11.

(a) To discover the investor's attitudes towards risk we can draw her utility function. To do so we need as many points as we can. From the data in the problem we already have two points $\{(R, U) : (0\%, 0)\,(10\%, 10)\}$.

We also have data on three investment projects and their certain equivalents, $C_X = 10\%$, $C_Y = 20\%$ and $C_Z = 30\%$, that can give us another three points.

Thus, for project X

$$U(C_X) = \mathbb{E}[U(R_X)]$$

$$U(10\%) = 0.5U(0\%) + 0.5U(30\%)$$

$$5 = 0.5U(30\%)$$

$$U(30\%) = 10$$

For project Y we have

$$U(C_Y) = \mathbb{E}[U(R_Y)]$$

$$U(20\%) = 0.4U(10\%) + 0.6U(30\%)$$

$$U(20\%) = 0.4 \times 5 + 0.6 \times 10$$

$$U(20\%) = 8$$

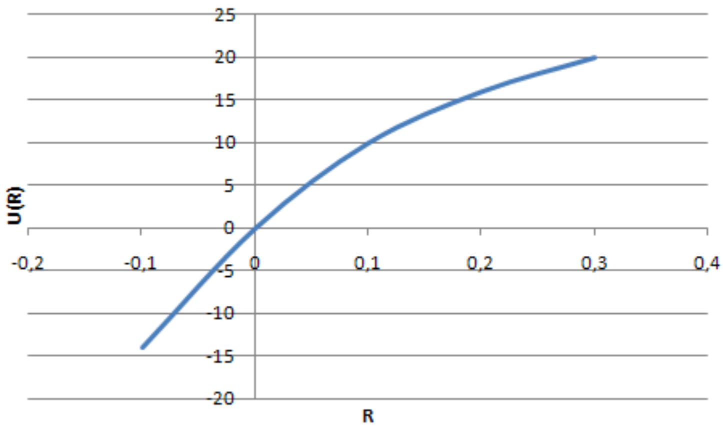


Figure 12: Exercise 3.11 - Utility Function

Finally, for project Z we have

$$U(C_Z) = \mathbb{E}[U(R_Z)]$$
$$U(10\%) = 0.2U(-10\%) + 0.8U(20\%)$$
$$5 = 0.2U(-10\%) + 0.8 \times 8$$
$$U(-10\%) = -7$$

With five points we can draw the utility function (see Figure 12) and observe the function is increasing and concave, therefore for equal increases in return the marginal utility is decreasing. Thus, the investor is risk averse.

(b) The risk premium associated with each of the projects is given by $\pi = E(R) - C$, where C is the certainty equivalent. We thus have

$$\begin{aligned} \mathbb{E}(R_X) = 15\% &\implies \pi_X = 15\% - 10\% = 5\% \\ \mathbb{E}(R_Y) = 22\% &\implies \pi_X = 22\% - 20\% = 2\% \\ \mathbb{E}(R_Z) = 14\% &\implies \pi_X = 14\% - 10\% = 4\% \end{aligned}$$

(c) The previous answer is based on the expected utility theorem and the utility function proprieties. The expected utility theorem states the rational rules to order different investment projects and basically it claims that the decision criterion is the maximization of the expected utility.

(d) To rank the three projects we need to compute their expected utilities. Using the results from (a) we get

$$\mathbb{E}(U(R_X)) = 5 \qquad \mathbb{E}(U(R_Y)) = 8 \qquad \mathbb{E}(U(R_Z)) = 5\ ,$$

so the investor prefers project Y to the other two projects and is indifferent between X and Z, i.e. $Y \succ X \sim Z$

(e) We know consider a game that pays 30% with probability h and 0% with probability $(1 - h)$. We need to find the probability level h that makes the investor indifferent between each project and this game.

$$\begin{aligned} h_X U(30\%) + (1-h_X)U(0\%) &= E(U(R_X)) \\ 10h_X &= 5 \\ h_X &= 0.5 \end{aligned}$$

$$\begin{aligned} h_Y U(30\%) + (1-h_Y)U(0\%) &= E(U(R_Y)) \\ 10h_Y &= 8 \\ h_Y &= 0.8 \end{aligned}$$

$$\begin{aligned} h_Z U(30\%) + (1-h_Z)U(0\%) &= E(U(R_Z)) \\ 10h_Z &= 5 \\ h_Z &= 0.5 \end{aligned}$$

From the above we get the exact same ranking as before: $Y \succ X \sim Z$.

Exercise 3.12.

(a) To find the absolute and relative risk aversion coefficients we first need to take the first and second derivative of the utility function

$$U'(W) = \frac{4}{W} > 0 \wedge U''(W) = -\frac{4}{W^2} < 0$$

Thus, she respects the nonsatiation assumption and is risk averse. About absolute and relative risk aversion we know

$$ARA(W) = -\frac{U''(W)}{U'(W)} = -\frac{-\frac{4}{W^2}}{\frac{4}{W}} = \frac{1}{W} \Rightarrow ARA'(W) = -\frac{1}{W^2} < 0, \ \forall \ W > 0$$

$$RRA(W) = -\frac{WU''(W)}{U'(W)} = -W\frac{-\frac{4}{W^2}}{\frac{4}{W}} = 1 \Rightarrow RRA'(W) = 0$$

Therefore, the investor exhibits decreasing absolute risk aversion and constant relative risk aversion, i.e. as her wealth increases she always keeps the same proportion invested in risky assets.

(b) We consider three projects X, Y, Z with only two possible outcomes, 201 and 1, and for each of them we know $\mathbb{E}(W_X) = 101$, $\mathbb{E}(W_Y) = 61$ and $\mathbb{E}(W_Z) = 71$.

(i) Let us consider p_X to be the real probability of the outcome 201 in project X and $(1 - p_X)$ to be the real probability of the outcome 1. Likewise use p_Y and p_Z when dealing with the other two projects. Then we have

$$\begin{aligned} \mathbb{E}(W_X) = 101 \quad &\Leftrightarrow \quad 201p_X + (1-p_X) = 101 \quad \Leftrightarrow \quad p_X = 0.5 \\ \mathbb{E}(W_Y) = \ \ 61 \quad &\Leftrightarrow \quad 201p_Y + (1-p_Y) = \ \ 61 \quad \Leftrightarrow \quad p_Y = 0.3 \\ \mathbb{E}(W_Z) = \ \ 71 \quad &\Leftrightarrow \quad 201p_Z + (1-p_Z) = \ \ 71 \quad \Leftrightarrow \quad p_Z = 0.35 \end{aligned}$$

(ii) Using the probabilities from (i) we can determine the expected utility associated with each project. We have,

$$\begin{aligned} \mathbb{E}\left[U(W_X)\right] &= (1-p_X)U(1) + p_X U(201) = 0.5 \times 2 + 0.5 \times 23.2132 = 12.6066 \\ \mathbb{E}\left[U(W_Y)\right] &= (1-p_Y)U(1) + p_Y U(201) = 0.7 \times 2 + 0.3 \times 23.2132 = 8.3640 \\ \mathbb{E}\left[U(W_Z)\right] &= (1-p_Z)U(1) + p_Z U(201) = 0.65 \times 2 + 0.35 \times 23.2132 = 9.4246 \end{aligned}$$

and the ranking is $X \succ Z \succ Y$.

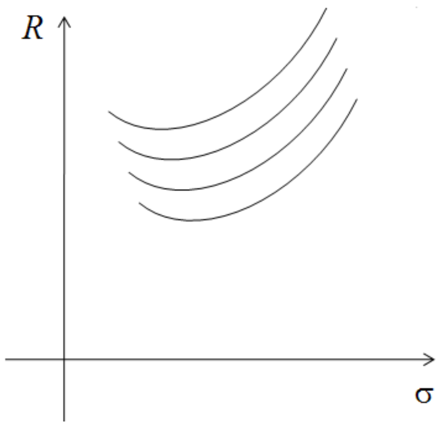


Figure 13: Exercise 3.13 - Indifference Curves

(iii) The certainty equivalent of project X, C_X is the certain amount that gives the investor the same utility as the expected utility of project X. Likewise for C_Y and C_Z for projects Y and Z, respectively. The risk premia is defined as $\pi_X = E(W_X) - C_X$ and likelwise for π_Y, π_Z.

$$U(C_X) = \mathbb{E}\left[U(W_X)\right] \quad \Leftrightarrow \quad 2 + 4\ln(C_X) = 12.6066 \quad \Leftrightarrow \quad C_X = e^{\frac{12.6066-2}{4}} = 14.1774$$
$$U(C_Y) = \mathbb{E}\left[U(W_Y)\right] \quad \Leftrightarrow \quad 2 + 4\ln(C_Y) = 8.3640 \quad \Leftrightarrow \quad C_Y = e^{\frac{8.3640-2}{4}} = 4.9086$$
$$U(C_Z) = \mathbb{E}\left[U(W_Z)\right] \quad \Leftrightarrow \quad 2 + 4\ln(C_Z) = 9.4246 \quad \Leftrightarrow \quad C_Z = e^{\frac{9.4246-2}{4}} = 6.3991$$

therefore, $\pi_X = 101 - 14.1774 = 86.8225$, $\pi_Y = 61 - 4.9086 = 56.0914$ and $\pi_Z = 71 - 6.3991 = 64.6009$.

(c) Since the new utility function is a linear transformation of the original function

$$V(W) = 2U(W) - 4 = 2\,(2 + 4\ln W) - 4 = 4 + 8\ln -4 = 8\ln W$$

and taking into account that the new information on expected payoffs is irrelevant because what matters are expected utilities, the three projects are now ordered exactly in the same way: $X \succ Y \succ Z$.

Exercise 3.13.

(a) To study her risk profile we need to take the first and the second derivative of the utility function $W - 6W^2$ with $W < 1/12$. So,

$$U'(W) = 1 - 12W > 0 \text{ for } W < 1/12, \qquad \text{and} \qquad U''(W) = -12 < 0\ .$$

Thus, the investor prefers more to less, as long as $W < 1/12$, and his risk averse. The indifference curves are plotted in Figure 13.

(b) Absolute and relative risk aversion are as follows

$$ARA(W) = -\frac{U''(W)}{U'(W)} = \frac{12}{1-12W} \qquad ARA'(W) = \frac{144}{(1-12W)^2} > 0$$

$$RRA(W) = -\frac{WU''(W)}{U'(W)} = \frac{12W}{1-12W} \qquad RRA'(W) = \frac{12}{(1-12W)^2} > 0$$

Therefore, the investor exhibits increasing absolute and relative risk aversion, i.e. as her wealth increases she reduces the amount and the proportion invested in risky assets.

(c) While absolute risk aversion measures the variation in the amount invested in risky assets as a function of wealth, the relative risk aversion measures the change in the proportion invested in risky assets provoked by a variation in wealth.

Exercise 3.14.

(a) The risk tolerance function (RTF) $f(\sigma, \bar{R})$ is nothing but the mean-variance representation of the expected value of the utility function $U(W)$.

Utility functions are defined in terms of final wealth, while RTF are defined in terms of returns, but we can always write $W = W_0(1 + R)$. For some utility functions we may not get a closed-form expression for $f(\sigma, \bar{R})$, that only happens in special cases or whenever returns follow a distribution for which $\bar{R}$ and σ are sufficient statistics.

Indifference curves are level curves of the RTF, i.e., curves along which the expected utility is constant $f(\sigma, \bar{R}) = K$.

(b) For $\bar{R} = \exp(0.7\sigma) + K$ we have

$$\left(\frac{\partial \bar{R}}{\partial \sigma}\right)_{IC} = 0.7 \exp(0.7\sigma) > 0$$

It is only possible to keep the same K level of expected utility is higher risk levels are associated with higher expected returns, so we can conclude the investor is risk-averse.

(c) If the efficient frontier is given by $\bar{R} = 0.05 + 0.8\sigma$, then to find the investor optimal we must find the point where the slopes of the indifference curves and the efficient frontier are the same.

$$\left(\frac{\partial \bar{R}}{\partial \sigma}\right)_{IC} = \left(\frac{\partial \bar{R}}{\partial \sigma}\right)_{EF}$$
$$0.7 \exp(0.7\sigma^*) = 0.8$$
$$\sigma^* = \frac{\log\left(\frac{0.8}{0.7}\right)}{0.7} = 0.1907$$

Exercise 3.15. Solved during lectures.

Exercise 3.16.

(a) For a two assets portfolio the risk is

$$\sigma_P^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A (1 - X_A) \sigma_{AB}$$

In this case we know $\sigma_{AB} = 0$ and we pretend $\sigma_P^2 = (9.22\%)^2$. Thus,

$$\sigma_P^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2$$

$$0.0085 = (10\%)^2 X_A^2 + (20\%)^2 (1 - X_A)^2$$

$$P_1 : \; X_A = 0.9 \wedge X_B = 0.1 \vee P_2 \; X_A = 0.7 \wedge X_B = 0.3$$

Since,

$$\bar{R}_{P_1} = 0.9 \times 8\% + 0.1 \times 12\% = 8.4\%$$

$$\bar{R}_{P_2} = 0.7 \times 8\% + 0.3 \times 12\% = 9.2\%$$

Only P_2 is efficient. Therefore, $\{(X_A, X_B) ; (0.7, 0.3)\}$ and $R_P = 9.2\%$.

(b) For a two assets portfolio the return is

$$\bar{R}_P = X_A \bar{R}_A + (1 - X_A)\,\bar{R}_B$$

In this case we want to find a portfolio with a return of 11%, so

$$\bar{R}_P = X_A \bar{R}_A + (1 - X_A)\,\bar{R}_B$$

$$11\% = 8\% X_A + 12\%\,(1 - X_A)$$

$$X_A = 0.25 \wedge X_B = 0.75$$

Consequently, the portfolio's variance is

$$\begin{aligned}\sigma_P^2 &= X_A^2 \sigma_A^2 + (1 - X_A)^2\,\sigma_B^2 \\ &= 0.25^2 \times (10\%)^2 + 0.75^2 \times (20\%)^2 \\ &= 0.023125\end{aligned}$$

and its risk is $\sigma_P = 15.21\%$.

(c) To find the tangent portfolio between the capital market line and the efficient frontier of risky assets we have to solve the following system of simultaneous equations in order to Z_i, $\forall i \geqslant 0$

$$\begin{cases} \bar{R}_1 - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13} + \cdots + Z_N\sigma_{1N} \\ \bar{R}_2 - R_F = Z_1\sigma_{21} + Z_2\sigma_2^2 + Z_3\sigma_{23} + \cdots + Z_N\sigma_{2N} \\ \bar{R}_3 - R_F = Z_1\sigma_{31} + Z_2\sigma_{32} + Z_3\sigma_3^2 + \cdots + Z_N\sigma_{3N} \\ \vdots \\ \bar{R}_N - R_F = Z_1\sigma_{NN}^2 + Z_2\sigma_{N2} + Z_3\sigma_{N3} + \cdots + Z_N\sigma_N^2 \end{cases}$$

which can be written using matricial notation

$$\left(\bar{R} - R_F 1\right) = \Sigma Z \qquad \Leftrightarrow \qquad \begin{pmatrix} \bar{8\%} - 4\% \\ \bar{12\%} - 4\% \\ \bar{15\%} - 4\% \end{pmatrix} = \begin{pmatrix} 0,01 & 0 & 0 \\ 0 & 0,04 & -0,03 \\ 0 & -0,03 & 0,0625 \end{pmatrix} Z$$

where we have used $\sigma_{BC} = \rho_{BC}\sigma_B\sigma_C = -0.6 \times 20\% \times 25\% = -0.03$.
Solving the above equation we get

$$Z = \Sigma^{-1}\left(\bar{R} - R_F 1\right)$$

where Σ^{-1} is the inverse covariance matrix, $\bar{R}$ is a column vector with the securities returns, R_F is a scalar and 1 is a column vector of 1s. Applying this last equation

$$Z = \Sigma^{-1}\left(R - R_F 1\right) = \begin{pmatrix} 100.0000 & 0.0000 & 0.0000 \\ 0.0000 & 39.0625 & 18.7500 \\ 0.0000 & 18.7500 & 25.0000 \end{pmatrix} \begin{pmatrix} 8\% - 4\% \\ 12\% - 4\% \\ 15\% - 4\% \end{pmatrix} = \begin{pmatrix} 4 \\ 5.1875 \\ 4.25 \end{pmatrix}$$

The Zs are proportional to the optimum amount to invest in each security. Then the optimum proportions to invest in stock k is X_k, where

$$X_k = \frac{Z_k}{\sum\limits_{i=1}^{N} Z_i}$$

Thus,

$$\begin{pmatrix} X_A \\ X_B \\ X_C \end{pmatrix} = \begin{pmatrix} 4/13.4375 \\ 5.1875/13.4375 \\ 4.25/13.4375 \end{pmatrix} = \begin{pmatrix} 29.77\% \\ 38.60\% \\ 31.63\% \end{pmatrix}$$

(d) The tangency portfolio's return is

$$\bar{R}_T = \sum_{i=1}^{3} X_i \bar{R}_i = 0.2977 \times 8\% + 0.386 \times 12\% + 0.3163 \times 15\% = 11.76\%$$

Since securities A and B and A and C are not correlated, the risk calculation is simplified

$$\begin{aligned}\sigma_T^2 &= \sigma_A^2 X_A^2 + \sigma_B^2 X_B^2 + \sigma_C^2 X_C^2 + 2X_B X_C \sigma_{BC} \\ &= 0.01 \times 0.2977^2 + 0.04 \times 0.3860^2 + 0.0625 \times 0.3163^2 + 2 \times 0.3860 \times 0.3163 \times (-0.03) \\ &= 0.005773\end{aligned}$$

Thus, the portfolio's risk is $\sigma_T = 7.60\%$.

The efficient frontier is given by the line:

$$\bar{R}_P = R_F + \frac{\bar{R}_T - R_F}{\sigma_T}\sigma_P = 4\% + \frac{11.76\% - 4\%}{7.59\%}\sigma_P = 4\% + 1.022\sigma_P$$

(e) (i) The indifference curves are give by $\bar{R} = 0.5\sigma^2 + 0.965\sigma + 0.01K$, and we have,

$$\frac{\partial \bar{R}^{IC}}{\partial \sigma} = \sigma + 0.965 > 0, \quad \text{for all } \sigma > 0.$$

Since the indifference curves are upward slopping in the space $(\sigma, \bar{R})$, we can conclude the investor is risk averse.

(ii) The investment decision criterion is to maximize the investor's expected utility subject to the efficient frontier. In this case we are given indifference curves, of each K level of expected utility. So we just need do equal the slopes of the indifference curves to the slope of the efficient frontier to find the optimal portfolio's risk. Let us denote the optimal portfolio with the letter P. Therefore,

$$\frac{\partial \bar{R}_T}{\partial \sigma_T} = \frac{\partial \bar{R}_P}{\partial \sigma_P}$$

$$1.022 = \sigma_P + 0.965$$

$$\sigma_P = 5.7\%$$

Remember that this optimal portfolio is compose by risk free and portfolio T, so that its risk is $\sigma_P = X_T\sigma_T$. Therefore, the weight of portfolio T in the optimal portfolio is

$$X_T = \frac{\sigma_P}{\sigma_T} = \frac{5.7\%}{7.60\%} = 0.75$$

And, of course, $X_F = 1 - X_T = 1 - 0.75 = 0.25$. Thus, she must invest 75% in portfolio T, which corresponds to

$$0.75X_T = 0.75\begin{pmatrix} 0.2977 \\ 0.3860 \\ 0.3163 \end{pmatrix} \Longrightarrow \begin{cases} X_A = 0.2233 \\ X_B = 0.2895 \\ X_C = 0.2372 \end{cases}$$

and 25% in the risk free asset. Therefore, she will invest

$$\text{Investment} = 400,000\begin{pmatrix} 0.2233 \\ 0.2895 \\ 0.2372 \\ 0.25 \end{pmatrix} \Longrightarrow \begin{cases} X_A = 89,302 \\ X_B = 115,814 \\ X_C = 94,884 \\ X_F = 100,000 \end{cases}$$

(iii) From the indifference curves $\bar{R} = 0.5\sigma^2 + 0.965\sigma + 0.01K$ we know K is the fixed expected utility level , for the three portfolios under analysis we have

$$\begin{aligned}
&\bar{R}_T = 0.5\sigma_T^2 + 0.965\sigma_T + 0.01K_T \\
&11.76\% = 0.5(7.60\%)^2 + 0.965(7.60\%) + 0.01K_T \quad \Longrightarrow K_T = 4,137
\end{aligned}$$

$$\begin{aligned}
&\bar{R}_O = 0.5\sigma_O^2 + 0.965\sigma_O + 0.01K_O \\
&9.82\% = 0.5(5.70\%)^2 + 0.965(5.70\%) + 0.01K_O \quad \Longrightarrow K_O = 4,156
\end{aligned}$$

$$\begin{aligned}
&\bar{R}_F = 0.5\sigma_F^2 + 0.965\sigma_F + 0.01K_F \\
&4\% = 0.5(0\%)^2 + 0.965(0\%) + 0.01K_F \quad \Longrightarrow K_F = 4
\end{aligned}$$

from what we can conclude the investor preferences are $O \succ T \succ F$.

(f) (i) The RTF is nothing but the expected value of the utility function, with domain in the space $(\sigma, \bar{R})$. For the log utility we have

$$\begin{aligned}
\mathbb{E}\left(U(W)\right) &= \mathbb{E}\left(\ln(W)\right) \\
&= \mathbb{E}\left(\ln(W_0(1+R))\right) \\
&= \ln(W_0) + \mathbb{E}\left(\ln(1+R)\right)
\end{aligned}$$

and, for a general distribution of R, the last expectation cannot be written in terms of $\sigma = Var(R)$ and $\bar{R} = \mathbb{E}(R)$.

(ii) Using a second-order Taylor approximation around W_0 we get

$$\begin{aligned}
U(W) &\approx U(W_0) + (W - W_0)U'(W_0) + \frac{1}{2}(W - W_0)^2 U''(W_0) \\
\ln(W) &\approx \ln(W_0) + \frac{W - W_0}{W_0} - \frac{1}{2}\frac{(W - W_0)^2}{W_0^2} \\
\ln(W) &\approx \ln(W_0) + R - \frac{1}{2}(R^2)
\end{aligned}$$

where we used $U'(W) = 1/W$ and $U''(W) = -1/W^2$ and $W = W_0(1+R)$.
The approximation to the RTF is thus

$$\begin{aligned}
f(\sigma, \bar{R}) &\approx \mathbb{E}\left[\ln(W_0) + R - \frac{1}{2}(R^2)\right] \\
&\approx \ln(W_0) + \bar{R} - \frac{1}{2}\mathbb{E}(R^2) \\
&\approx \ln(W_0) + \bar{R} - \frac{1}{2}(\sigma^2 + \bar{R}^2)
\end{aligned}$$

(iii) Recall the efficient frontier is

$$\bar{R}_P = 4\% + 1.022\sigma_P$$

The optimum to the log investor is to maximize the approximation to his RTF which is equivalent to

$$\begin{aligned}
\max_P \quad & \bar{R}_P - \frac{1}{2}(\sigma_P^2 + \bar{R}_p^2) \\
\text{s.t.} \quad & \bar{R}_P = 4\% + 1.022\sigma_P
\end{aligned}$$

Using the restriction in the objective function we get

$$\max_{\sigma_P} \quad (4\% + 1.022\sigma_P) - \frac{1}{2}(\sigma_P^2 + (4\% + 1.022\sigma_P)^2)$$

From the FCO we get

$$\begin{aligned} 1.022 - \frac{1}{2}\left(2\sigma_P^* + 2(4\% + 1.022\sigma_P^*)1.022\right) = 0 \\ 1.022(1 - 0.04) - (1 + (1.022)^2)\sigma_P^* = 0 \\ \sigma_P^* = 0.4799 \end{aligned}$$

So, the log-investor has an optimal risk level of 47.99% and thus he should invest

$$x = \frac{47.99\%}{7.60\%} = 631.57\% \implies x_F = -531.57\% \ ,$$

assuming he faces no limits on borrowing, the optimal is to borrow 531.57% to invest 631.57% in the tangent portfolio.

(iv) Indifference curves are curves of fixed expected utility, i.e. fixed levels of the RTF, $f(\sigma, \bar{R}) = K$. Using the Taylor approximation in (ii) we have

$$\ln(W_0) + \bar{R} - \frac{1}{2}(\sigma^2 + \bar{R}^2) = K$$

Solving w.r.t. $\bar{R}$ would give us a quadratic form, so in this case it is easier to solve w.r.t. σ^2. We get

$$IC: \qquad \sigma^2 = 2\left(\ln(W_0) - K\right) + 2\bar{R} - \bar{R}^2$$

(v) Now we need to re-write the efficient frontier also w.r.t. σ^2, so we can compare its slope with the slope of the IC above.

$$EF: \qquad \bar{R} = 0.04 + 1.022\sigma \qquad \implies \qquad \sigma^2 = \left(\frac{\bar{R} - 0.04}{1.022}\right)^2$$

The two curves will have the same slope at

$$\begin{aligned} \left(\frac{\partial \sigma^2}{\partial \bar{R}}\right)_{IC} &= \left(\frac{\partial \sigma^2}{\partial \bar{R}}\right)_{EF} \\ 2 - 2\bar{R}^* &= 2\frac{\bar{R}^* - 0.04}{1.022}\frac{1}{1.022} \\ \bar{R}^* &= \frac{(1.022)^2 + 0.04}{1 + (1.022)^2} = 53\% \end{aligned}$$

An expected return of 53% is only possible if we leverage a lot to invest in T, concretely

$$53\% = (1 - x)4\% + x * 11.76\% \qquad \implies x = 631,57\%.$$

As expected we get exactly the same optimum as in (iii).

(g) Any investor who is risk neutral, cares only about maximising the expected return of investments. In the market situation of the exercise, when we can both lend and borrow at the same rate R_F without limits, it is always possible to borrow a bit more to increase the expected return. Without loss of generality – as the investor is indifferent between all investments with the same $\bar{R}$, we can focus on the efficient frontier to show the optimal risk level is $\sigma^*_{neutral} = +\infty$.

To see this note that

$$\max_P \quad \bar{R}_P \qquad \Leftrightarrow \qquad \max_{\sigma_p} \quad 4\% + 1.022\sigma_p \qquad \implies \qquad \sigma^*_{neutral} = +\infty$$

s.t. EF

(h) In the case of the risk lover we can focus on efficient portfolios, because for any fixed risk level, those are the ones that maximize expected return and a risk lover likes both risk and expected return. His optimum can be understood as, first maximize risk and then for the maximal risk maximize expected return. Or, maximize risk along the efficient frontier.

Recall the efficient frontier can be written both in terms of $\bar{R}_P = 0.04 + 1.022\sigma_P$ or $\sigma_P = \dfrac{\bar{R}_P - 0.04}{1.022}$.

Formally we can write

$$\max_{P} \quad \sigma_P \qquad \Leftrightarrow \qquad \max_{\bar{R}_P} \quad \frac{\bar{R}_P - 4\%}{1.022} \qquad \Longrightarrow \qquad \bar{R}^*_{lover} = +\infty$$

$$\text{s.t. EF}$$

3.2 Alternatives Techniques

Exercise 3.17.

(a) The geometric mean is given by

$$\bar{R}_j^G = \prod_{i=1}^{N} \left(1 + \bar{R}_{ij}\right)^{P_{ij}} - 1$$

Therefore, the geometric mean returns of the outcomes shown in Exercise **??** (assuming an initial investment of 100) are:

$$\bar{R}_A^G = \prod_{i=1}^{3} \left(1 + \bar{R}_{iA}\right)^{P_{iA}} - 1 = 1.05^{1/3} \times 1.06^{1/3} \times 1.09^{1/3} - 1 = 0.0665$$

$$\bar{R}_B^G = \prod_{i=1}^{3} \left(1 + \bar{R}_{iB}\right)^{P_{iB}} - 1 = 1.04^{1/4} \times 1.07^{1/2} \times 1.10^{1/4} - 1 = 0.0698$$

$$\bar{R}_C^G = \prod_{i=1}^{3} \left(1 + \bar{R}_{iC}\right)^{P_{iC}} - 1 = 1.01^{1/5} \times 1.09^{3/5} \times 1.18^{1/5} - 1 = 0.0907$$

Thus $C \succ B \succ A$.

(b) The idea of maximizing the geometric mean return to chose the optimal portfolio is supported by two main arguments:

1. has the highest return probability of reaching, or exceeding, any given wealth level in the shortest possible time; and
2. has the highest probability of exceeding any given wealth level over any given period of time.

Exercise 3.18.

(a) To use the stochastic dominance criterion we need to calculate the accumulated probability (first order stochastic dominance - FOSD) and the sum of accumulated probabilities (second order stochastic dominance - SOSD). Table 5 exhibits the accumulated and sum of accumulates probability.

Thus, using the accumulated probability we cannot find any FOSD. However, when we consider the sum of accumulated probability, the SOSD allows us to rank the projects, such that $C \succ B \succ A$.

	Accumulated Probability			Sum of Accumulated Probability		
Return	A	B	C	A	B	C
4%	0.2	0.0	0.0	0.2	0.0	0.0
5%	0.2	0.1	0.0	0.4	0.1	0.0
6%	0.5	0.4	0.4	0.9	0.5	0.4
7%	0.5	0.6	0.7	1.4	1.1	1.1
8%	0.9	0.9	0.9	2.3	2.0	2.0
9%	0.9	1.0	0.9	3.2	3.0	2.9
10%	1.0	1.0	1.0	4.2	4.0	3.9

Table 5: Exercise 3.18 - FOSD and SOSD

(b) Any risk averse investor would choose the same ranking as above. So any utility function with $U'(.) > 0$ and $U''(.) < 0$ would do. Log, negative exponencial, etc.

(c) Roy's safety first criterion is to minimize $Prob\,(R_P < R_L)$. Then,

$$Prob\,(R_A < 5\%) = 0.2; \quad Prob\,(R_B < 5\%) = 0.0; \quad Prob\,(R_C < 5\%) = 0.0$$

Therefore, under this decision criterion, investments B and C are preferable than investment A, but to the investor investments B and C are indifferent, $B \backsim C \succ A$.

(c) Kataoka's safety first criterion is to maximize R_L subject to $Prob\,(R_P < R_L) \leqslant \alpha$. For $\alpha = 10\%$, maximum R_L for each of the three possible investments is

$$I_A: \; R_L = 4\%; \quad I_B: \; R_L = 6\%; \quad I_C: \; R_L = 6\%$$

As before B and C are preferable to A, but B and C are indifferent, $B \backsim C \succ A$.

(d) Telser's safety first criterion is maximize $\bar{R}_P$ subject to $Prob\,(R_P \leqslant R_L) \leqslant \alpha$. In this problem, the restriction is $Prob\,(R_P \leqslant 0.5) \leqslant 0.1$, which excludes investment A, because $Prob\,(R_A \leqslant 0.5) = 0.2$ what does not respect the restriction. Investments B and C respect the restriction ($Prob\,(R_B \leqslant 0.5) = 0.1 \wedge Prob\,(R_C \leqslant 0.5) = 0.0$). However, these two investments are not indifferent as before. Actually, Telser's objective is to maximize $\bar{R}_P$, so that we must chose the investment with higher expected return. Thus,

$$\bar{R}_B = \sum_{i=1}^{5} P_{B_i} R_{B_i} = 0.1 \times 5 + 0.3 \times 6 + 0.2 \times 7 + 0.3 \times 8 + 0.1 \times 9 = 7$$

$$\bar{R}_C = \sum_{i=1}^{4} P_{C_i} R_{C_i} = 0.4 \times 6 + 0.3 \times 7 + 0.2 \times 8 + 0.1 \times 10 = 7.1$$

Then $\bar{R}_C > \bar{R}_B \Rightarrow C \succ B$.

(e) The geometric mean is given by

$$\bar{R}_j^G = \prod_{i=1}^{N} \left(1+\bar{R}_{ij}\right)^{P_{ij}} - 1$$

Therefore, the geometric mean returns of the outcomes are:

$$\bar{R}_A^G = \prod_{i=1}^{4} \left(1+\bar{R}_{A_i}\right)^{P_{A_i}} - 1 = 1.04^{0.2} \times 1.06^{0.3} \times 1.08^{0.4} \times 1.1^{0.1} - 1 = 0.0678$$

$$\bar{R}_B^G = \prod_{i=1}^{5} \left(1+\bar{R}_{B_i}\right)^{P_{B_i}} - 1 = 1.05^{0.1} \times 1.06^{0.3} \times 1.07^{0.2} \times 1.08^{0.3} \times 1.09^{0.1} - 1 = 0.0699$$

$$\bar{R}_C^G = \prod_{i=1}^{4} \left(1+\bar{R}_{C_i}\right)^{P_{C_i}} - 1 = 1.06^{0.4} \times 1.07^{0.3} \times 1.08^{0.2} \times 1.1^{0.1} - 1 = 0.0709$$

Thus $C \succ B \succ A$.

Exercise 3.19.

(a) The solution to this exercise is similar to that one of Exercise 3.18. However, we now have a continuous distribution what makes the calculations considerably more nasty if done with bare hands and qualifies the exercise to be solved using Excel or a similar software. So you may want to ask your instructor the excel file with the solution. Nevertheless we present the charts with the FOSD and SOSD (see Figure 14), from which we can conclude that none of these investments show FOSD or SOSD over the remaining ones.

(b) Recall that Roy's safety first criterion is to minimize $Prob\left(R_P < R_L\right)$. Therefore we want to calculate the following probabilities and rank them accordingly

$$\Pr\left(R_A < 5\%\right); \quad \Pr\left(R_B < 5\%\right); \quad \Pr\left(R_C < 5\%\right)$$

Since, the returns follow normal distributions that are not standardised, we need to standardise them. Recall that,

$$\frac{R_A - \bar{R}_A}{\sigma_A} = Z \backsim N\left(0,1\right)$$

Then,

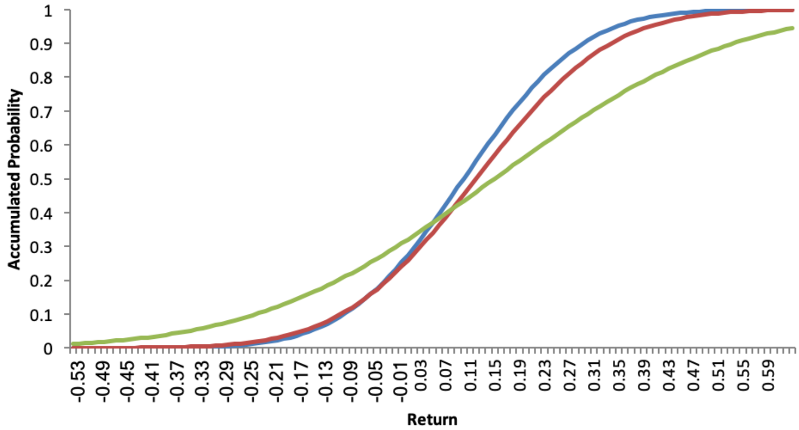
$$\Pr\left(R_A < 5\%\right) = \Pr\left(\frac{R_A - \bar{R}_A}{\sigma_A} < \frac{0.05-0.1}{0.15}\right) = \Pr\left(Z_A < -\frac{1}{3}\right) = N\left(-\frac{1}{3}\right) = 0.3694$$

$$\Pr\left(R_B < 5\%\right) = \Pr\left(\frac{R_B - \bar{R}_B}{\sigma_B} < \frac{0.05-0.12}{0.17}\right) = \Pr\left(Z_B < -0.41176\right) = N\left(-0.41176\right) = 0.3400$$

$$\Pr\left(R_C < 5\%\right) = \Pr\left(\frac{R_C - \bar{R}_C}{\sigma_C} < \frac{0.05-0.15}{0.30}\right) = \Pr\left(Z_C < -\frac{1}{3}\right) = N\left(-\frac{1}{3}\right) = 0.3694$$

Therefore, under this decision criterion, investments B is preferable than investment A and C, which are indifferent, $B \succ A \backsim C$.

(a)



(b)

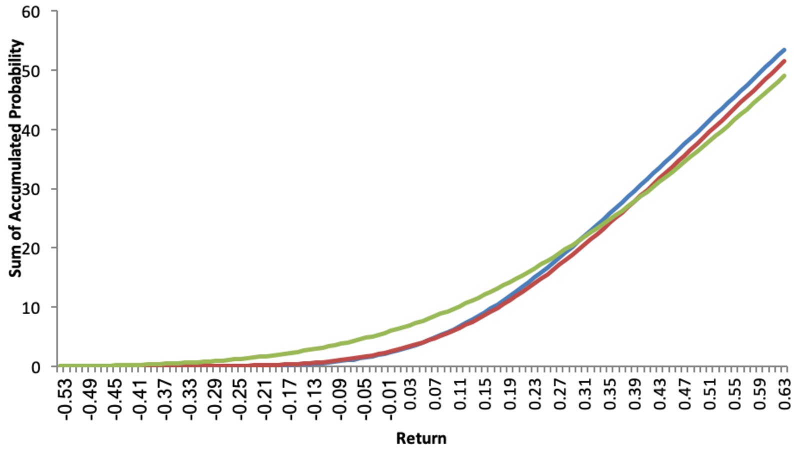


Figure 14: Exercise 3.19 – first (a) and second-order (b) stochastic dominance graphs.

(c) Kataoka's safety first criterion is to maximize R_L subject to $Prob\,(R_P < R_L) \leqslant \alpha$. For $\alpha = 10\%$, maximum R_L for each of the three possible investments is:

– Investment A

$$Prob\,(R_A \leqslant R_{L_A}) \leqslant \alpha$$

$$Prob\left(Z_A \leqslant \frac{R_{L_A} - \bar{R}_A}{\sigma_A}\right) \leqslant \alpha$$

$$Prob\left(Z_A \leqslant \frac{R_{L_A} - 0.1}{0.15}\right) \leqslant 0.1$$

$$\frac{R_{L_A} - 0.1}{0.15} \geqslant -1.282$$

$$R_{L_A} \geqslant -0.0923$$

$$R_{L_A} = -0.0922$$

– Investment B

$$Prob\,(R_B \leqslant R_{L_B}) \leqslant \alpha$$

$$Prob\left(Z_B \leqslant \frac{R_{L_B} - \bar{R}_B}{\sigma_B}\right) \leqslant \alpha$$

$$Prob\left(Z_B \leqslant \frac{R_{L_B} - 0.12}{0.17}\right) \leqslant 0.1$$

$$\frac{R_{L_B} - 0.12}{0.17} \geqslant -1.282$$

$$R_{L_B} \geqslant -0.0979$$

$$R_{L_B} = -0.0978$$

– Investment C

$$Prob\,(R_C \leqslant R_{L_C}) \leqslant \alpha$$

$$Prob\left(Z_C \leqslant \frac{R_{L_C} - \bar{R}_C}{\sigma_C}\right) \leqslant \alpha$$

$$Prob\left(Z_C \leqslant \frac{R_{L_C} - 0.15}{0.30}\right) \leqslant 0.1$$

$$\frac{R_{L_C} - 0.15}{0.30} \geqslant -1.282$$

$$R_{L_C} \geqslant -0.2346$$

$$R_{L_C} = -0.2345$$

Thus, A is preferable to B that is preferable to C, $A \succ B \succ C$.

(d) Telser's safety first criterion is maximize $\bar{R}_P$ subject to $Prob\,(R_P \leqslant R_L) \leqslant \alpha$. In this problem, the restriction is $Prob\,(R_P \leqslant 0.5) \leqslant 0.1$, which excludes the three investments, since

$$Prob\,(R_A \leqslant 0.5) = 0.3694 \nleqslant 0.1$$

$$Prob\,(R_B \leqslant 0.5) = 0.3400 \nleqslant 0.1$$

$$Prob\,(R_C \leqslant 0.5) = 0.3694 \nleqslant 0.1$$

(e) The Value at Risk is given by $\bar{R}_i - Z_\alpha \sigma_i$. Since we set $\alpha = 0.025$ we have $Z_{0.025} = 1.96$. Therefore,

$$VaR_A = \bar{R}_A - 1.96\sigma_A = 0.1 - 1.96 \times 0.15 = -0.196$$

$$VaR_B = \bar{R}_B - 1.96\sigma_B = 0.12 - 1.96 \times 0.17 = -0.2139$$

$$VaR_C = \bar{R}_C - 1.96\sigma_C = 0.15 - 1.96 \times 0.30 = -0.4392$$

Thus, A is preferable to B that is preferable to C, $A \succ B \succ C$.

4 Equilibrium in Financial Markets

4.1 CAPM

Exercise 4.1.

(a) Using the single-index model, the risk of a security i is given by $\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_{e_i}^2$, where the first term is the systematic risk and the second term is the specific risk. Using in the expression the values given in the problem

$$\sigma_A^2 = \beta_A^2\sigma_m^2 + \sigma_{e_A}^2 = 1.5^2 + 0.5^2 + 0.05 = 0.6125$$

Therefore the risk is $\sigma_A = 0.783$

(b) If the specific risk is null, then $\sigma_{eC}^2 = 0$. Security's C variance is $\sigma_C^2 = 0.75$. Thus, using the single-index model the β of C is

$$\begin{aligned}\sigma_C^2 &= \beta_C^2\sigma_m^2 + \sigma_{eC}^2\\ 0.75 &= \beta_C^2 \times 0.25 + 0\\ \beta_C &= 1.73205\end{aligned}$$

(c) From CAPM we know the return of a security is $R_A = R_f + \beta\,(R_m - R_f)$. From the data we know $R_A = 20\%$ and security B is risk-free ($\beta = 0$), so that the risk-free interest rate is 10%. Thus,

$$\begin{aligned}\bar{R}_A &= R_f + \beta\left(\bar{R}_m - R_f\right)\\ 0.2 &= 0.1 + 1.5(\bar{R}_m - 0.1)\\ \bar{R}_m &= \frac{0.25}{1.5}\\ &= 0.1667\end{aligned}$$

(d) These assumptions are those of CAPM. See your notes or the textbook.

Exercise 4.2.

(a) From CAPM we know the return of a security is $R_A = R_f + \beta\,(R_m - R_f)$ and its β is $\beta = \frac{\sigma_{i,m}}{\sigma_m^2}$. Since the market risk is 0.1, its variance is $\sigma_m^2 = 0.01$. The covariance between asset's i return and the market return is given by $\sigma_{i,m} = \sigma_i\sigma_m\rho_{i,m}$. Finally, $\rho_{i,m} = 1$, since security i is perfectly correlated with the market. So, using the given data, $\sigma_{i,m} = 0.2 \times 0.1 \times 1 = 0.02$. Thus,

$$\beta = \frac{\sigma_{i,m}}{\sigma_m^2} = \frac{0.02}{0.01} = 2$$

and

$$\begin{aligned}R_i &= R_f + \beta\,(R_m - R_f)\\ &= 0.05 + 2\,(0.1 - 0.05)\\ &= 0.15\end{aligned}$$

(b) The request line is given by the single-index model $R_i = \alpha_i + \beta_i\bar{R}_m$. We know β_i and $\bar{R}_m$. To draw the line we need to find α_i, which is given by the expression $\alpha_i = R_i - \beta_i\bar{R}_m$. In this case, $\alpha_i = 0.15 - 2 \times 0.1 = -0.05$. The line is represented in Figure 15.

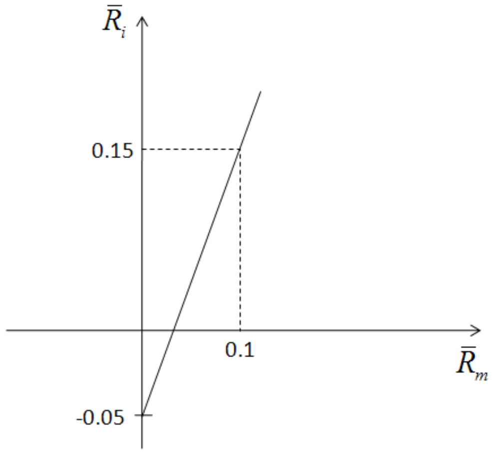


Figure 15: Exercise 4.2 - Characteristic line

Exercise 4.3.

(a) Using CAPM to calculate the expected return

$$\bar{R}_X = R_f + \beta_X \left(\bar{R}_m - R_f\right) = 0.07 + \beta\,(0.09 - 0.07)$$

β_X can be found using $\beta_X = \frac{\sigma_{Xm}}{\sigma_m^2}$. Thus

$$\beta_X = \frac{\sigma_{Xm}}{\sigma_m^2} = \frac{0.02}{0.025} = 0.8$$

Finally,

$$\bar{R}_X = 0.07 + 0.8\,(0.09 - 0.07) = 0.086$$

(b) If $\bar{R}_m = 0.12$ then the expected return is

$$\bar{R}_X = R_f + \beta_X \left(\bar{R}_m - R_f\right) = 0.07 + 0.08\,(0.12 - 0.07) = 0.11$$

Since the CAPM's expected return is lower than the market expected return, the price is underpriced.

Exercise 4.4. To know the return of each portfolio to look for an arbitrage opportunity we need to find each portfolio β, which is the weighted average of each security's β, and each portfolio's expected return. Thus

$$\begin{aligned}\beta_1 &= x_{1A}\beta_A + x_{1B}\beta_B + x_{1C}\beta_C \\ &= -0.5 \times 1.5 + 0 \times 1 + 1.5 \times 0.5 \\ &= 0\end{aligned}$$

$$\begin{aligned}\beta_2 &= x_{2A}\beta_A + x_{2B}\beta_B + x_{2C}\beta_C \\ &= 0 \times 1.5 - 1 \times 1 + 2 \times 0.5 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\bar{R}_1 &= x_{1A}\bar{R}_A + x_{1B}\bar{R}_B + x_{1C}\bar{R}_C \\ &= -0.5 \times 0.12 + 0 \times 0.1 + 1.5 \times 0.05 \\ &= 0.015\end{aligned}$$

$$\begin{aligned}\bar{R}_2 &= x_{2A}\bar{R}_A + x_{2B}\bar{R}_B + x_{2C}\bar{R}_C \\ &= 0 \times 0.12 - 1 \times 0.1 + 2 \times 0.05 \\ &= 0\end{aligned}$$

Therefore, we have two risk-free portfolios with different expected returns, implying an arbitrage opportunity. So, without investing a single penny we can short-sale portfolio 2 and buy portfolio 1, earning an arbitrage profit of 1.5%.

Exercise 4.5.

(a) To fill the table given in the exercise we need to find β_m, β_c, $\bar{R_A}$ and $\bar{R_B}$. By definition, $\beta_m = 1$. Since security C is risk-free, its β is null and $R_f = 0.02$. Thus the expected return of securities A and B is

$$\begin{aligned}\bar{R}_A &= R_f + \beta_A \left(\bar{R}_m - R_f\right) \\ &= 0.02 + 0.08 \times 0.5 \\ &= 0.06\end{aligned}$$

$$\begin{aligned}\bar{R}_B &= R_f + \beta_B \left(\bar{R}_m - R_f\right) \\ &= 0.02 + 0.08 \times (-0.1) \\ &= 0.012\end{aligned}$$

(b) Accordingly to the single-index model total risk is

$$\sigma_i^2 = \underbrace{\beta_i^2 \sigma_m^2}_{\text{Systematic Variance}} + \underbrace{\sigma_{e_i}^2}_{\text{Specific Variance}}$$

Thus, for security A the systematic variance is $\beta_A^2 \sigma_m^2 = 0.5^2 \times 0.04^2 = 0.0004$ and the specific variance is $\sigma_{e_A}^2 = \sigma_A^2 - \beta_A^2 \sigma_m^2 = 0.12^2 - 0.0004 = 0.014$. Thus, systematic risk é $\sqrt{0.0004} = 0.02$ and specific risk is $\sqrt{0.014} = 0.1183$. For security B the systematic risk is $\beta_B^2 \sigma_m^2 = (-0.1)^2 \times 0.04^2 = 0.000016$ and the specific risk is $\sigma_{e_B}^2 = \sigma_B^2 - \beta_B^2 \sigma_m^2 = 0.12^2 - 0.000016 = 0.014384$. Thus, systematic risk é $\sqrt{0.000016} = 0.004$ and specific risk is $\sqrt{0.014384} = 0.1199$.

(c) If CAPM holds any investor has always incentives to compose a portfolio with a risk-free asset and the market portfolio. By holding the market portfolio, well diversified by definition, the investor will eliminate the portfolio's specific risk. If CAPM holds, expectations are homogeneous meaning that all investors share the same expectations, which should imply a very low level of trading. If, for some reason the expected return in the market for a given security is the predict by CAPM, it should means the security is not rewarding properly its systematic risk, therefore, it is not an equilibrium return and we have an arbitrage opportunity. In this case, expectations are temporarily heterogenous, until the market adjust to its equilibrium on the security market line.

Exercise 4.6.

(a) The equation for the security market line is $\bar{R}_i = R_f + \beta_i \left(\bar{R}_m - R_f\right)$. Thus, from the data in the problem we have:

$$\begin{cases} \bar{R}_1 = R_f + \beta_1 \left(\bar{R}_m - R_f\right) \\ \bar{R}_2 = R_f + \beta_2 \left(\bar{R}_m - R_f\right) \end{cases} \Leftrightarrow \begin{cases} 0.06 = R_f + 0.5 \left(\bar{R}_m - R_f\right) \\ 0.12 = R_f + 1.5 \left(\bar{R}_m - R_f\right) \end{cases}$$

Solving in order to $\bar{R}_m$ and R_f,

$$\begin{cases} \bar{R}_m = 0.09 \\ R_f = 0.03 \end{cases}$$

Finally, the the security market line is

$$\bar{R}_i = 0.03 + 0.06\beta_i$$

(b) Using the above security market line, an asset with a beta of 2 would have an expected return of:

$$\bar{R}_i = 0.03 + 0.06\beta_i \qquad = 0.03 + 0.06 \times 2 = 0.15$$

(c) To exploit an arbitrage strategy we need to find a portfolio with asset 1 and asset 2 that replicates the risk ($\beta_p = 1.2$) of the given asset, but with a different return. since the β of a portfolio is the weighted average of each security β and the weights of asset 1 and asset 2 must sum 1, it comes

$$\begin{cases} x_1 + x_2 = 1 \\ \beta_p = x_1\beta_1 + x_2\beta_2 \end{cases} \Leftrightarrow \begin{cases} x_2 = 1 - x_1 \\ 1.2 = 0.5x_1 + 1.5(1 - x_2) \end{cases} \Leftrightarrow \begin{cases} x_1 = 0.3 \\ x_2 = 0.7 \end{cases}$$

The return of this replication portfolio is $R_p = 0.3 \times 0.06 + 0.7 \times 0.12 = 0.102$. Therefore, we have an arbitrage opportunity that can be exploited by short-selling the replication portfolio and buying asset 3, making an arbitrage profit of $0.15 - 0.102 = 0.048$.

Exercise 4.7.

Given the security market line in this problem, for the two stocks to be fairly priced their expected returns must be:

$$\bar{R}_X = 0.04 + 0.08 \times 0.5 = 0.08$$
$$\bar{R}_X = 0.04 + 0.08 \times 2 = 0.2$$

If the expected return on either stock is higher than its return given above, the stock is a good buy.

Exercise 4.8.

Given the security market line in this problem, the two funds' expected returns would be:

$$\bar{R}_A = 0.04 + 0.19 \times 0.8 = 0.192 > 0.1 \rightarrow \text{ bad performance}$$

$$\bar{R}_B = 0.04 + 0.19 \times 1.2 = 0.268 > 0.15 \rightarrow \text{ bad performance}$$

Comparing the above returns to the funds' actual returns, we see that both funds performed poorly, since their actual returns were below those expected given their beta risk.

Exercise 4.9. Part (a) and Part (b) can be answered simultaneously.
The security market line is:

$$\bar{R}_i = R_f + \beta \left(\bar{R}_m - R_f\right)$$

Substituting the given values for assets 1 and 2 gives two equations with two unknowns and solving simultaneously gives:

$$\begin{cases} 0.094 = R_f + 0.8\left(\bar{R}_m - R_f\right) \\ 0.134 = R_f + 1.3\left(\bar{R}_m - R_f\right) \end{cases} \Leftrightarrow \begin{cases} \bar{R}_f = 0.03 \\ \bar{R}_m = 0.11 \end{cases}$$

Exercise 4.10. [OBS: this exercise is out of place, it should be in the APT subsection]

A general equilibrium relationship for security returns must imply absence of arbitrage. In this case we consider systematic risk to be concerned with market risk and interest rate risk. So it would be interesting to find an expression that explain returns with two risk factors: market risk; and interest rate risk. To do so, we need to create an arbitrage portfolio as follows:

$$\sum_i X_i^{ARB} \times 1 = 0 \tag{10}$$

$$a_{ARB} = \sum_i X_i^{ARB} a_i = 0 \tag{11}$$

$$b_{ARB} = \sum_i X_i^{ARB} b_i = 0 \tag{12}$$

Since the above portfolio has zero net investment and zero risk with respect to the given two-factor model, by the force of arbitrage its expected return must also be zero:

$$\bar{R}_{ARB} = \sum_i X_i^{ARB} \bar{R}_i = 0 \tag{13}$$

From a theorem of linear algebra, since the above orthogonality conditions (10), (11) and (12) with respect to the X_i^{ARB} result in orthogonality condition (13) with respect to the X_i^{ARB}, $\bar{R}_i$ can be expressed as a linear combination of 1, a_i and b_i:

$$\bar{R}_i = \lambda_0 \times 1 + \lambda_1 a_i + \lambda_2 b_i \tag{14}$$

We can create a zero-risk investment portfolio (without systematic risk) to find λ_0 as follows:

$$\sum_i X_i^Z = 1$$

$$a_Z = \sum_i X_i^Z a_i = 0$$

$$b_Z = \sum_i X_i^Z b_i = 0$$

Substituting the above equations into equation (14) gives:

$$\bar{R}_Z = \sum_i X_i^Z \bar{R}_i = \lambda_0 \sum_i X_i^Z + \lambda_1 \sum_i X_i^Z a_i + \lambda_2 \sum_i X_i^Z b_i = \lambda_0$$

Then, we can create a strictly market-risk investment portfolio to find λ_1 as follows:

$$\sum_i X_i^M = 1$$

$$a_M = \sum_i X_i^M a_i = 1$$

$$b_M = \sum_i X_i^M b_i = 0$$

Substituting the above equations into equation (14) gives:

$$\bar{R}_M = \sum_i X_i^M \bar{R}_i = \lambda_0 \sum_i X_i^M + \lambda_1 \sum_i X_i^M a_i + \lambda_2 \sum_i X_i^M b_i = \lambda_0 + \lambda_1$$

or

$$\lambda_1 = \bar{R}_M - \lambda_0 = \bar{R}_M - \bar{R}_Z$$

Finally, we can create a strictly interest rate-risk investment portfolio to find λ_2 as follows:

$$\sum_i X_i^C = 1$$

$$a_C = \sum_i X_i^C a_i = 0$$

$$b_C = \sum_i X_i^C b_i = 1$$

Substituting the above equations into equation (14) gives:

$$\bar{R}_C = \sum_i X_i^C \bar{R}_i = \lambda_0 \sum_i X_i^C + \lambda_1 \sum_i X_i^C a_i + \lambda_2 \sum_i X_i^C b_i = \lambda_0 + \lambda_2$$

or

$$\lambda_2 = \bar{R}_C - \lambda_0 = \bar{R}_C - \bar{R}_Z$$

Substituting the derived values for λ_0, λ_1 and λ_2 into equation (14), we have:

$$\bar{R}_i = \bar{R}_Z + \left(\bar{R}_M - \bar{R}_Z\right) \times a_i + \left(\bar{R}_C - \bar{R}_Z\right) \times b_i$$

Exercise 4.11.

(a) In the graph (see Figure 16) , the efficient frontier with riskless lending but no riskless borrowing is the ray extending from R_F to the tangent portfolio L and then along the minimum-variance curve through the market portfolio M and out toward infinity (assuming unlimited short sales). All investors who wish to lend will hold tangent portfolio L in some combination with the riskless asset, since no other portfolio offers a higher slope. Furthermore, unless all investors lend or invest solely in portfolio L, the market portfolio M will be along the minimum-variance curve to the right of portfolio L, since the market portfolio is a wealth-weighted average of all the efficient risky-asset portfolios held by investors, and no rational investor would hold a risky-asset portfolio along the curve to the left of L.

The expected return on a zero-beta asset is the intercept of a line tangent to the market portfolio, and the zero-beta portfolio on the minimum-variance frontier must be below the global minimum variance portfolio of risky assets by the geometry of the graph. Furthermore, by the geometry of the graph, since the risk-free lending rate is the intercept of the line tangent to portfolio L, and since L is to the left of M on the minimum-variance curve, the risk-free lending rate must be below the expected return on a zero-beta asset.

(b) The zero-beta security market line is the line in the graph (see Figure 17) extend from the expected return on a zero-beta asset through the market portfolio and out toward infinity (assuming unlimited short sales). The expected return-beta relationships of all risky securities risky-asset portfolios (including the market portfolio M and portfolio L) are described by that line. The other line from the risk-free lending rate to portfolio L only

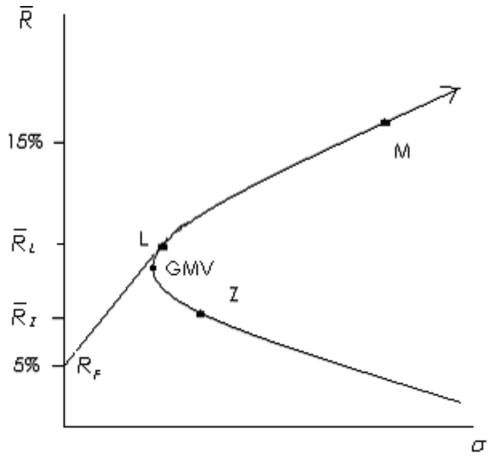


Figure 16: Exercise 4.11 - Efficient Frontier

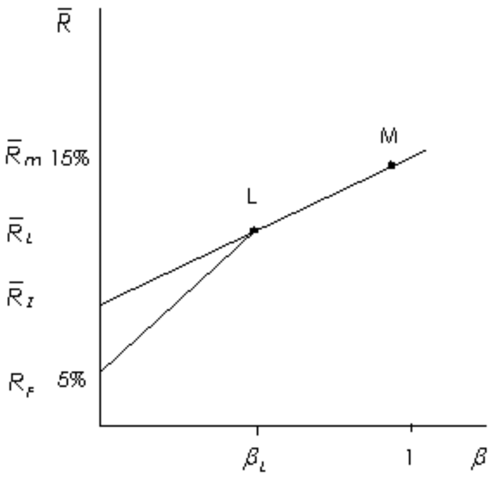


Figure 17: Exercise 4.11 - Zero-Beta Security Market Line

describes the expected return-beta relationships of combination portfolios of the risk-free asset and portfolio L; those combination portfolios are not described by the zero-beta security market line.

Exercise 4.12. If the post-tax form of the equilibrium pricing model holds, then:

$$\bar{R}_i = R_F + \left[\left(\bar{R}_m - R_F\right) - \left(\delta_m - R_F\right)\tau\right]\beta_i + \left(\delta_i - R_F\right)\tau$$

If the standard CAPM model holds, then:

$$\bar{R}_i = R_F + \left(\bar{R}_m - R_F\right)\beta_i$$

Let us assume that the post-tax model holds instead of the standard model, and $\delta_m = R_F$.

Then, for a stock with $(\delta_i - R_F)\,\tau > 0$, if you are right and use the post-tax model, you would correctly believe that the stock has a higher expected return than the stock's return expected by the other investor using the standard model.

Similarly, for a stock with $(\delta_i - R_F)\,\tau < 0$, you would correctly believe the stock has a lower expected return than the stock's return expected by the other investor using again the standard model.

Therefore, if you manage two securities, one with $(\delta_i - R_F)\,\tau > 0$ and the other with $(\delta_i - R_F)\,\tau < 0$, you can swap them with the other investor. Since you both have heterogenous expectations, each one of you will believe that are making an excess return.

Now consider a specific example using the following data for stocks A and B, the market portfolio and the riskless asset:

$$\beta_A = 1.0;\ \delta_A = 8\%;\ \beta_B = 1.0;\ \delta_B = 0\%;\ \bar{R}_M = 14\%;\ \delta_m = 4\%;\ R_F = 4\%;\ \tau = 0.25$$

If the post-tax model holds, then you would correctly believe that the equilibrium expected returns for the two stocks are:

$$\begin{cases} \bar{R}_A = 4 + ((14-4) - (4-4)\times 0.25)\times 1.0 + (8-4)\times 0.25 \\ \bar{R}_B = 4 + ((14-4) - (4-4)\times 0.25)\times 1.0 + (0-4)\times 0.25 \end{cases} \Leftrightarrow \begin{cases} \bar{R}_A = 15\% \\ \bar{R}_B = 13\% \end{cases}$$

While the other investor using the standard model would incorrectly believe that the stocks' equilibrium expected returns are:

$$\begin{cases} \bar{R}_A = 4 + (14-4)\times 1.0 \\ \bar{R}_B = 4 + (14-4)\times 1.0 \end{cases} \Leftrightarrow \begin{cases} \bar{R}_A = 14\% \\ \bar{R}_B = 14\% \end{cases}$$

You would tend to buy stock A and sell stock B short. Of course, residual risk puts a limit to the amount of unbalancing you would do. But by some unbalancing, you earn an excess return. At the same time the other investor using the standard model would be indifferent between the two stocks. If your tax factor was below the aggregate tax factor (τ lower than 0.25) then you should buy stock B from the other investor and sell that investor stock A. The fact that this will lead to higher after-tax cash flows for you is straightforward.

4.2 APT

Exercise 4.13.

(a) If APT's model holds, returns are generated by a multi-index model such that

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

Where,

λ_j is the risk premium associated to the risk factor I_j, $j = 1, 2$
b_{ik} is the sensitivity of security i to the risk factor I_j, $j = 1, 2$

To derive the equilibrium model we need to calculate λ_j. Since we know the expected returns for three portfolios X, Y and Z and the sensitivity of each to the risk factors, we can build a equation system with three equations and three variables:

$$\begin{cases} \bar{R}_X = \lambda_0 + \lambda_1 b_{X1} + \lambda_2 b_{X2} \\ \bar{R}_Y = \lambda_0 + \lambda_1 b_{Y1} + \lambda_2 b_{Y2} \\ \bar{R}_Z = \lambda_0 + \lambda_1 b_{Z1} + \lambda_2 b_{Z2} \end{cases} \Leftrightarrow \begin{cases} 0.16 = \lambda_0 + \lambda_1 1 + \lambda_2 0.7 \\ 0.14 = \lambda_0 + \lambda_1 0.6 + \lambda_2 1 \\ 0.11 = \lambda_0 + \lambda_1 0.5 + \lambda_2 1.5 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0.095929 \\ \lambda_1 = 0.0572816 \\ \lambda_2 = 0.009709 \end{cases}$$

Finally,

$$\bar{R}_i = 0.0959 + 0.0573 b_{i1} + 0.0097 b_{i2}$$

(b) If this portfolio does not respect the equilibrium conditions defined in part a, we will find an arbitrage opportunity. Thus, first we need to check the non arbitrage expected return for portfolio W:

$$\begin{aligned} \bar{R}^e_W &= 0.0959 + 0.0573 b_{i1} + 0.0097 b_{i2} \\ &= 0.0959 + 0.0573 \times 1 + 0.0097 \times 0 \\ &= 0.1532 \end{aligned}$$

Since, $\bar{R}^e_W = 0.1489 > \bar{R}_W = 0.13$, this portfolio W is not at equilibrium, allowing the existence of arbitrage opportunities. The low level of the market expected return implies that the current market price is too high, meaning portfolio W is overpriced. Thus, we would like to short sell it and buy a fairly priced portfolio that replicates W's cash flows and risk. The subsequent increase in W's supply will force its price to fall until reach a non arbitrage price, such that $\bar{R}^e_W = \bar{R}_W$.

(c) Recall that APT equilibrium model with a risk-free asset is

$$\bar{R}_i = R_F + b_{i1}\lambda_1 + b_{i2}\lambda_2 \tag{15}$$

and that if the CAPM is the equilibrium model, it holds for all securities, as well as all portfolios of securities. Assume the indexes can be represented by portfolios of securities. Then, if the CAPM holds, the equilibrium return on each λ_j is given by the CAPM or

$$\lambda_1 = \beta_{\lambda 1}\left(\bar{R}_m - R_F\right)$$

$$\lambda_2 = \beta_{\lambda 2}\left(\bar{R}_m - R_F\right)$$

Substituting into Equation (15) yields

$$\begin{aligned} \bar{R}_i &= R_F + b_{i1}\beta_{\lambda 1}\left(\bar{R}_m - R_F\right) + b_{i2}\beta_{\lambda 2}\left(\bar{R}_m - R_F\right) \\ &= R_F + \left(b_{i1}\beta_{\lambda 1} + b_{i2}\beta_{\lambda 2}\right)\left(\bar{R}_m - R_F\right) \end{aligned}$$

Defining β_i as $(b_{i1}\beta_{\lambda 1} + b_{i2}\beta_{\lambda 2})$ results in the expected return of $\bar{R}_i$ being priced by the CAPM:

$$\bar{R}_i = R_F + \beta_i\left(\bar{R}_m - R_F\right)$$

Which is a solution with multiple factors fully consistent with CAPM.

Exercise 4.14.

(a) (i) As in the previous exercise, if APT's model holds, returns are generated by a multi-index model such that

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

Thus, to find the equation that holds with these three securities we should proceed as before

$$\begin{cases} \bar{R}_X = \lambda_0 + \lambda_1 b_{X1} + \lambda_2 b_{X2} \\ \bar{R}_Y = \lambda_0 + \lambda_1 b_{Y1} + \lambda_2 b_{Y2} \\ \bar{R}_Z = \lambda_0 + \lambda_1 b_{Z1} + \lambda_2 b_{Z2} \end{cases} \Leftrightarrow \begin{cases} 0.10 = \lambda_0 + \lambda_1 0.5 + \lambda_2 1 \\ 0.12 = \lambda_0 + \lambda_1 1 + \lambda_2 1.5 \\ 0.11 = \lambda_0 + \lambda_1 0.5 + \lambda_2 2 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0.0675 \\ \lambda_1 = 0.0015 \\ \lambda_2 = 0.025 \end{cases}$$

Finally,

$$\bar{R}_i = 0.0675 + 0.015 b_{i1} + 0.025 b_{i2}$$

(ii) The risk-free rate is given by λ_0, thus $R_F = 0,0675$.

(b) Security D will be at equilibrium if its equilibrium expected return rate equals its market expected return rate. Thus, we first need to compute the equilibrium expected return using our APT model,

$$\begin{aligned} \bar{R}^e_D &= 0.0675 + 0.015 b_{i1} + 0.025 b_{i2} \\ &= 0.0675 + 0.015 \times 2 + 0.025 \times 0.5 \\ &= 0.1075 \end{aligned}$$

Since, $\bar{R}^e_D = 0.1075 < \bar{R}_D = 0.12$, this portfolio D is not at equilibrium, allowing the existence of arbitrage opportunities. The high level of market expected return implies that the current market price is too low, meaning portfolio D is underpriced. Thus, we would like to buy it and short sell a fairly priced portfolio that replicates D's cash flows and risk. The subsequent increase in D's demand will force its price to increase until reach a non arbitrage price, such that $\bar{R}^e_D = \bar{R}_D$.

(c) As long as we can manage to find the right proportions to invest in each security, it should be possible to build the replication portfolio with securities A, B and C. This new portfolio sensitivity to factor 1 and 2 must equal the sensitivity of security D to these same risk factors. Since, the portfolio sensitivity is given by the weighted average of each security sensitivity and the proportions invested in the three securities must sum 1, it comes

$$\begin{cases} b_{D1} = x_A b_{A1} + x_B b_{B1} + x_C b_{C1} \\ b_{D2} = x_A b_{A2} + x_B b_{B2} + x_C b_{C2} \\ x_A + x_B + x_C = 1 \end{cases} \Leftrightarrow \begin{cases} 2 = x_A 0.5 + x_B 1 - x_C 0.5 \\ 0.5 = x_A 1 + x_B 1.5 + x_C 2 \\ x_A + x_B + x_C = 1 \end{cases} \Leftrightarrow \begin{cases} x_A = 1 \\ x_B = 1 \\ x_C = -1 \end{cases}$$

Exercise 4.15.

(a) To create an arbitrage opportunity, it must be possible to make a profit without investment and risk, which means

$$\begin{cases} \sum_{1=1}^{3} x_i = 0 \\ \sum_{1=1}^{3} x_i b_{i,1} = 0 \end{cases}$$

A possible portfolio that respects these conditions is

$$\begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = 1 \end{cases}$$

Its expected return is $\bar{R}_p = \sum_{i=1}^{3} x_i \bar{R}_i = 1 \times 12 - 2 \times 15 + 1 \times 40 = 22$.

(b) The equilibrium relationship associated to the arbitrage pricing model is

$$\begin{cases} 0.10 = \lambda_0 + \lambda_1 \times 1 \\ 0.20 = \lambda_0 + \lambda_1 \times 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0 \\ \lambda_1 = 0.1 \end{cases}$$

Therefore, the APT line is

$$\bar{R}_i = 0 + 0.1 b_{i1} = 0.1 b_{i1}$$

Thus, the missing value is $\bar{R}_3 = 0.1 b_{i1} = 0.1 \times 3 = 0.3$

If we compare the expected returns with the equilibrium returns we can conclude

- Since $\bar{R}_1 = 12\% > \bar{R}_1^e = 10\%$, if you buy it you will get a return higher than what you would receive in equilibrium because Security 1 is underpriced. Therefore you should buy it
- Since $\bar{R}_2 = 15\% < \bar{R}_2^e = 20\%$, if you buy it you will get a return lower than what you would receive in equilibrium because Security 1 is overpriced. Therefore you should (short) sell it
- Since $\bar{R}_3 = 40\% > \bar{R}_3^e = 30\%$, if you buy it you will get a return higher than what you would receive in equilibrium because Security 1 is underpriced. Therefore you should buy it

(c) Without transaction costs, a linear relationship between βs and returns implies that any point outside this line represents an arbitrage opportunity and a abnormal return. However, if we consider transaction costs, the expected return in equilibrium must be corrected, falling by the amount they assume. If transaction costs were not constant, the relationship between βs and returns will not be linear at all. But, if the abnormal return and the transaction costs occur at the same time, they may cancel or at least be lower than transactions costs, reaching a new equilibrium outside the original line, since one cannot earn abnormal returns. Thus, transaction costs may imply a non linear relationship, which still respects the law of one price and the non arbitragem assumption.

Exercise 4.16.

(a) From the relationship between CAPM and APT we know that $\lambda_j = \left(\bar{R}_m - R_F\right) \beta_{\lambda j}$. Thus, to have consistency between CAPM and the data we need to observe

$$\begin{cases} \lambda_1 = \left(\bar{R}_m - R_F\right) \beta_{\lambda 1} \\ \lambda_2 = \left(\bar{R}_m - R_F\right) \beta_{\lambda 2} \end{cases} \Leftrightarrow \begin{cases} \beta_{\lambda 1} = \dfrac{\lambda_1}{\left(\bar{R}_m - R_F\right)} \\ \beta_{\lambda 2} = \dfrac{\lambda_2}{\left(\bar{R}_m - R_F\right)} \end{cases}$$

From the data in the problem we know $\bar{R}_m - R_F = 0,04$, so we have to calculate λ_1, λ_2 and λ_3, using the previously used equilibrium condition $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$. Thus,

$$\begin{cases} \bar{R}_A = \lambda_0 + \lambda_1 b_{A1} + \lambda_2 b_{A2} \\ \bar{R}_B = \lambda_0 + \lambda_1 b_{B1} + \lambda_2 b_{B2} \\ \bar{R}_C = \lambda_0 + \lambda_1 b_{C1} + \lambda_2 b_{C2} \end{cases} \Leftrightarrow \begin{cases} 0.12 = \lambda_0 + \lambda_1 1 + \lambda_2 0.5 \\ 0.134 = \lambda_0 + \lambda_1 3 + \lambda_2 0.2 \\ 0.12 = \lambda_0 + \lambda_1 3 - \lambda_2 0.5 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0.1 \\ \lambda_1 = 0.01 \\ \lambda_2 = 0.02 \end{cases}$$

Finally,

$$\begin{cases} \beta_{\lambda 1} = \dfrac{\lambda_1}{(\bar{R}_m - R_F)} \\ \beta_{\lambda 2} = \dfrac{\lambda_2}{(\bar{R}_m - R_F)} \end{cases} \Leftrightarrow \begin{cases} \beta_{\lambda 1} = \dfrac{0.01}{0.04} = 0.25 \\ \beta_{\lambda 2} = \dfrac{0.02}{0.04} = 0.5 \end{cases}$$

(b) Again, from the relationship between CAPM and APT, the β of each portfolio is given by $\beta_i = (b_{i1}\beta_{\lambda 1} + b_{i2}\beta_{\lambda 2})$. Thus

$$\begin{cases} \beta_A = (b_{A1}\beta_{\lambda 1} + b_{A2}\beta_{\lambda 2}) \\ \beta_B = (b_{B1}\beta_{\lambda 1} + b_{B2}\beta_{\lambda 2}) \\ \beta_C = (b_{C1}\beta_{\lambda 1} + b_{C2}\beta_{\lambda 2}) \end{cases} \Leftrightarrow \begin{cases} \beta_A = 1 \times 0.25 + 0.5 \times 0.5 \\ \beta_B = 3 \times 0.25 + 0.2 \times 0.5 \\ \beta_C = 3 \times 0.25 - 0.5 \times 0.5 \end{cases} \Leftrightarrow \begin{cases} \beta_A = 0.5 \\ \beta_B = 0.85 \\ \beta_C = 0.5 \end{cases}$$

(c) Since $\lambda_0 = R_F$ and $\lambda_0 = 0.1$, then $R_F = 0.1$

Exercise 4.17.

(a) If the APT assumptions hold then, in equilibrium, all securities are in the same plane $b_1/b_2/\bar{R}$. Thus, we can use deduce the equilibrium condition $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$ solving the equation system, as before

$$\begin{cases} \bar{R}_X = \lambda_0 + \lambda_1 b_{X1} + \lambda_2 b_{X2} \\ \bar{R}_Y = \lambda_0 + \lambda_1 b_{Y1} + \lambda_2 b_{Y2} \\ \bar{R}_Z = \lambda_0 + \lambda_1 b_{Z1} + \lambda_2 b_{Z2} \end{cases} \Leftrightarrow \begin{cases} 0.19 = \lambda_0 + \lambda_1 1 + \lambda_2 0.5 \\ 0.14 = \lambda_0 + \lambda_1 1.4 + \lambda_2 0 \\ 0.08 = \lambda_0 + \lambda_1 3 - \lambda_2 1 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = 0.07 \\ \lambda_1 = 0.05 \\ \lambda_2 = 0.14 \end{cases}$$

Thus,

$$\begin{aligned} R_F &= \lambda_0 = 0.07 \\ \bar{R}_{I_1} &= \lambda_1 + R_F = 0.05 + 0.07 = 0.12 \\ \bar{R}_{I_2} &= \lambda_2 + R_F = 0.14 + 0.07 = 0.21 \end{aligned}$$

(b) The expected return for portfolio W at equilibrium is given by $\bar{R}_W^e = 0.07 + 0.05b_{i1} + 0.14b_{i2} = 0.07 + 0.05 \times 1 + 0.14 \times 0 = 0.12$. Since $\mathbb{E}[R_W] = 0.13 > \bar{R}_w^e = 0.12$ we know the security is underpriced being an interesting investment to make (we should buy). Portfolio's W risk is similar to the risk of factor I_1 $(b_1 = 1 \wedge b_2 = 0)$, so that a possible arbitrage strategy is to short sell the index factor (assuming you could do so) and buy portfolio W.

An alternative is to form a new portfolio P using portfolios A, B and C, such that $b_1 = 1 \wedge b_2 = 0$:

$$\begin{cases} b_{P1} = b_1 x + b_1 y + b_1 z \\ b_{P2} = b_2 x + b_2 y + b_2 z \\ x + y + z = 1 \end{cases} \Leftrightarrow \begin{cases} 1 = 1x + 1.4y + 3z \\ 0 = 0.5x - z \\ \end{cases} \Leftrightarrow \begin{cases} x = -1 \\ y = 2.5 \\ z = -0.5 \end{cases}$$

To compose Portfolio P you would short sell X and Z to buy Y, in the proportions just computed.

(c) To evaluate the fund's performance, we need to compare the equilibrium expected return with the actual return. The equilibrium expected return is calculated as $\bar{R}_W^e = 0.07 +$

$0.05b_{i1} + 0.14b_{i2} = 0.07 + 0.05 \times 1.2 + 0.14 \times 0.2 = 0.158$. Now, to find the actual return we can use the Sharpe's Ratio (SR), defined as

$$SR = \frac{\bar{R}_{Fund} - R_F}{\sigma_{Fund}}$$

$$\bar{R}_{Fund} = SR \times \sigma_{Fund} + R_F \tag{16}$$

σ_{Fund} is not given, but if this portfolio is fully diversified it only faces systematic risk, such that the correlation between the two factors is null and, therefore,

$$\begin{aligned} \sigma^2_{Fund} &= b^2_{1Fund}\sigma^2_{I_1} + b^2_{2Fund}\sigma^2_{I_2} \\ &= (1.2)^2(0.1)^2 + (0.2)^2(0.25)^2 \\ &= 0.0169 \end{aligned}$$

And $\sigma_{Fund} = 0.13$. Applying it in (16), it comes

$$\bar{R}_{Fund} = 0.75 \times 0.13 + 0.07 = 0.1675$$

Thus, the fund has achieved a performance higher than what was expected under equilibrium.

(d) It is possible since the indexes' returns I_1 and I_2 can be explain by CAPM. In that case, APT and CAPM are equivalents, as shown in a previous exercise. In this case $\bar{R}_{I_1} = R_F + \beta_{I_1}\left(\bar{R}_m - R_F\right)$ and $\lambda_1 = \bar{R}_{I_1} - R_F$ such that

$$\beta_{I_1} = \frac{\lambda_1}{\bar{R}_{I_1} - R_F} = \frac{0.05}{0.15 - 0.07} = 0.625$$

and

$$\beta_{I_2} = \frac{\lambda_2}{\bar{R}_{I_2} - R_F} = \frac{0.14}{0.15 - 0.07} = 1.75$$

To calculate the portfolios' βs we know that, in general, $\beta_i = b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2}$, then

$$\beta_X = 1 \times 0.625 + 0.5 \times 1.75 = 1,5$$

$$\beta_Y = 1.4 \times 0.625 + 0 \times 1.75 = 0.875$$

$$\beta_Z = 3 \times 0.625 - 1 \times 1.75 = 0.125$$

Exercise 4.18.

(a) CAPM and APT pretend to explain expected returns, although through with quite different assumptions. CAPM is a general equilibrium model with very strong assumptions like homogeneous expectations, while APT only assumes the absence of arbitrage. APT is also a must more general model than CAPM in the sense it allows returns to be explained by a set of variables that can help to better explain returns. Nevertheless, under certain circunstancies (APT's risk factors being explained by CAPM's market portfolio) the two models are equivalent. From an empirical point of view, both models face a major drawback. CAPM's market portfolio is impossible to capture since it englobes all possible and imaginable assets, including non tradable assets like our home. APT can be used with all kind of variables, however what are the relevant variables no one really knows and eventually we may not have databases for them.

(b) If APT holds, then the two indexes returns are also explained by APT $\left(\bar{R}_i = 0.07 + 0.03b_{1i} + 0.05b_{2i}\right)$, but with one singularity: each index only shows sensitivity to one risk factor b. Thus

$$\{I_1 \therefore b_1 = 1 \wedge b_2 = 0\}$$

$$\{I_2 \therefore b_1 = 0 \wedge b_2 = 1\}$$

and

$$\bar{R}_{I_1} = 0.07 + 0.03 \times 1 + 0.05 \times 0 = 0.1$$

$$\bar{R}_{I_2} = 0.07 + 0.03 \times 0 + 0.05 \times 1 = 0.12$$

Finally, it should be straight forward to you that $R_F = 0.07$.

(c) If CAPM holds, then the equilibrium condition is given by $\bar{R}_i = R_F + \beta_i\left(\bar{R}_m - R_F\right)$, where $\beta_i\left(\bar{R}_m - R_F\right)$ captures the systematic risk premium appropriate to security i. The model does not reward specific risk because we assume fully diversified portfolios. Thus, it must apply to the securities described in the problem. Using the data given it comes

$$\begin{cases} 0.304 = R_F + 1.8\left(\bar{R}_m - R_F\right) \\ 0.135 = R_F + 0.5\left(\bar{R}_m - R_F\right) \end{cases} \Leftrightarrow \begin{cases} R_F = 0.07 \\ R_m = 0.2 \end{cases}$$

(d) Yes. CAPM and APT are equivalent if the indexes' returns were explained by CAPM. In this case,

$$\begin{cases} \bar{R}_{I_1} = R_F + \beta_{I_1}\left(\bar{R}_m - R_F\right) \\ \bar{R}_{I_2} = R_F + \beta_{I_2}\left(\bar{R}_m - R_F\right) \end{cases} \Leftrightarrow \begin{cases} \beta_{I_1} = \dfrac{\bar{R}_{I_1}}{\bar{R}_m - R_F} \\ \beta_{I_2} = \dfrac{\bar{R}_{I_2}}{\bar{R}_m - R_F} \end{cases} \Leftrightarrow \begin{cases} \beta_{I_1} = \dfrac{0.10 - 0.07}{0.20 - 0.07} = 0.23 \\ \beta_{I_2} = \dfrac{0.12 - 0.07}{0.20 - 0.07} = 0.385 \end{cases}$$

and the indexes I_1 and I_2 βs are given by the expression $\beta_i = b_{i1}\beta_{I_1} + b_{i2}\beta_{I_2}$. Thus,

$$\beta_i = b_{i1}\beta_{I_1} + b_{i2}\beta_{I_2} = 0.23b_{i1} + 0.385b_{i2}$$

5 Portfolio Management

Exercise 5.1.

(a) Volatility is not always judged as a good risk measure since it considers both systematic and unsystematic risk. Actually, unsystematic or specific risk can be fully diversified, therefore the systematic risk should be the only one rewarded, what explains why measures of systematic risk are more often judged as better risk measures.

(b) Using standard deviation as the measure for variability, the reward-to-variability ratio for a fund is the fund's excess return (average return over the riskless rate) divided by the standard deviation of return, i.e., the fund's Sharpe ratio. E.g., for fund A we have:

$$\frac{\bar{R}_A - R_F}{\sigma_A} = \frac{14-3}{6} = 1.833$$

See Table 6 for the remaining funds' Sharpe ratios.

(c) A fund's differential return, using standard deviation as the measure of risk, is the fund's average return minus the return on a naïve portfolio, consisting of the market portfolio and the riskless asset, with the same standard deviation of return as the fund's. E.g., for fund A we have:

$$\bar{R}_A - \left(R_F + \frac{\bar{R}_m - R_F}{\sigma_m} \times \sigma_A\right) = 14 - \left(3 + \frac{13-3}{5} \times 6\right) = -1\%$$

See Table 6 for the remaining funds' differential returns based on standard deviation.

(d) A fund's differential return, using beta as the measure of risk, is the fund's average return minus the return on a naïve portfolio, consisting of the market portfolio and the riskless asset, with the same beta as the fund's. This measure is often called "Jensen's alpha". E.g., for fund A we have:

$$\bar{R}_A - \big(R_F - \big(\bar{R}_m - R_F\big) \times \beta_A\big) = 14 - (3 + (13-3) \times 1.5) = -4\%$$

See Table 6 for the remaining funds' Jensen alphas.

(e) Treynor's ratio is quite similar to Sharpe's Ratio, but considering β as the appropriate risk measure. E.g., for fund A we have:

$$\frac{\bar{R}_A - R_F}{\beta_A} = \frac{14-3}{1.5} = 7.833$$

(f) This differential return measure is similar to Jensen's Alpha, except that the riskless rate is replaced with the average return on a zero-beta asset. E.g., for fund A we have:

$$\bar{R}_A - \big(\bar{R}_Z - \big(\bar{R}_m - \bar{R}_Z\big) \times \beta_A\big) = 14 - (4 + (13-4) \times 1.5) = -3.5\%$$

(g) Fund B shows a better performance than Fund A when considering Sharpe's Ratio. To invert this the following relationship should hold

$$\frac{\bar{R}_A - R_F}{\sigma_A} \geqslant \frac{\bar{R}_B - R_F}{\sigma_B} \Leftrightarrow 1.833 \geqslant \frac{\bar{R}_B - R_F}{\sigma_B} = \frac{\bar{R}_B - 3}{4} \Leftrightarrow \bar{R} \geqslant 10.33$$

So, for the ranking to be reversed, Fund B's average return would have to be lower than 10.33%.

Fund	Sharpe Ratio	Treynor Ratio	Differential Return (sigma)	Jensen's Alpha	Differential Return (Beta and $\bar{R}_Z$)
A	1.833	7.333	-1%	-4%	-3.5%
B	2.250	18.000	2%	4%	3.5%
C	1.625	13.000	-3%	3%	3.0%
D	1.063	14.000	-5%	2%	1.5%
E	1.700	8.500	-3%	-3%	-2.0%

Table 6: Exercise 5.1 - Answers (b to f)

Exercise 5.2. To compute Sharpe's ratio (SR), defined as the fund's excess return (average return over the riskless rate) divided by the standard deviation of return, we need to know the funds' volatility, which we can calculate using the single index model. Thus

$$\sigma_A = \sqrt{\beta_A^2 \sigma_m^2 + \sigma_{e_A} = 1,3^2 \times 0.3^2 + 0.003} = 0.3938$$

$$\sigma_B = \sqrt{\beta_B^2 \sigma_m^2 + \sigma_{e_B} = 0.9^2 \times 0.3^2 + 0.04} = 0.336$$

Therefore,

$$SR_A = \frac{\bar{R}_A - R_F}{\sigma_A} = \frac{0.15 - 0.05}{0.3938} = 0.2539$$

$$SR_B = \frac{\bar{R}_B - R_F}{\sigma_B} = \frac{0.09 - 0.05}{0.336} = 0.119$$

6 Miscellaneous

Exercise 6.1.

a. (i) Since in this country it is possible to both deposit and lend at the same interest rate $R_F = 4\%$, we know the efficient frontier in the plan risk/expected return is a straight line passing trough the risk free asset and the so-called tangent portfolio (that is the only portfolio composed only of risky investments that is efficient). Thus, the efficient frontier in this country is given by

$$\bar{R}_p = R_F + \frac{\bar{R}_T - R_F}{\sigma_T}\sigma_p \quad \Leftrightarrow \quad \bar{R}_p = 4\% + \frac{4}{3}\sigma_p$$

where p is an efficient portfolio.
To check whether A is efficient or not we must see if it is on the straight line above

$$\bar{R}_A = 4\% + \frac{4}{3}\sigma_A \quad \Leftrightarrow \quad 8\% = 4\% + \frac{4}{3}3\% \quad \Leftrightarrow \quad 8\% = 8\%$$

and we can conclude portfolio A belong to the efficient frontier and, thus, is an efficient portfolio.

(ii) The optimal portfolio for a *super averse* investor is the portfolio that maximizes the risk tolerance function $U(R) = 12\bar{R} - \bar{R}^2 - \sigma^2$ subject to the restriction it must be an efficient portfolio so, $\bar{R}_p = 4\% + \frac{4}{3}\sigma_p$. To get the optimal portfolio we need to

$$\max U(R,\sigma) = 12\bar{R}_p - \bar{R}^2 - \sigma_p^2 \quad s.t \quad \bar{R}_p = 4\% + \frac{4}{3}\sigma_p$$

which is equivalent to the following unrestricted problem

$$\max \tilde{U}(\sigma) = 12\left(4\% + \frac{4}{3}\sigma_p\right) - \left(4\% + \frac{4}{3}\sigma_p\right)^2 - \sigma_p^2$$

The FOC of the problem is

$$\frac{\partial \tilde{U}}{\partial \sigma} = 0 \quad \Leftrightarrow \quad 12 \times \frac{4}{3} - 2\left(4\% + \frac{4}{3}\sigma_p\right)\frac{4}{3} = 0 \quad \Leftrightarrow \quad \sigma_O = 0.96\% \ .$$

The expected return of the optimal portfolio O is then given by

$$\bar{R}_O = 4\% + \frac{4}{3}0.96\% = 5.297\%.$$

To obtain the optimal portfolio's composition we must rely on the fact the optimal portfolio is efficient and any efficient portfolio is a combination of the risk free asset with the tangent portfolio. Thus

$$\bar{R}_O = x_F R_F + (1 - x_f)\,\bar{R}_T \quad \Leftrightarrow \quad 5.297\% = 4\% x_F + 12\%\,(1 - x_F) \quad \Leftrightarrow \quad x_F = 84\% \ .$$

So, the optimal portfolio for a *super averse* investor requires depositing 84% of the initial amount and investing the remaining 16% in the tangent portfolio T.

(iii) If the simply averse invest 120% in the tangent portfolio that means they are leveraging themselves and taking a loan equivalent to 20% of their initial amount. Thus, they are shortselling the risk free asset, i.e. $x_F = -20\%$. Their expected return is

$$\bar{R}_{simply} = 4\% \times (-20\%) + 12\% \times 120\% = 13.6\% \ .$$

Since this point must also be on the efficient frontier we also have optimal risk level must satisfy

$$\bar{R}_{simply} = 4\% + \frac{4}{3}\sigma_{simply} \quad \Leftrightarrow \quad 13.6\% = 4\% + \frac{4}{3}\sigma_{simply} \quad \Leftrightarrow \quad \sigma_{simply} = 7.2\% \ .$$

(iv) Total amount deposited = 1 million super averse × 1000 euros × 84% = 840 000 euros
Total amount of loans = 4 million simply averse × 2000 euros × 20% = 1 600 000 euros Since 1600000 ≠ 840000 we conclude the market is *not* in equilibrium.

b. (i) We now know two efficient portfolios: T and B both belonging to the hyperbola that by the envelop theorem is the frontier of the investment opportunity set of combinations of risky assets. By the Merton theorem we also know two portfolios are enough to derive the entire frontier, so the minimum variance portfolio MV is also a combination of T and B.
The variance of any combination of T and B is given by

$$\sigma^2 = x_T\sigma_T^2 + (1-x_T)^2\,\sigma_B^2 + 2x_T\,(1-x_T)\,\sigma_T\sigma_B\rho_{TB}\ .$$

The minimum variance portfolio minimizes is the only with the lowest possible risk, so it is it is the one that

$$\begin{aligned}\min \quad & x_T\sigma_T^2 + (1-x_T)^2\,\sigma_B^2 + 2x_T\,(1-x_T)\,\sigma_T\sigma_B\rho_{TB}\\ \Leftrightarrow & \\ \min \quad & x_T(6\%)^2 + (1-x_T)^2\,(12\%)^2 + 2x_T\,(1-x_T)\,(6\%)(12\%)0.6\end{aligned}$$

From the FOC we get

$$\begin{aligned}\frac{\partial\sigma^2}{\partial x_T} = 0 \quad & \Leftrightarrow \quad (6\%)^2 - 2(12\%)^2\,(1-x_T) + 2(6\%)(12\%)0.6 - 4x_T(6\%)(12\%)0.6 = 0\\ & \Leftrightarrow \quad x_T = 107.69\%\ ,\end{aligned}$$

and the minimum variance portfolio involves shortselling portfolio B ($x_B = -7.69\%$) to invest more than 100% in portfolio T ($x_T = 107.69\%$).

(ii) See slides from classes for the sketch.
In this case the efficient frontier has three branches: (i) a segment of a straight line from the deposit rate to the first tangent portfolios; (ii) a portion of the envelope hyperbola (between the two tangent portfolios) and (iii) another segment of a line for risk levels higher than the risk of the second tangent portfolio (the tangent obtained using the active interest rate).

(iii) If the optimal risk levels do not change, then we know $\sigma^*_{super} = 0.96\%$ (from the exercise) and $\sigma^*_{super} = 7.2\%$ (from point a(iii)).
For the super averse investor nothing changes since their optimal risk level is below the risk of portfolio T and the deposit rate did not change. So they still invest 84% in the risk free asset and 16% in portfolio T.
For the simply averse investors we only know their optimal risk level is higher than σ_T, but we do not know whether it is bellow risk level of the tangent portfolio when we take the intersection with the yy-axis to be 7%, the portfolio usually denoted by T'.
We thus need first to determine portfolio T'. This portfolio must also be a combination of T and B and is the portfolio that

$$\max_{x_T,x_B} \frac{\bar{x}_T\bar{R}_T + x_B\bar{R}_B - 7\%}{\sqrt{x_T^2\sigma_T^2 + x_B^2\sigma_B^2 + 2x_Tx_B\sigma_T\sigma_B\rho_{TB}}} \qquad s.t. \quad x_T + x_B = 1.$$

The first order conditions are equivalent to the following system of linear equations in z_T, Z_B and we know the z's are proportional to the x's,

$$\begin{cases}\bar{R}_T - 7\% = \sigma_T^2 z_T + \sigma_{TB}z_B\\ \bar{R}_B - 7\% = \sigma_{TB}z_T + \sigma_B^2 z_B\end{cases} \Leftrightarrow \begin{cases}\bar{1}2\% - 7\% = (6\%)^2 z_T + (6\%)(12\%)0.6z_B\\ \bar{1}5\% - 7\% = (6\%)(12\%)0.6z_T + (12\%)^2 z_B\end{cases} \Leftrightarrow$$

$$\begin{cases}\bar{z}_T = 11.28472\\ \bar{z}_B = 2.170139\end{cases} \Leftrightarrow \begin{cases}\bar{x}_T = 83.87\%\\ \bar{x}_B = 16.13\%\end{cases}$$

Portfolio T' requires investing 83.87% in portfolio T and 16.13% in portfolio B. The expected return and risk of T' are given by

$$\bar{R}_{T'} = 83.37\% \times 12\% + 16.13\% \times 15\% = 12.48\%$$
$$\sigma_{T'} = \sqrt{(83.37\%)^2 \times (6\%)^2 + (16.13\%)^2(12\%)^2 + 2(83.37\%)(16.13\%)(6\%)(12\%)0.6} = 6.38\%$$

Since the optimal risk level of the simply averse is higher than $\sigma_{T'}$, we know simply averse investors will take a loan to leverage themselves, even with the higher rate of 7% and invest more than 100% in T'.
The expected return is $\bar{R}_{simply} = 7\% + \frac{12.48\% - 7\%}{6.38\%} 7.2\% = 13.18\%$ and therefore we can see how much is the leverage:

$$13.18\% = 7\% x_F + 12.48\%(1 - x_F) \quad \Leftrightarrow \quad x_F = -12.77\% \quad \Rightarrow \quad x_{T'} = 112.77\% \ .$$

Simply averse investor take a loan to increase their capital by 12.77% and invest all their money in portfolio T'.

Exercise 6.2.

a. Since the expression for the efficient frontier is a straight line we know

$$\bar{R}_p = R_F + \frac{\bar{R}_T - R_F}{\sigma_T}\sigma_p \ ,$$

which tells us that: (i) in this market there is a risk-free asset and that borrowing and lending is possible at the exact same rate $R_F = 3.5\%$, also (ii) since the slope of the efficient frontier equals the Sharpe ratio of the tangent portfolio we have $SR_T = \dfrac{\bar{R}_T - R_F}{\sigma_T} =$ 0.3436

b. (i) Mr. Silva has a quadratic utility function. For his particular function we have:

- $U'(W) = 50 - 2(0.01)W > 0$ for wealth levels that satisfy $W < \dfrac{50}{0.02den} = 2500$. So, for a interval big enough around his initial wealth he prefers more to less.
- $U''(W) = -0.02 < 0$. From this we conclude Mr. Silva is risk averse.
- $A(W) = -\dfrac{U''(W)}{U'(W)} = \dfrac{0.02}{50 - 0.02W}$. Evaluating this function at the initial wealth $W_0 = 1000$ we get his absolute risk aversion coefficient before investment $A(1000) = \frac{0.02}{50 - 0.02 \times 1000} = \frac{0.02}{30}$. Taking the first derivative of the absolute risk aversion function we get $A'(W) = \frac{0.0004}{(50 - 0.02W)^2} > 0$ and we can conclude Mr. Silva has increasing absolute risk aversion, i.e. when his wealth increases he will decrease the amount of euros invested in risky assets.
- $R(W) = A(W)W = \dfrac{0.02W}{50 - 0.02W}$. Evaluating this function at the initial wealth $W_0 = 1000$ we get his relative risk aversion coefficient before investment $R(1000) = \frac{0.02 \times 1000}{50 - 0.02 \times 1000} = \frac{20}{30}$. Taking the first derivative of the relative risk aversion function we get $R'(W) = \frac{50}{(50 - 0.02W)^2} > 0$. Not surprisingly (given his increasing absolute risk aversion) Mr.Silva also has increasing relative risk aversion, i.e. when his wealth increases he keeps a smaller portion of his wealth in risky assets.

(ii) The risk tolerance function gives us for eah pair of volatility and expected return, $(\sigma, \bar{R})$, the expected utility of an investor.

To derive Mr. Silva's risk tolerance function we need to compute the expected value of his utility function rewriting it in terms of returns, instead of wealth. Note that by definition of what wealth W and return R are, we get $W = W_0(1+R)$.

$$\begin{aligned} f(\sigma,\bar{R}) &= \mathbb{E}[U(W)] = \mathbb{E}[U(W_0(1+R))] \\ &= \mathbb{E}\left[50W_0(1+R) - 0.01W_0^2(1+R)^2\right] \\ &= 50W_0(1+\mathbb{E}(R)) - 0.01W_0^2\mathbb{E}\left[(1+R)^2\right] \\ &= 50W_0(1+\mathbb{E}(R)) - 0.01W_0^2\left[1 + 2\bar{R} + \underbrace{\mathbb{E}(R^2)}_{\sigma^2+\bar{R}^2}\right] \end{aligned}$$

Using $W_0 = 1000$ and simplifying we have Mr.Silva risk tolerance function

$$f(\sigma,\bar{R}) = 40000 + 30000\bar{R} - 10000\sigma^2 - 10000\bar{R}^2$$

(iii) To find Mr.Silva's optimal risk level we have to maximize his risk tolerance function, subject to the efficient frontier.

$$\max_{\sigma,\bar{R}} f(\sigma,\bar{R}) \quad s.t. \quad \bar{R} = 3.5\% + 0.3436\sigma$$

Including the restriction in the objective function we get

$$f(\sigma,\bar{R})|_{R=3.5\%+0.3436\sigma} = 40000+30000(3.5\%+0.3436\sigma)-10000\sigma^2-10000(3.5\%+0.3436\sigma)^2$$

This new restricted f function, depends only on σ. So to get its maximum we need to take its first derivative w.r.t. σ and set it to zero

$$\begin{aligned} \frac{\partial f}{\partial \sigma^*} &= 0 \\ 30000 \times 0.03436 - 20000\sigma^* - 20000(0.035 + 0.03436\sigma^*)0.3436 &= 0 \\ 3 \times 0.3436 - 2\sigma^* - 2 \times 0.3436\left[0.035 + 0.3436\sigma^*\right] &= 0 \\ \sigma^* &= 23.13\% \end{aligned}$$

c. (i) We start by computing the inputs to mean-variance theory

$$\begin{aligned} \bar{R}_1 &= 0.25(-5\%) + 0.5(0\%) + 0.25(50\%) = 11.25\% \\ \bar{R}_2 &= 0.25(10\%) + 0.5(-5\%) + 0.25(35\%) = 8.75\% \\ \sigma_1^2 &= 0.25(-5\% - 11.25\%)^2 + 0.5(0\% - 11.25\%)^2 + 0.25(50\% - 11.25\%)^2 = 0.05047 \\ &\quad \Rightarrow \quad \sigma_1 22.46\% \\ \sigma_2^2 &= 0.25(10\% - 8.75\%)^2 + 0.5(-5\% - 8.75\%)^2 + 0.25(35\% - 8.75\%)^2 = 0.02672 \\ &\quad \Rightarrow \quad \sigma_2 = 16.35\% \\ \sigma_{12} &= 0.25(-5\% - 11.25\%)(-5\% - 11.25\%) + 0.5(0\% - 11.25\%)(-5\% - 8.75\%) + \\ &\quad +0.25(50\% - 11.25\%)(35\% - 8.75\%) = 0.03265 \end{aligned}$$

From before we also know there is a risk-free asset with $R_F = 3.5\%$. The tangent portfolio is the one that maximizes the Sharpe ratio which is the same as solving a linear system of equations in z_1, z_2 which are proportional to the optimal weights

$$\begin{cases} \bar{R}_1 - R_f = \sigma_1^2 z_1 + \sigma_{12} z_2 \\ \bar{R}_2 - R_f = \sigma_{12} z_1 + \sigma_2^2 z_2 \end{cases} \Rightarrow \begin{cases} 11.25\% - 3.5\% = 0.05047 z_1 + 0.03265 z_2 \\ \bar{R}_2 - R_f = 0.03265 z_1 + 0.02672 z_2 \end{cases} \Leftrightarrow \begin{cases} z_1 = 1.263158 \\ z_2 = 0.421053 \end{cases}$$

Since z_1, z_2 are proportional to the tangent portfolio weights we can easily find them

$$x_1^T = \frac{z_1}{z_1 + z_2} = \frac{1.263158}{1.263158 + 0.421053} = 75\% \qquad x_2^T = \frac{z_2}{z_1 + z_2}\frac{0.421053}{1.263158 + 0.421053} = 25\%$$

The expected return as risk of the tangent portfolio are as follows

$$\begin{aligned}
\bar{R}_T &= 0.75 \times 11.25\% + 0.25 \times 8.75\% = 10.625\% \\
\sigma_T^2 &= 0.75^2 \times 0.05047 + 0.25^2 \times 0.02672 + 2 \times 0.75 \times 0.25 \times 0.03265 = 0.0423 \\
\sigma_T &= 20.57\%
\end{aligned}$$

An alternative to compute the tangent portfolio's volatility weould be to use its expected return $\bar{R}_T$ and the equation for the efficient frontier

$$10.625\% = 3.5\% + 0.3436\sigma_T \Leftrightarrow \sigma_T = 20.57\% \ .$$

(ii) From before we know the optimal risk level of Mr. Silva is 23.13%. This is a point in the efficient frontier, so the optimal portfolio expected return is

$$\bar{R}^* = 3.5\% + 0.3436 \times 23.13\% = 11.51\% \ .$$

The optimal portfolio is a particular combination of the risk-free asset and the tangent portfolio. We find out the exact composition by solving

$$11.51\% = 3.5\% x_F + (1 - x_F)10.625\% \quad \Leftrightarrow \quad x_F = -12.45\% \quad \Rightarrow x_T = 112.45\% \ .$$

The optimal for Mr.Silva is to take a loan (of about 12.45% of his initial investment) to leverage a bit his position and invest 112.45% in the tangent portfolio.

(iii) Yes it would change since the current optimal portfolio involves taking a loan. Possibly at the new active rate he is no longer interested in taking a loan. His new optimum is most likely a combination of the tangent portfolio with a second portfolio belonging to the hyperbola that is the frontier of the investment opportunity set of risky assets.

Exercise 6.3.

a. (i) *We are in a scenario were the correlation between the returns of any two assets is constant. So the tangent portfolio can be computed using a cut-off method.*

However, since shortselling is allowed, one can also simply use the general mean-variance theory. The inputs are:

$$\bar{R} = \begin{pmatrix} 8\% \\ 12\% \\ 15\% \end{pmatrix} \qquad V = \begin{pmatrix} 0.01 & 0.01 & 0.0125 \\ 0.01 & 0.04 & 0.025 \\ 0.0125 & 0.025 & 0.0625 \end{pmatrix}$$

where all covariances are obtained by multiplying each pair of individual assets volatility by the constant correlation of +0.5.

To find the tangent portfolio we need to solve the system $\left[\bar{R} - R_F\right] = VZ$

$$\begin{pmatrix} 5\% \\ 9\% \\ 12\% \end{pmatrix} = \begin{pmatrix} 0.01 & 0.01 & 0.0125 \\ 0.01 & 0.04 & 0.025 \\ 0.0125 & 0.025 & 0.0625 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \Leftrightarrow Z = V^{-1}\left[\bar{R} - R_F\right] = \begin{pmatrix} 2.85 \\ 0.95 \\ 0.95 \end{pmatrix} \Rightarrow X = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix}$$

(ii) The expected return and risk of the tangent portfolio are:

$$\begin{aligned}
\bar{R}_T &= X'\bar{R} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 8\% \\ 12\% \\ 15\% \end{pmatrix} = 10.22\% \\
\sigma_T^2 &= X'VX = \begin{pmatrix} 0.6 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 0.01 & 0.01 & 0.0125 \\ 0.01 & 0.04 & 0.025 \\ 0.0125 & 0.025 & 0.0625 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix} = 0.0151 \\
&\Rightarrow \quad \sigma_T = 12.323\%
\end{aligned}$$

(iii) Since it is possible to deposit and borrow at the same rate $R_F = 3\%$, the efficient frontier is a straight line tangent to the investment opportunity set of risky assets. This line passes trough the risk-free point and the tangent portfolio, thus

$$\bar{R}_P = R_F + \frac{\bar{R}_T - R_F}{\sigma_T}\sigma_p \text{ , in our case, } \bar{R}_P = 3\% + \frac{10.22\% - 3\%}{12.323\%}\sigma_p \Leftrightarrow \bar{R}_P = 3\% + 0.586\sigma_p$$

b. (i) The optimal risk level is attained at the point where the some indifference curve is tangent to the efficient frontier. I.e., they both have the same slope at that point

$$\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{\text{EF}} = \left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{\text{IC}}$$

The efficient frontier is $\bar{R}_P = 3\% + 0.586\sigma_p$, and we have $\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{\text{EF}} = 0.586$ Differentiating the indifference curves we get $\left.\frac{\partial \bar{R}_p}{\partial \sigma_p}\right|_{\text{IC}} = 2\sigma_p + 0.415$

The optimal is thus $0.586 = 2\sigma_p^* + 0.415 \quad \Leftrightarrow \sigma_p^* = 8.55\%$.

(ii) Given the optimal risk level $\sigma_p^* = 8.55\%$ and the efficient frontier equation, we get the optimal expected return

$$\bar{R}^* = 3\% + 0.586 \times 8.55\% = 8\%$$

This is attainable by depositing part of the initial wealth and investing the remaining in the tangent portfolio

$$8\% = x_F 3\% + (1 - x_F)10.22\% \qquad \Leftrightarrow \qquad x_F = 30\% \quad \Rightarrow \quad x_T = 70\% \ .$$

The optimal for this investor s to deposit 30% of his wealth and to invest the remaining 70% in the tangent portfolio.

c. (i) Nothing changes. It is still possible to deposit and borrow at the same rate, which means portfolio T is the only combination of risky assets that is efficient. The exact same portfolio T is feasible because it does not involve shortselling.

(ii) The optimal portfolio remains the same, for the same reason, portfolio T is feasible even in a world with restrictions to shortsell.

d. The ranking of assets according to Roy ranks higher assets with lower probability of undesirable returns. In this case those are returns lower than $R_L = 5\%$.

When returns follow normal distributions we know that

$$\min \Pr(\bar{R} < 5\%) \qquad \Leftrightarrow \qquad \max \frac{\bar{R} - 5\%}{\sigma}$$

and the ranking of the three assets is

$$C : \frac{15\% - 5\%}{25\%} = 0.4 \quad > \quad B : \frac{12\% - 5\%}{20\%} = 0.35 \quad > \quad A : \frac{8\% - 5\%}{10\%} = 0.35$$

The best, according to Roy, is C, than B, than A.