Master in Actuarial Science

Models in Finance

03-01-2019

Time allowed: Two and a half hours (150 minutes)

Instructions:

1. This paper contains 6 questions and comprises 4 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used.

1. Let S_t be a stochastic process which is the solution of the Stochastic Differential Equation (SDE)

$$dS_t = \sigma S_t dB_t + e^{-t} S_t dt,$$

where B_t is a standard Brownian motion and σ is a constant.

   (a) Solve the stochastic differential equation and calculate $\mathbb{E}[S_t]$. (16)

   (b) Consider that S_t is the share price of a non-dividend paying security. Let $\sigma = 0.2$ and consider the time variable is measured in years. Calculate the probability that, over a 3 year period, the share return will at be least 20%. (12)

2. Consider the continuous time lognormal model for the market price of an investment S_t.

   (a) What are the advantages and disadvantages of the normal distribution assumption of the lognormal model and what kind of models can be used in order to obtain non-normal distributions? (12)

   (b) Discuss the empirical evidence and the theoretical arguments about the behaviour of stock prices with respect to: (i) the time evolution of the expected value of the returns; (ii) the time evolution of the volatility of returns; (iii) the mean reverting effect of the stock prices. Discuss also how well the continuous time lognormal model describes these empirical features (16)

3. Consider European put and call options written on a non-dividend paying share.

   (a) Explain how and why do the strike price and the interest rate affect the price of the European call and put options. (12)

   (b) Considering an appropriate portfolio, derive a lower bound for the price of the call option. (14)

   (c) Calculate the price of a put option (at the money), knowing that a call option (also at the money) and with the same expiry date has a price $c_t = 1$€, the current price of the share is 15€, the time to expiry is 18 months and the continuously compounded interest rate is 4%. (12)

4. Consider a 3-period recombining binomial model for the non-dividend paying share with price process S_t such that over each time period the stock price can move down by a factor $d = 0.92$ or up by a factor $u = \frac{1}{d}$. Assume that the (continuously compounded) risk-free interest rate is 5% per period and that $S_0 = 10$€.

(a) Construct the binomial tree for the 3-period model and calculate the price of a European financial derivative with payoff given by $max\left\{exp\left(\frac{S_T}{5}\right)-,0\right\}$ with maturity date in 3 periods.€. (18)

(b) Consider the binomial model under the risk-neutral measure Q and the lognormal model (also under Q) such that

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(r-\frac{1}{2}\sigma^2\right)(t-t_0),\sigma^2(t-t_0)\right),$$

where r is the interest rate per year. Consider that the binomial model has a time step given by δt. If we calibrate the binomial model in a way that the return and the variance of the log-return over the time interval δt of the binomial model and the lognormal model are equal, deduce that (for small δt)

$$q = \frac{e^{r\delta t}-d}{u-d}$$

and

$$u = exp\left(\sigma\sqrt{\delta t}\right).$$

(Hint: assume that $\left\{\mathbb{E}_Q\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right]\right\}^2 \approx 0$ when δt is small.) (16)

5. Consider the Black-Scholes model.

(a) List the assumptions underlying the Black-Scholes model and state the general risk-neutral valuation formula for the price, at time $t < T$, of a derivative security with payoff X at the expiry date T. (14)

(b) Consider a financial derivative with the payoff

$$Y = \frac{1}{T-t_0}\int_{t_0}^{T} S_u du,$$

where T is the maturity date, $t_0 < T$, and S_u is the price of the underlying non-dividend paying share at time u. Show that the price of this financial derivative at time $t < t_0$ is given by

$$V_t = \frac{S_t}{r(T-t_0)}\left[1-\exp\left(-r(T-t_0)\right)\right].$$

(18)

6. Consider the zero-coupon bond market.

(a) State the formulas that relate the zero-coupon bond price at time t of a zero-coupon bond paying 1€ at time T (and denoted by $B(t,T)$) and the

(i) Spot rate curve $R(t,T)$

(ii) instantaneous forward rate curve $f(t,T)$ (14)

(b) Consider that the instantaneous forward rate is modelled by

$$f(t,T) = 0.04e^{-0.3(T-t)} + 0.08(1 - e^{-0.3(T-t)}).$$

Sketch the graph of $f(t,T)$ as a function of T and derive the expressions for $B(t,T)$ and for $R(t,T)$. (14)

(c) Discuss the main differences and advantages/disadvantages between the Hull-White model and the one-factor Vacisek model (12)