Master in Actuarial Science

Models in Finance

03-01-2019

Time allowed: Two and a half hours (150 minutes)

Solutions

1. .

(a) Define $Y_t = \ln(S_t)$. By Itô's formula applied to $f(x) = \ln(x)$:

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial x}(S_t) dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t)(dS_t)^2 \\ &= e^{-t} dt + \sigma dB_t - \frac{1}{2}\sigma^2 dt \\ &= \left(e^{-t} - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t. \end{aligned}$$

where we have used $(dB_t)^2 = dt$. Therefore

$$Y_t = \ln(S_0) + \int_0^t \left(e^{-s} - \frac{1}{2}\sigma^2\right) ds + \sigma B_t,$$

and

$$S_t = S_0 \exp\left(1 - e^{-t} - \frac{1}{2}\sigma^2 t + \sigma B_t\right).$$

Moreover,

$$\mathbb{E}[S_t] = S_0 \exp\left(1 - e^{-t}\right).$$

(b) We have

$$S_3 = S_0 \exp(0.8902 + 0.2B_3)$$

Since $B_t \sim N(0;t)$, then $X := \log\left(\frac{S_3}{S_0}\right) \sim N(0.8902; 0.12)$, we have

$$\begin{aligned} P\left(\frac{S_3}{S_0} \geq 1.20\right) &= P(\exp(X) \geq 1.20) \\ &= P(X \geq \ln(1.20)) = 0.9795. \end{aligned}$$

2. .

(a) Normality assumption: market crashes appear more often than one would expect from a normal distribution of the log-returns (the empirical distribution has fat tails when compared to the Normal). Moreover, days with very small changes also happen more often than the normal distribution suggests (more peaked distribution). The main advantage of considering the normal distribution is its mathematical tractability.

The fat tails and jumps justify the consideration of Lévy processes (associated with fat tails) for modelling security prices.

(b) (i) There are good theoretical reasons to suppose that the expected returns per time unit should vary over time. It is reasonable to suppose that investors will require a risk premium on equities relative to bonds. As a result, if interest rates are high, we might expect the expected value of returns to be high as well. However, it is not easy to test this argument empirically.

(ii) Empirical data shows that volatility parameter is not constant in time. The implied volatility obtained from option prices and the examination of historic option prices suggests that volatility expectations fluctuate markedly over time.

(iii) One unsettled empirical question is whether markets are mean reverting, or not. A mean reverting market is one where rises are more likely following a market fall, and falls are more likely following a rise. There appears to be some evidence for this, but the evidence rests heavily on the aftermath of a small number of dramatic crashes. Furthermore, there also appears to be some evidence of momentum effects, which imply that a rise one day is more likely to be followed by another rise the next day.

(i) In the lognormal model, the expected value of returns per time unit, or drift, is constant, which does not agree with the theoretical argument given in (a). However, in this case it is difficult to test empirically if it is really necessary to assume a non-constant drift.

(ii) In the lognormal model, the volatility is assumed to be constant, in contradiction with empirical evidence.

(iii) The lognormal model is not mean reverting. However, there is no strong empirical evidence of mean-reversion effects in stock prices.

One class of models with the feature of non-constant volatility are the ARCH models. Models with non-normal returns or stochastic volatility models also satisfy this property.

3. .

(a) Strike:

Call option: a higher strike price means a lower intrinsic value. A lower intrinsic value means a lower premium.

Put option: a higher strike price will mean a higher intrinsic value and a higher premium.

Interest rates:

Call option: an increase in the risk-free rate of interest will result in a higher value for the option because the money saved by purchasing the option rather than the underlying share can be invested at this higher rate of interest, thus increasing the value of the option.

Put option: higher interest means a lower value (put options can be purchased as a way of deferring the sale of a share: the money is tied up for longer)

(b) For the call option, at time t, consider portfolio A: one European call + cash $Ke^{-r(T-t)}$. At time T, the value of A is equal to $S_T - K + K = S_T$ if $S_T > K$. If $S_T < K$ then the payoff from portfolio A is $0+K > S_T$. Therefore the portfolio payoff $\geq S_T$ and this implies, by the no arbitrage principle, that $c_t + Ke^{-r(T-t)} \geq S_t$ and the lower bound for the price of European call is

$$c_t \geq S_t - Ke^{-r(T-t)}.$$

(c) From the put-call parity (note that $K = S_t = 15€$), we have that

$$1 + 15e^{-0.04\left(\frac{18}{12}\right)} = p_t + 15.$$

Therefore,

$$\begin{aligned} p_t &= -14 + 15e^{-0.06} \\ &= 0.1265€. \end{aligned}$$

4. .

(a) $u = \frac{1}{d} = \frac{1}{0.92} = 1.087$. In order to obtain an arbitrage free model, we must have $d < e^r < u$. Therefore

$$\ln(0.92) < r < \ln(1.087).$$

Or

$$-0.0834 < r < 0.0834.$$

Since $r = 0.05$, the model is arbitrage free. Binomial tree values: 10; 10.87, 9.2; 11.8157, 10, 8.464; 12.8437, 10.87, 9.2, 7.7869. If $r = 5\%$, then the risk-neutral probability for an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 0.92}{1.087 - 0.92} = 0.7861$$

Payoff function of the Financial Derivative:

$$max\left\{exp\left(\frac{S_T}{5}\right) - 8, 0\right\}$$

Payoff: $V_3\left(u^3\right) = 5.0494$, $V_3\left(u^2 d\right) = 0.7934$, $V_3\left(ud^2\right) = 0$, $V_3\left(d^3\right) = 0$

Using the usual backward procedure with $r = 0.05$ and $q = 0.7861$

At time 2: $V_2\left(u^2\right) = \exp(-r)\left[qV_3\left(u^3\right) + (1-q)V_3\left(u^2 d\right)\right] = 3.9372$, $V_2\left(ud\right) = \exp(-r)\left[qV_3\left(u^2 d\right) + (1-q)V_3\left(ud^2\right)\right] = 0.5933$, $V_2\left(d^2\right) = 0$,

At time 1: $V_1\left(u\right) = \exp(-r)\left[qV_2\left(u^2\right) + (1-q)V_2\left(ud\right)\right] = 3.0648$, $V_1\left(d\right) = \exp(-r)\left[qV_2\left(ud\right) + (1-q)V_2\left(d^2\right)\right] = 0.4436$,

At time 0, the price is $V_0 = \exp(-r)\left[qV_1\left(u\right) + (1-q)V_1\left(d\right)\right] = 2.382$.

(b) The conditions that must be met are (where δt is the time interval of each step in the binomial model):

$$E_Q\left[\frac{S_{t+\delta t}}{S_t}\right] = \exp\left(r\delta t\right), \tag{1}$$

$$\text{var}_Q\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right] = \sigma^2 \delta t \tag{2}$$

Note also that in the binomial model:

$$E_Q\left[\frac{S_{t+\delta t}}{S_t}\right] = qu + (1-q)\,d.$$

And from Eq. (1), we get

$$q = \frac{e^{r\delta t} - d}{u - d}. \tag{3}$$

If we use Eq. (2) and the assumption $u = 1/d$, we obtain:

$$\begin{aligned}\text{var}_Q\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right] &= q\left(\ln u\right)^2 + (1-q)\left(-\ln u\right)^2 - \left\{E\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right]\right\}^2 \\ &= \left(\ln u\right)^2 - \left\{E\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right]\right\}^2\end{aligned}$$

The last term involves terms of higher order than δt, i.e. $\left\{E\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right]\right\}^2 \approx 0$ which tends to zero as $\delta t \to 0$ (assumption in the hint)
So, ignoring the terms of order higher than δt, we obtain:

$$\left(\ln u\right)^2 = \sigma^2 \delta t.$$

Solving, we obtain (σ is the volatility):

$$u = \exp\left(\sigma\sqrt{\delta t}\right), \tag{4}$$

$$d = \exp\left(-\sigma\sqrt{\delta t}\right). \tag{5}$$

5. .

(a) The assumptions underlying the Black-Scholes model are as follows:
1. The price of the underlying share follows a geometric Brownian motion.
2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
4. Unlimited short selling (that is, negative holdings) is allowed.
5. There are no taxes or transaction costs.
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.
The general risk-neutral valuation formula for a derivative with payoff X is

$$V_t = e^{-r(T-t)}\mathbb{E}_Q\left[X|\mathcal{F}_t\right],$$

where Q is the risk-neutral measure (or equivalent martingale measure) and r is the risk-free interest rate.

(b) The price is given by

$$\begin{aligned}V_t &= e^{-r(T-t)}\mathbb{E}_Q\left[\frac{1}{T-t_0}\int_{t_0}^T S_u du|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\frac{1}{T-t_0}\int_{t_0}^T \mathbb{E}_Q\left[S_u|\mathcal{F}_t\right]du\end{aligned}$$

The dynamics of the stock prices S_t under Q is given by the SDE

$$\begin{aligned} dS_u &= r\, S_u\, du + \sigma\, S_u\, d\widetilde{Z}_u, \quad u > t \\ S_t &= s \end{aligned}$$

This is a geometric Brownian motion and the solution is such that:

$$S_u = s \exp\left[\left(r - \frac{\sigma^2}{2}\right)(u-t) + \sigma\left(\widetilde{Z}_u - \widetilde{Z}_t\right)\right],$$

and

$$\mathbb{E}_Q\left[S_u | \mathcal{F}_t\right] = \mathbb{E}_Q\left[S_u | S_t\right] = S_t e^{r(u-t)}.$$

Therefore

$$\begin{aligned} V_t &= \frac{e^{-r(T-t)}}{T-t_0} \int_{t_0}^{T} S_t e^{r(u-t)} du \\ &= \frac{e^{-r(T-t)} S_t}{(T-t_0)\, r}\left[e^{r(T-t)} - e^{r(t_0-t)}\right] \\ &= \frac{S_t}{r\,(T-t_0)}\left[1 - \exp\left(-r\,(T-t_0)\right)\right]. \end{aligned}$$

6. .

(a) We have that zero-coupon bond prices are related to the spot-rate and instantaneous forward-rate by:

$$R(t,T) = \frac{-1}{T-t} \log B(t,T) \quad \text{if } t < T$$

or

$$B(t,T) = \exp\left[-R(t,T)(T-t)\right].$$

and

$$f(t,T) = \lim_{S \to T} F(t,T,S) = -\frac{\partial}{\partial T} \log B\,(t,T).$$

or (integrating):

$$B\,(t,T) = \exp\left[-\int_t^T f\,(t,u)\, du\right].$$

By $F(t,T,S)$ we represent the forward rate

$$F(t,T,S) = \frac{1}{S-T} \log \frac{B\,(t,T)}{B\,(t,S)} \quad \text{for } t < T < S.$$

(b) The bond price is given by

$$B\,(t,T) = \exp\left[-\int_t^T f\,(t,u)\,du\right]$$
$$= \exp\left[-\int_t^T \left(0.04e^{-0.3(u-t)} + 0.08(1-e^{-0.3(u-t)})\right)du\right]$$

$$= \exp\left[-\int_t^T \left(0.08 - 0.04e^{-0.3(u-t)})\right)du\right]$$
$$= \exp\left[-0.08\,(T-t) - 0.133e^{-0.3(T-t)} + 0.133\right].$$

Moreover, $R(t,T) = \frac{-1}{T-t}\left[-0.08\,(T-t) - 0.133e^{-0.3(T-t)} + 0.133\right] = 0.08 - 0.133\left[\frac{1-e^{-0.3(T-t)}}{T-t}\right]$.

(c) The SDEs for the Vasicek model gives us a time-homogeneous model. This implies lack of flexibility for pricing related contracts. A simple way to get theoretical prices to match observed market prices is to introduce some elements of time-inhomogeneity into the model. The Hull & White (HW) model does this. This model is similar to Vasicek model but now $\mu\,(t)$ is no longer a constant. The HW model can even be extended to include a time-varying deterministic $\sigma\,(t)$. This allows us to calibrate the model to traded option prices as well as zero-coupon bond prices. Moreover, since $\mu\,(t)$ is deterministic, the HW model is as tractable as the Vasicek model.

The HW model suffers from the same drawback as the Vasicek model: interest rates might become negative.

Plot for 6(b):