# Financial Markets and Instruments

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*11th January 2019*

**Duration: 2.5h**

# Suggested Solutions

**GROUP I ( 30 points)**

1. State the assumptions underlying mean-variance theory (MVT) and explain why, even in the presence of risk neutral or risk loving investors, we can focus on the so-called efficient frontier. Finally, discuss equilibrium under MVT, if there would be only risk neutral and/or risk lover investors. .......... [15p]

   **Answer:**

   The main assumptions of mean-variance theory (MVT) are: (i) that investors care only about expected returns and volatilities of portfolios, and (ii) that volatility – standard deviation of returns – is a good measure os risk.

   MVT focus of efficient portfolios because they are the ones that, for any given risk level, guarantee the highest possible expected return and the vast majority of investors in financial markets are risk averse, i.e they like expected returns and dislike volatility (risk).

   Nonetheless, even if we consider risk-neutral investors we can focus on the efficient frontier as they would be indifferent between any portfolio with the same level of expected return, so for any expected return level, we can always decide to pick the efficient one. For the risk lover investors we know they like both expected return and risk. Efficient frontiers have increasing expected returns for increasing levels of risk, and for each level of risk it guarantees the maximum possible expected return. So, the risk lovers problem can be understood choosing the efficient portfolio with the highest possible risk .

   In terms of equilibrium, if there would be only risk neutral and/or risk lover investors, equilibrium would not be possible. All risk-neutral investors would choose highest possible expected return portfolio. All risk loving investors would choose the highest possible volatility on the efficient frontier. Thus, all investors end up choosing exactly the same extreme portfolio, opting to leverage up as much as possible or taking extreme shortselling positions if leverage would not be possible. Either way this could not be possibly an equilibrium as to some investors to borrow, cash or assets, there must be other investor wishing to lend and that can only happen in the presence of risk-averse investors.

2 Choose <u>ONE</u> of the following statements and discuss whether they are true or false. ..... [15p]

I. *Most return generating models are based upon unrealistic assumptions, thus, there is no sound ground for applying them in practice.*

**Comment:** FALSE

It is true that return generating models such as constant correlation models, single factor models or multi-factor models, rely on assumptions that may, or may not, hold in practice, thus their use may lead to the introduction of *model risk* in mean-variance optimisation. However, that does not mean they are useless as they also contribute to reduce another important risk associated with the application of mean-variance theory - *estimation risk.*

Indeed, without any model, the application of mean-variance theory requires estimation of all its inputs: all expected returns, $\bar{R}_i$, all volatilities $\sigma_i$ and all possible correlations $\rho_{ij}$. This means that for $n$ risky assets the number of parameters to estimate are

$$n + n + \frac{n(n-1)}{2}: \quad \bar{R}_i, \quad \sigma_i, \quad \rho_i j, \quad \forall \quad i = 1, \cdots, n \quad , \quad j \neq i, \quad j = 1, \cdots, n.$$

Note the number of expected returns and volatilities grows linearly with the number of risky assets, but the number of correlations grows in a quadratic way. In fact, the number of correlations to be estimated is huge and even small error on each estimate may lead to quite different conclusions in the end.

Using return generating models, the number of parameters to estimate tends to be much smaller and to grow either linearly with the number of assets – that is the case of constant correlation models and single factor models – or to be proportional to $n \times K$ for a relatively small $K$ that stands for the number of factors in multi-factor models. The parameters associated with return generating models are not only less but also easier to estimate in a robust way, so using models help eliminating part of the estimation risk which may more the compensate the possible introduction of model risk.

II. *An investor worried with safety is indifferent between the optimal portfolios according to Roy, Kataoka or Telsser.*

**Comment:** FALSE.

Roy, Kataoka and Telser define alternative safety criteria so the optimal portfolios according to each of them may differ.

Roy criterium is appropriate for investors who are extremely averse to returns below a limit $R_L$ and wish to minimize the probability of occurrence of that event.

Kataoka criterium should be used for investors that express their concerns in terms of the worst $\alpha\%$ outcomes/scenarios and choose portfolios that maximize the $\alpha\%$ quantile of the returns distribution.

Finally, Telser criterium should be applied whenever investors like to say both $R_L$ and $\alpha\%$ requiring one should only consider portfolios that have a probability of returns lower or equal to $R_L$ smaller than $\alpha\%$. In Telser's case if more than one portfolio satisfies the safety constraint one should then pick the one with maximum expected return, since the investor's concern about risk was already considered.

**GROUP II (30 points)**

Explain the relation between the utility function $U(W)$, the risk tolerance function $f(\sigma, \bar{R})$, and the indifference curves of the risk tolerance function. ............................................. [10p]

Show that the second order Taylor approximation of any risk tolerance function is *equivalent* to

$$f\left(\sigma, \bar{R}\right) = \bar{R} - \frac{1}{2} RRA(W_0)\left(\bar{R}^2 + \sigma^2\right).$$

.................................................................................... [20p]

**Solution:**

The so called risk tolerance function $f(\sigma_p, \bar{R}_p)$ is nothing but the expected value of the utility function, so $f(\sigma_p, \bar{R}_p) = \mathbb{E}\left[U(W)\right]$. For investors who verify the Von Newman-Morgensten axioms the choice of the optimal portfolio can be made by maximizing the risk tolerance function over the efficient frontier. Whenever $f(\sigma_p, \bar{R}_p)$ exists in closed-form one can draw it on the plan $\left(\sigma_p, \bar{R}_p\right)$ using the associated indifferent curves, i.e., curves in constant expected utility level $f(\sigma_p, \bar{R}_p) = K$ for $K \in \mathbb{R}$.

We start by taking a second-order Taylor approximation of the $U(W)$ around the initial investment $W_0$ and use $W = W_0(1+R)$.

$$\begin{aligned} U(W) &\approx U(W_0) + U'(W_0)(W - W_0) + \frac{1}{2}U''(W)(W - W_0)^2 \\ &\approx U(W_0) + U'(W_0)W_0R + \frac{1}{2}U''(W)W_0^2R^2 \end{aligned}$$

Any linear transformation of the above expression with a positive "'slope" coefficient, leads to an *equivalent* RTF, i.e. an RTF that represents the same prfrences.

Using the linear transformation

$$\begin{aligned} V(W) &= U(W_0)\underbrace{\left(\frac{1}{U'(W_0)W_0}\right)}_{>0} - \frac{U(W_0)}{U'(W_0)W_0} \\ &= \frac{U'(W_0)W_0R}{U'(W_0)W_0} + \frac{1}{2}\frac{U''(W)W_0R^2}{U'(W_0)} \\ &= R - \frac{1}{2}RRA(W_0)R^2 \end{aligned}$$

we can then conclude

$$\begin{aligned} E(U(W)) \quad \Leftrightarrow \quad E(V(W)) &\approx \mathbb{E}\left[R - \frac{1}{2}RRA(W_0)R^2\right] \\ &\approx \bar{R} - \frac{1}{2}RRA(W_0)(\bar{R}^2 + \sigma^2) \ = f(\sigma, \bar{R}) \ . \end{aligned}$$

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## GROUP III

**Problem 1 (90 points)**

Consider the assumptions of a single factor model (SFM), where for the common factor we have $\bar{R}_m = 15\%$, $\sigma_m = 20\%$. Furthermore, assume there exists a riskless asset that can be used to both lend and borrow with $R_f = 5\%$ and the following information about 6 risky assets.

| | $\bar{R}_i$ | $\beta_i$ | $\sigma^2_{ei}$ |
|---|---|---|---|
| 1 | 25,1% | 2 | 0,002 |
| 2 | 19,8% | 1.5 | 0,003 |
| 3 | 17,0% | 1.2 | 0,004 |
| 4 | 14,8% | 1 | 0,005 |
| 5 | 12,8% | 0.8 | 0,006 |
| 6 | 12,0% | 0.7 | 0,007 |

1. Consider the single factor used for the SFM model is a good proxy to the market portfolio of CAPM.

   (a) Write down the CAPM equilibrium equation and interpret............................[5p]
   **Solution:**
   The CAPM equilibrium equation is

   $$\bar{R}^e_i = R_f + \beta_i \left[\bar{R}_m - R_f\right] \qquad\qquad \bar{R}^e_i = 0.05 + 0.1\beta_i \ .$$

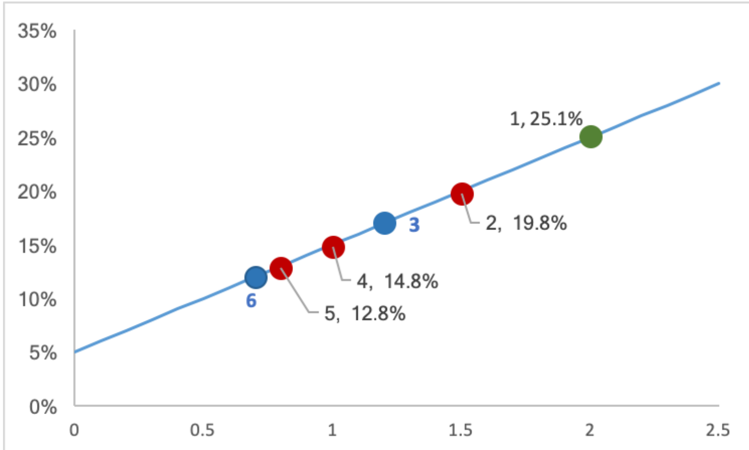
   In *equilibrium* expected returns are explain by the risk-free rate $R_f$ and the overall market price of risk $\bar{R}_m - R_f$ weighted by the $\beta_i$, that measures systematic risk.

   (b) Show that only assets 3 and 6 are in equilibrium. ..................................[5p]
   **Solution:**
   Using the equilibrium equation from before, we have

   $$\begin{aligned}
   \bar{R}^e_1 &= 0.05 + 0.1 \times 2 = 25\% & < \quad & \bar{R}_1 = 25.1\% \\
   \bar{R}^e_2 &= 0.05 + 0.1 \times 1.5 = 20\% & > \quad & \bar{R}_2 = 19.8\% \\
   \bar{R}^e_3 &= 0.05 + 0.1 \times 1.2 = 17\% & = \quad & \bar{R}_3 = 17\% \\
   \bar{R}^e_4 &= 0.05 + 0.1 \times 1 = 15\% & > \quad & \bar{R}_4 = 14.8\% \\
   \bar{R}^e_5 &= 0.05 + 0.1 \times 0.8 = 13\% & > \quad & \bar{R}_5 = 12.8\% \\
   \bar{R}^e_6 &= 0.05 + 0.1 \times 0.7 = 12\% & = \quad & \bar{R}_6 = 12\% \ ,
   \end{aligned}$$

   so we can conclude that only assets 3 and 6 are in equilibrium.

   (c) Represent the security market line and the 6 assets in the $\left(\beta, \bar{R}^e\right)$ space in and give your buy/sell recommendations. ........................................................[10p]
   **Solution:**
   Assets above the SML are underpriced, while under are overpriced. Thus the recommendation is to buy asset 1, and sell assets 2, 4 and 5.




2. Suppose Mr. Capm would like to consider only combinations of the two risky assets that are in equilibrium.

   (a) Determinine the MVT inputs for the two assets in equilibrium. . . . . . . . . . . . . . . . . . . . . . . [5p]
   **Solution:**
   For the two assets in equilibrium – asset 3 and 6 – the vector of expected returns , $\bar{R}$ and the variance-covariance matrix, $V$, are

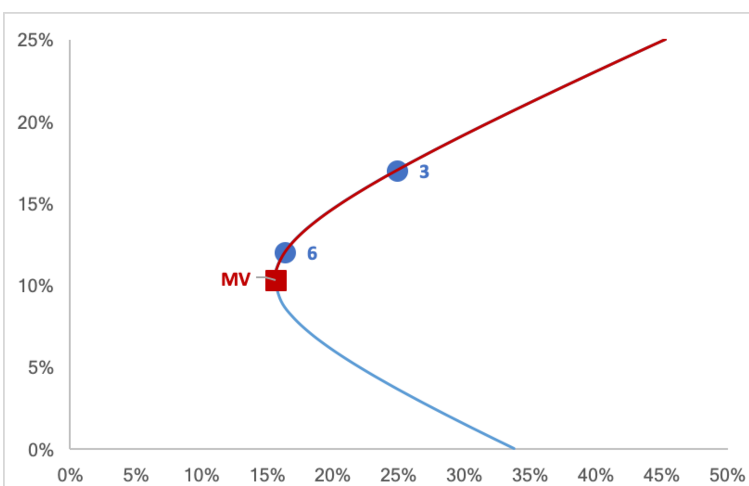
$$\bar{R} = \begin{pmatrix} 17\% \\ 12\% \end{pmatrix} \qquad V = \begin{pmatrix} 0.0616 & 0.0336 \\ 0.0336 & 0.0266 \end{pmatrix}$$

   where we have used $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$ to get $\sigma_3^2 = 0.0616$ and $\sigma_6^2 = 0.0266$ and $\sigma_{ij} = \beta_i \beta_j \sigma_m^2$ to obtain $\sigma_{36} = 0.0366$.

   (b) What is the geometric shape of the set representing all *efficient* combinations of the two assets. Explain and sketch it in mean-variance space $(\sigma, \bar{R})$. . . . . . . . . . . . . . . . . . . . . . . . [10p]
   **Solution:**
   Since $\sigma_{36} = 0.0366 \implies \rho_{36} = 0.83$ the IOS is an hyperbola passing trough both assets. The minimum variance portfolio can easily be obtained by

$$x_3^{MV} = \frac{\sigma_6^2 - \sigma_{36}}{\sigma_3^2 + \sigma_6^2 - 2\sigma_{36}} = -33.33\% \implies x_6^{MV} = 133.33\% \ ,$$

   for which we have $\bar{R}^{MV} = 10.33\%$ and $\sigma^{MV} = 15.58\%$.
   The efficient frontier is the upper part of the hyperbola, from the MV portfolio upwards.



Analytically the EF is $\sigma_p^2 = 8.4\bar{R}_p^2 - 1.736\bar{R}_p + 0.11396$, for $\bar{R} \geq 10.33\%$.

(c) If Mr. Capm would consider, in addition, the risk free asset. What would then be:

(i) the investment opportunity set ............................................... [5p]

**Solution**:

If we can use the risk free asset to both deposit and borrow, combinations of a risk-free asset with any portfolio of just risky assets (hyperbola point) are represented by straight lines connecting the risk free with that point. All possible such combinations are represented by a cone with vertice in the risk free asset and limit lines $\bar{R}_p = R_f \pm SR_T\sigma_p$ where $SR_T$ is the highest possible Sharpe ratio of an hyperbola point.

(ii) the combination of the two risky assets with the highest Sharpe ratio and its SR value. [10p]

**Solution**:

The combination of risky assets with the highest Sharpe ratio is the tangent portfolio and can be determined by

$$\begin{aligned} \max_{x_3,x_6} \quad & \frac{x_3\bar{R}_3 + x_6\bar{R}_6 - R_f}{\sqrt{x_3^2\sigma_3^2 + x_6^2\sigma_6^2 + 2x_3x_6\sigma_{36}}} \\ s.t. \quad & x_3 + x_6 = 1 \end{aligned} \quad .$$

Solving the above optimatization problem is equivalent to solving

$$Z = V^{-1}\left[\bar{R} - R_f\mathbb{1}\right] = \begin{pmatrix} 52.1978 & -65.9341 \\ -65.9341 & 120.8791 \end{pmatrix}\begin{pmatrix} 0.12 \\ 0.07 \end{pmatrix} = \begin{pmatrix} 1.6484 \\ 0.5495 \end{pmatrix} \implies X_T = \begin{pmatrix} 75\% \\ 25\% \end{pmatrix}.$$

So the tangent portfolio requires investing 75% in asset 3 and 25%i asset 6. Its Sharpe ratio value is $SR_T = \frac{\bar{R}_T - R_f}{\sigma_T} = \frac{15.75\% - 5\%}{22.12\%} = 0.4861$ .

(iii) the efficient frontier. ............................................... [2.5p]

**Solution**:

The EF is the straight line $\bar{R}_p = R_f + SR_T\sigma_p$. Is this case we have $\bar{R}_p = 0.05 + 0.4861\sigma_p$.

(d) Do you think Mr. Capm should consider also the risk free asset? Why or why not? .. [5p]

**Solution**:

Yes hw should, by using the risk-free asset he goes from an IOS that is just the hyperbola, to an IOS that is an entire cone that contains the hyperbola. So he can never be worst off by considering the risk-free asset.

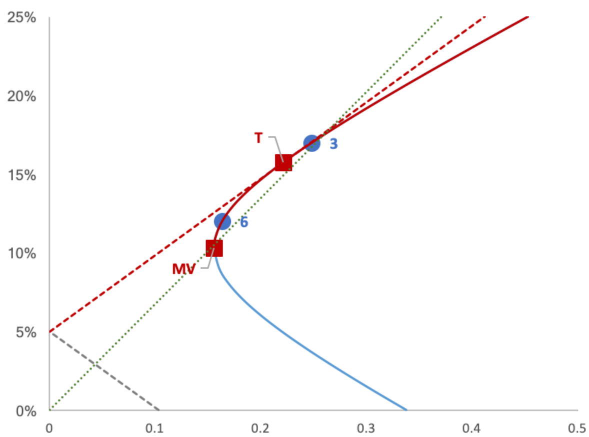
From the equation of the EF above we also see that all efficient portfolios are combinations of the risk free asset with the tangent portfolio. So, unless his optimal portfolio would be tangent portfolio itself, he would always be better off considering the risk free asset.

3. Assume now the risky returns are approximately Gaussian.

(a) Identify all portfolios that have at most 25% probability of negative returns. Derive the mathematical expression and represent it in $(\sigma, \bar{R})$ space. ........................ [12.5p]
[For $z \sim N(0,1)$ recall $\mathbb{P}(z \leq -0.6745) = 0.25$.]

**Solution**:
All portfolios that have at most 25% probability of negative returns satisfy

$$\begin{aligned}
\mathbb{P}\left[R_p \le 0\right] &\le 0.25 \\
\mathbb{P}\left[\frac{R_p - \bar{R}_p}{\sigma_p} \le \frac{0 - \bar{R}_p}{\sigma_p}\right] &\le 0.25 \\
\Phi\left(\frac{0 - \bar{R}_p}{\sigma_p}\right) &\le 0.25 \\
-\bar{R}_p &\le \Phi^{-1}(0.25) \times \sigma_p \\
\bar{R}_P &\ge 0.6745 \times \sigma_p
\end{aligned}$$




All portfolios with expected returns on or above the so-called Telser line (represented in the graph as the dashed green line) verify the Telser condition for an $R_L = 0\%$ and $\alpha = 25\%$.

(b) Suppose Mr. Capm from Question 2 is a Telser investor. What would you recommend

(i) Considering only combinations of the two risky assets. ........................ [7.5p]

**Solution**:
The Telser portfolio is the one with the highest possible expected return, $\bar{R}_p$ provided the Telser condition is satisfied.
When we consider only the hyperbola is the upper crossing point between the hyperbola and the Telser line.
Using the hyperbola equation $\sigma_p^2 = 8.4\bar{R}_p^2 - 1.736\bar{R}_p + 0.11396$ and the Telser line $\bar{R}_p = 0.6745 \times \sigma_p$ the intersection points are

$$\sigma_p^2 = 8.4(.6745)^2\sigma_p^2 - 1.736 \times 0.6745\sigma_p + 0.1139 \qquad \sigma_p = 15.59\% \text{ or } \sigma_p = 25.91\%.$$

The Telser crossing point (highest expected return) is reached with $\sigma_{Tel} = 25.91\%$ and a return of $\bar{R}_{Tel} = 0.6745 \times 0.2591 = 17.48\%$ .
Using just assets 3 and 6 the combination that gives us that expected return is

$$17.48\% = 17\%x_3 + 12\%(1 - x_3) \quad \Leftrightarrow \quad x_3^{Tel} = 109.57\% ,$$

and we should recommend shortselling 9.57% of asset 6 to invest 109.57% in asset 3.

(ii) Considering in addition the risk-free asset. Why? ............................ [2.5p]

**Solution**:
If there is a risk-free asset the safest according to Telser can be obtain by the intersection between the EF $R_p = 0.05 + 0.4861_p$ and the Telser line $\bar{R}_p = 0.6745 \times \sigma_p$, setting

$$0.05 + 0.4861_p = 0.6745 \times \sigma_p \qquad \Leftrightarrow \qquad \sigma = 26.54\% ,$$

thus, we should recommend investing $x_T = \frac{\sigma_{Tel}}{\sigma_T} = \frac{26.54\%}{22.12\%} = 120\%$ which requires taking a loan of 20% as $x_f = -20\%$.

4. Using the cut-off method and assuming shortselling is not allowed, what is your opinion about the fact that Mr. Capm is only considering the two risky assets in equilibrium? I.e., would you recommend considering any other risky asset? Explain your answer. ..................... [10p]

**Solution**:
The cut-off methof when shortselling is not allowed requires, first ranking of the assets, and then iterative computations of $C_1, C_2, \ldots$ until we get consistency in the tangent portfolio composition.

Since we are in the context of a SFM the ranking should be in terms of excess return towards beta and the cut off levels arte is determined by

$$C_k = \frac{\sigma_m^2 \sum_{i=1}^{k} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{e_i}^2}}{1 + \sigma_m^2 \sum_{i=1}^{k} \frac{\beta_i^2}{\sigma_{e_i}^2}}.$$

The computations are summarised in the table below where the assets already show up ranked.

| Rank | $\bar{R}_i$ | $\beta_i$ | $\sigma_{ei}^2$ | $\frac{\bar{R}_i - R_f}{\beta_i}$ | $\frac{(\bar{R}_i - Rf)\beta_i}{\sigma_{ei}^2}$ | $\beta_i^2/\sigma_{ei}^2$ | C |
|---|---|---|---|---|---|---|---|
| 1 | 0.251 | 2 | 0.002 | 0.1005 | 201 | 2000 | 0.09925 |
| 3 | 0.17 | 1.2 | 0.004 | 0.1 | 36 | 360 | 0.09937 |
| 6 | 0.12 | 0.7 | 0.007 | 0.1 | 7 | 70 | **0.09938** |
| 2 | 0.198 | 1.5 | 0,003 | **0.098** | | | |
| 4 | 0.148 | 1 | 0.005 | 0.098 | | | |
| 5 | 0.128 | 0.8 | 0.006 | 0.0975 | | | |

We can conclude that only three asserts would belong to the tangent portfolio in that case: assets 1, 3 and 6. Since Mr. Capm was already considering assets 3 and 6 we would just recommend him to add asset 1.

### Problem 2 (60 points)

In a country *NearByTheSea* the efficient mean-variance frontier is given by

$$\begin{cases} \bar{R}_p = 0.03 + 1.2\sigma_p & \sigma_p \leq 0.10 \\ \sigma_p^2 = 5.56\bar{R}_p^2 - 1.50\bar{R}p + 0.11 & 0.10 < \sigma_p < 0.20 \\ \bar{R}_p = 0.114 + 0.48\sigma_p & 0.20 \leq \sigma_p \leq 0.35 \end{cases}$$

1. Based upon the above information:
   (a) Determine the expected returns of the efficient portfolios, $T1$ and $T2$ with 10% and 20% volatility, respectively. .................................................................. [5p]
   **Solution:**
   The expected returns of $T1$ can be computed using the first straightline (as it is the tangent between that line and the envelop hyperbola) $\bar{R}_{T1} = 0.03 + 1.2 \times 10\% = 15\%$.
   The same logic applies to $T2$ so we have $\bar{R}_{T2} = 0.114 + 0.48 \times 20\% = 21\%$.

(b) Knowing the minimum variance portfolio can be seen as the combination of $T1$ and $T2$ where we invest 125% in $T1$, find out the implicit correlation between the returns of portfolios $T1$ and $T2$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [10p]

**Solution:**

If the minimum variance portfolio has $x_{T1}^{MV} = 125\%$

$$x_{T1}^{MV} = \frac{\sigma_{T2}^2 - \sigma_{T1,T2}}{\sigma_{T1}^2 + \sigma_{T2}^2 - 2\sigma_{T1,T2}} \quad \Leftrightarrow \quad 1.25 = \frac{(0.20)^2 - \sigma_{T1,T2}}{(0.1)^2 + (0.2)^2 - 2\sigma_{T1,T2}} \quad \Leftrightarrow \quad \sigma_{T1,T2} = 0.015$$
$$\Rightarrow \quad \rho_{T1,T2} = \frac{0.015}{0.1 \times 0.3} = 0.75 \ .$$

(c) From the shape of the efficient frontier, what can you conclude about: (i) the existence or not of a riskless asset, (ii) passive and active interest rates, (iii) possible shortselling restrictions, (iv) possible borrowing limits . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [10p]

**Answers:**

(i) Since it is possible to have an investment with zero risk, the riskless asset exist.

(ii) Because the EF has two different segnments of lines, with different slopes, we conclude the passive and active interest rates must be different. From the equations we get $R_f^p = 3\%$ and $R_f^a = 11.4\%$.

(iii) Because it is not possible to have more than 35% volatility, we conclude the envelop hyperbola must be bounded somehow, and have a maximum volatility point at a level less or equal than 35%. Otherwise there would always exist a combination of at least just risky assets for any volatility level (upper hyperbola is increasing for all $\sigma_p$ values). Thus, there must exist shortselling restrictions.

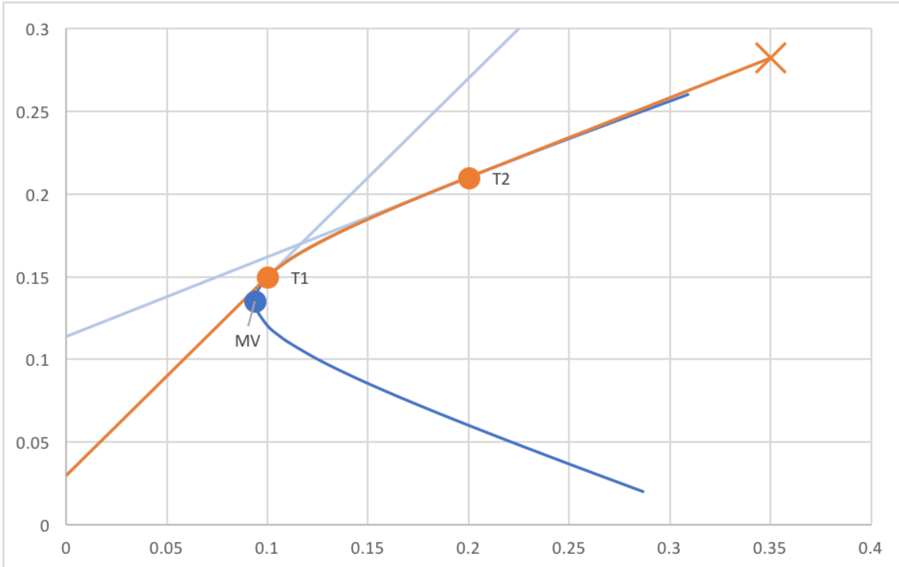
(iv) Because the maximum volatility feasible on the EF is 35% and the volatility of $T2$ is 20% we conclude the maximum vol portfolio is achieved by investing in $T2$ , $x_{T2} = 35\%/20\% = 175\%$, which means borrowing is limited to 75% of the initial investment.

(d) Sketch the shape of the investment opportunity set (IOS) set and of the efficient frontier (EF) in the mean-variance $(\sigma, \bar{R})$ space and describe how each efficient point could be achieved. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [10p]

**Solution:**

From the above we can conclude the IOS is delimited from above by the efficient frontier itself (i.e by segments of the envelop hyperbola and of the wo straight lines). From below the IOS it is delimited by the line $\bar{R}_p = R_f^p - SR_{T1}\sigma_p$ because although not efficient it is possible to shortsell portfolio $T1$ to invest more than 100% in deposit. So, the IOS is a constrained cone, with vertice on the passive risk free interest rate.

The efficient frontier has three segments. The first segment – part of a line for volatilities lower than 10% – can be understood as combinations of deposit with the first tangent portfolio $T1$. The second segment – part of the envelop hyperbola between $T1$ and $T2$ – can be understood as combinations of $T1$ and $T2$. Finally, the third segment – part of a second line, valid for volatilities between 20% and 35% – can be interpreted as leveraging up (borrowing) to invest more than 100% of the initial wealth in $T2$.




2. Consider Mr. Quelhas has an utility function $U(W) = -e^{bW}$, with $b < 0$.

   (a) Evaluate Mr. Quelhas risk profile, interpreting your conclusions about his absolute and relative risk aversion. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [10p]
   **Solution:**
   From Mr. Quelhas utility function we get

$$U(W) = -e^{bW} \qquad U'(W) = -be^{bW} > 0 \qquad U''(W) = -b^2e^{bW} < 0 \quad \text{for} \quad b < 0,$$

   and conclude (i) he prefers more to less – $U'(W) > 0$ – and he is risk averse – $U''(W) < 0$. (ii) Also, we can derive his absolute and relative risk aversion functions

$$\begin{aligned} ARA(W) &= -\frac{U''(W)}{U'(W)} = -\frac{b^2e^{bW}}{be^{bW}} = -b & \qquad ARA'(W) &= 0 \\ RRA(W) &= WA(W) \ = -bW & \qquad RRA'(W) &= -b > 0 \quad \text{for} \quad b < 0\ , \end{aligned}$$

   and conclude Mr. Quelhas has constant absolute risk aversion with$-b > 0$ his coefficient of absolute risk aversion. This means that if the money available to invest would increase he would not invest any additional amount in risky assets, nor would he disinvest. Consistently, with a constant absolute risk aversion, the investor has a increasing relative risk aversion, which means that for increasing amounts of money he decides to risk lower percentages of that money in risky assets.

   (b) Take $W_0 = 1$ and use the second order Taylor approximation to the risk tolerance function of Mr. Quelhas to determine for which levels of the parameter $b$, his optimal volatility is the maximum allowed volatility $\sigma^* = 35\%$. Explain all steps of your solution. . . . . . . [15p]
   **Solution:**
   The second order Taylor approximation of any RTF is given by

$$f(\sigma, \bar{R}) \approx \bar{R} - \frac{RRA(W_0)}{2}(\bar{R}^2 + \sigma^2)\ ,$$

   where $RRA(\cdot)$ is the relative risk aversion function and $W_0$ is the initial wealth to be invested.

In the case of Mr. Quelhas we have $RRA(W) = -bW$, so

$$f(\sigma, \bar{R}) \approx \bar{R} + \frac{bW_0}{2}(\bar{R}^2 + \sigma^2) \ .$$

Also for optimal volatilities higher than $\sigma_{T2} = 20\%$, we know all efficient portfolios lie on the second line $\bar{R}_p = 0.114 + 0.48\sigma_p$.

Since Mr. Quelhas would always choose and efficient portfolio we can replace the relevante equation of the EF on the RTF. Using also $W_0 = 1$, we get RTF it in terms of $\sigma$ alone,

$$f(\sigma) \approx (0.114 + 0.48\sigma) + \frac{b}{2}\left[(0.114 + 0.48\sigma)^2 + \sigma^2\right] \ .$$

To find the portfolio that maximizes the RTF we must set $f'(\sigma) = 0$. Since,

$$f'(\sigma) \approx 0.48 + \frac{b}{2}\left[2\,(0.114 + 0.48\sigma)\,0.48 + 2\sigma\right] \ ,$$

$$\text{we get} \quad 0.48 + b\left[0.48\,(0.114 + 0.48\sigma^*)\,0.48 + \sigma^*\right] = 0 \quad \Leftrightarrow \quad \sigma^* = -\frac{0.48(1 + 0.114b)}{(0.48^2 + 1)b} \ .$$

Given the efficient frontier has a maximum volatility of 35%, for any $\sigma^* \geq 35\%$ the optimal is the extreme point, thus,

$$\begin{aligned}
-\frac{0.48(1 + 0.114\ b)}{(0.48^2 + 1)b} &\geq 0.35 \\
-0.48(1 + 0.114\ b) &\leq 0.35\ b(1 + 0.48^2) \\
b &\geq -\frac{0.48}{0.114 \times 0.48 + 0.35 \times (1 + 0.48^2)} \\
b &\geq -0.989
\end{aligned}$$

Thus, the coefficient of absolute risk aversion $-b \leq 0.989$ . For higher degrees of risk aversion the optimal volatility of Mr. Quelhas would be lower than 35%.