LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Mathematics Department
Undergraduate Degrees in Economics and Management
Mathematics 2

Normal Period Exam - January 9, 2018

Duration: 1h15

Name:	**Number:**

1. Consider the two variable function defined by

$$f(x,y) = 4x^2 + y^2.$$

a. [1,0 points] Sketch the level curves of f corresponding to the values $k = 1$, $k = 4$ and $k = 0$.

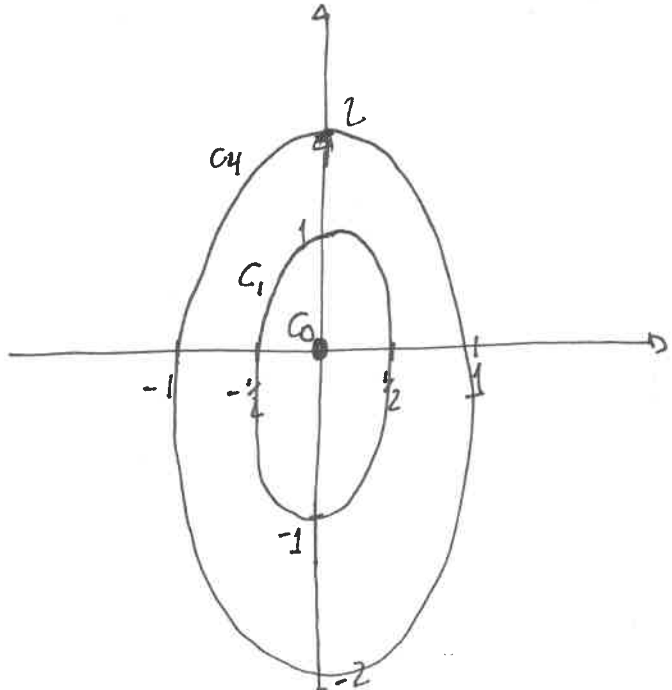
$C_1 = \{(x,y) : 4x^2 + y^2 = 1\}$

$4x^2 + y^2 = 1 \iff \dfrac{x^2}{\left(\frac{1}{2}\right)^2} + \dfrac{y^2}{1^2} = 1$

$C_4 = \{(x,y) : 4x^2 + y^2 = 4\}$

$4x^2 + y^2 = 4 \iff \dfrac{x^2}{1^2} + \dfrac{y^2}{2^2} = 1$

$C_0 = \{(x,y) : 4x^2 + y^2 = 0\} = \{(0,0)\}$



b. [1,0 points] Consider the ellipse C of ...

2. [2,0 points] Compute and classify the critical points of the function defined in $\mathbb{R}^2$ by

$$g(x,y) = x^3 + 3xy^2 - 15x - 12y.$$

(x,y) is a critical point of g if $\frac{\partial g}{\partial x}(x,y) = 0$ and $\frac{\partial g}{\partial y}(x,y) = 0$,

iff $\begin{cases} 3x^2 + 3y^2 - 15 = 0 \\ 6xy - 12 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 - 5 = 0 \\ y = \frac{2}{x} \end{cases} \Leftrightarrow \begin{cases} x^2 + \frac{4}{x^2} - 5 = 0 \\ y = \frac{2}{x} \end{cases}$

$$x^2 + \frac{4}{x^2} - 5 = 0 \overset{t=x^2}{\Leftrightarrow} t^2 + 4 - 5t = 0 \Leftrightarrow t = 1 \lor t = 4$$

$$\Leftrightarrow \cancel{x} = 1 \lor \cancel{x} = -1 \lor \cancel{x} = 2 \lor \cancel{x} = -2$$

The four critical points are $(1,2)$, $(-1,-2)$, $(2,1)$, $(-2,-1)$

$$\text{Hess } f(x,y) = \begin{bmatrix} 6x & 6y \\ 6y & 6x \end{bmatrix}$$

$$\text{Hess } f(-1,-2) = \begin{bmatrix} -6 & -12 \\ -12 & -6 \end{bmatrix} \quad \Delta_1 = -6, \quad \Delta_2 = 36 - 12^2 < 0$$

$(-1,-2)$ is a saddle point

$$\text{Hess } f(1,2) = \begin{bmatrix} 6 & 12 \\ 12 & 6 \end{bmatrix}$$

$\Delta_1 = 36 > 0$

$\Delta_2 = 36 - 12^2 < 0$

$(1,2)$ is a saddle point

$$\text{Hess } f(2,1) = \begin{bmatrix} 12 & 6 \\ 6 & 12 \end{bmatrix} \quad \Delta_1 = 12, \quad \Delta_2 = 12^2 - 6^2 > 0$$

$(2,1)$ is a local minimum

$$\text{Hess } f(-2,-1) = \begin{bmatrix} -12 & -6 \\ -6 & -12 \end{bmatrix} \quad \Delta_1 = -12 < 0, \quad \Delta_2 = 12^2 - 6^2 > 0$$

$(-2,-1)$ is a local maximum.

3. [2,0 points] Find the maximum and the minimum of the function defined in $\mathbb{R}^2$ by

$$f(x,y) = x^2 y$$

over the set $M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Let $g(x,y) = x^2 + y^2$

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ x^2+y^2=1 \end{cases} \Leftrightarrow \begin{cases} 2xy = 2\lambda x \\ x^2 = 2\lambda y \\ x^2+y^2=1 \end{cases} \Leftrightarrow \begin{cases} x(y-\lambda)=0 \\ x^2 = 2\lambda y \\ x^2+y^2=1 \end{cases}$$

$x(y-\lambda) = 0 \Leftrightarrow x=0 \vee y=\lambda$

1st case: $x=0$ $\begin{cases} 0=0 \\ 2\lambda y = 0 \\ y^2=1 \end{cases}$: $(x,y) = (0,1)$ or $(x,y)=(0,-1)$

2nd case $y=\lambda$ $\begin{cases} 0=0 \\ x^2 = 2\lambda^2 \\ x^2+\lambda^2=1 \end{cases} \Leftrightarrow \begin{cases} 0=0 \\ x^2 = 2\lambda^2 \\ 3\lambda^2 = 1 \end{cases} \Leftrightarrow \begin{cases} x^2 = 2/3 \\ \lambda^2 = \frac{1}{3} \end{cases} \Leftrightarrow \begin{array}{l} x = \pm\sqrt{\frac{2}{3}} \\ y=\lambda = \pm\sqrt{1/3} \end{array}$

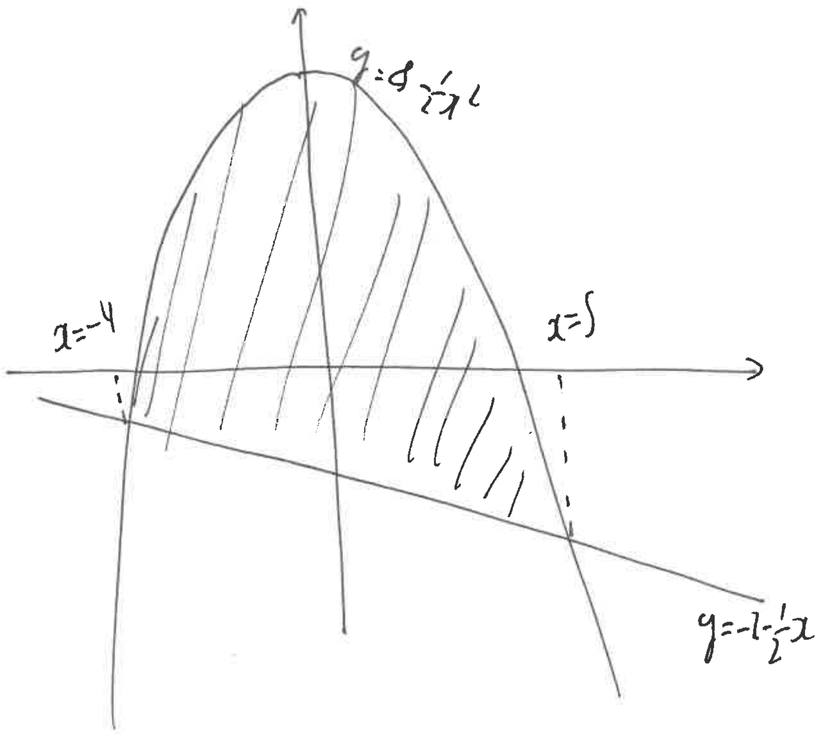
$(x,y) = (\sqrt{2/3}, \sqrt{1/3})$ or $(x,y) = (-\sqrt{\frac{2}{3}}, \sqrt{1/3})$ or $(x,y) = (\sqrt{\frac{2}{3}}, -\sqrt{1/3})$
or $(x,y) = (-\sqrt{2/3}, -\sqrt{1/3})$

$f(\sqrt{\frac{2}{3}}, \sqrt{1/3}) = f(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}) = \frac{2}{3\sqrt{3}}$ maximum

$f(\sqrt{\frac{2}{3}}, -\sqrt{1/3}) = f(-\sqrt{2/3}, -\sqrt{1/3}) = -\frac{2}{3\sqrt{3}}$ minimum

4. [2,0 points] Using a double integral, compute the area of the region of the plane defined by

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : y \geq -2 - \frac{1}{2}x \wedge y \leq 8 - \frac{1}{2}x^2\}.$$



Intersection:

$$-2-\frac{1}{2}x = 8-\frac{1}{2}x^2$$

$$\Leftrightarrow$$

$$-4-x = 16-x^2$$

$$\Leftrightarrow x^2-x-20=0$$

$$x = \frac{1 \pm \sqrt{1+80}}{2} = \frac{1\pm\sqrt{81}}{2}$$

$$\Leftrightarrow$$

$$x = 5 \vee x = -4$$

$$\mathcal{B} = \{(x,y) : -4 \leq x \leq 5 \wedge -2-\frac{1}{2}x \leq y \leq 8-\frac{1}{2}x^2\}$$

$$A(\mathcal{B}) = \int_{-4}^{5}\int_{-2-\frac{1}{2}x}^{8-\frac{1}{2}x^2} dy\,dx = \int_{-4}^{5} \left(8-\frac{1}{2}x^2+2+\frac{1}{2}x\right)dx$$

$$= \left[10x + \frac{1}{4}x^2 - \frac{1}{6}x^3\right]_{-4}^{5} = \frac{243}{4}.$$

5. [2,0 points] Compute the general solution of the differential equation

$$y''(x) - 4y'(x) + 4y(x) = 4x^2$$

Homogeneous equation:

$$\lambda^2 - 4\lambda + 4 = 0 \iff \lambda = 2$$

$$y_H(x) = Ae^{2x} + Bxe^{2x} \qquad A, B \in \mathbb{R}.$$

For the particular solution, let's try a second degree polynomial:

$$y_p(x) = ax^2 + bx + c$$

$$y_p'(x) = 2ax + b$$

$$y_p''(x) = 2a$$

$$y_p'' - 4y_p' + 4y_p = 4x^2 \iff 2a - 4(2ax + b) + 4(ax^2 + bx + c) = 4x^2$$

This equality is valid for all $x \in \mathbb{R}$ if

$$\begin{cases} 4a = 4 \\ -8a + 4b = 0 \\ 2a - 4b + 4c = 0 \end{cases}$$

$$\iff \begin{cases} a = 1 \\ b = 2 \\ 4c = 6 \iff c = 3/2 \end{cases}$$

The general solution is $y(x) = y_H(x) + y_p(x)$

$$= Ae^{2x} + Bxe^{2x} + x^2 + 2x + \frac{3}{2}.$$

$A, B \in \mathbb{R}$