LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Master in Actuarial Science
Rate Making and Experience Rating

Exam 1, 10/01/2019, 9:00
Time allowed: 2:30

Instructions:

1. This paper contains 4 groups of questions and comprises 4 pages including the title page and Annex;
2. Enter all requested details on the cover sheet;
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so;
4. Number the pages of the paper where you are going to write your answers;
5. Attempt all questions;
6. Begin your answer to each of the 4 question groups on a new page;
7. Marks are shown in brackets. Total marks: 200;
8. Show calculations where appropriate;
9. An approved calculator may be used. No mobile phones or other communication devices are permitted;
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Consider a certain insurance portfolio where each risk can be individualized by a specific characteristic. Denote the risk by the random variable X and the particular characteristic by parameter θ which quantifies that characteristic. For a given θ, let $X|\theta \frown \exp(\theta)$, with mean $1/\theta$. Let θ be the outcome of a random variable $\Theta \frown Gamma(5, 0.002)$, with mean 0.01. That is **[80]**

$$\begin{aligned} f_X(x|\theta) &= \theta e^{-\theta x}, \;\; x > 0 \;\; (\theta > 0)\,, \\ \pi(\theta) &= \theta^4 e^{-500\theta} 500^5/4!, \; \theta > 0\,. \end{aligned}$$

Consider that in the past three consecutive years a certain risk had reported claim amounts of 100, 250 and 150. For the next year consider the prediction or estimation of claim X_4

Assume that the usual hypothesis (H_1 and H_2) in credibility theory are applicable to the risk group under study (with the additional given by Bühlmann's model, where appropriate).

Bühlmann's credibility formula for a given risk X in a homogeneous portfolio, for a coming year of exposure, is given by

$$P_c = z\bar{X} + (1-z)\mu_X,$$

where $z = n/(n+k)$, $k = \upsilon/a$, $\mu = E[\mu(\theta)]$, $\upsilon = E[\upsilon(\theta)]$, $a = Var[\mu(\theta)]$, $\mu(\theta)$ and $\upsilon(\theta)$ are the risk mean and variance, respectively, n is the number of years in force for that risk, and $\bar{X}$ is its sample mean.

   (a) Determine the (unconditional) mean and variance of X. (15)

   (b) Show that the joint density $f_X(100, 250, 150)$ is given by

$$f(100, 250, 150) = \frac{500^5}{4!}\frac{7!}{1\,000^8};$$

   (15)

   (c) Show that the 4-year joint density $f(100, 250, 150, x_4)$ is given by (10)

$$f(100, 250, 150, x_4) = \frac{500^5}{4!}\frac{8!}{(1\,000 + x_4)^9}.$$

   (d) Show that the predictive density, for x_4, is a $Pareto(8; 1\,000)$ distribution. (10)

   (e) Calculate the sample mean, the colective and the Bayesian premia. Compare the values found. (10)

   (f) Determine Bühlmann's credibility premium, P_c. (10)

   (g) Calculate the posterior distribution. Comment briefly. (10)

2. Data in the table below are observations of aggregate claims of some insurer's portfolio with five risk groups that were observed along five consecutive years. Figures are in €10^3 units and are deflated. **[40]**

		Year				
		1	2	3	4	5
Risk	1	80	67	65	93	118
	2	112	136	60	93	104
	3	80	75	90	111	116
	4	135	121	155	123	77
	5	112	136	93	71	104

Assume that Bühlmann's H_1 and H_2 hypothesis are verified by the portfolio of risks.

   (a) Estimate the Credibility Premium for each of the above risks. (15)

   (b) Average the premia calculated and check that the credibility estimator is balanced. Explain briefly. (10)

   (c) Let X_{kj} and $\bar{X}_j$ be the observation in year k and the sample mean of Risk j, respectively. Show that the estimator (15)

$$\hat{\upsilon} = \frac{1}{5}\sum_{j=1}^{5} S'^2_j\,, \text{ with } S'^2_j = \frac{\sum_{k=1}^{5}\left(X_{kj} - \bar{X}_j\right)^2}{4}$$

   is an unbiased estimator of $\upsilon = E[\upsilon(\Theta)]$, with $\upsilon(\theta) = V[X_j|\Theta = \theta]$.

3. A certain insurer is considering a *bonus-malus* system (briefly BMS) based on the individual's annual claim count record to rate each individual risk in a given motor insurance portfolio. **[50]**

Class number	Premium level %	New step after claims 0	1	2	3	4	5	6+
1	250	2	1	1	1	1	1	1
2	200	3	1	1	1	1	1	1
3	150	4	1	1	1	1	1	1
4	125	5	2	1	1	1	1	1
5	100	6	3	1	1	1	1	1
6	90	7	4	2	1	1	1	1
7	80	8	5	3	1	1	1	1
8	70	9	6	4	2	1	1	1
9	60	10	7	5	3	1	1	1
10	50	11	8	6	4	2	1	1
11	40	12	9	7	5	3	1	1
12	37.5	12	10	8	6	4	2	1

Table 1: Rules and premium percentages

(a) Bonus-malus systems based on credibility models are not those usually chosen by ratemakers for motor insurance, despite some interesting properties. Describe briefly those properties.

(b) Consider a BMS that evolves according to what is shown in Table 1. Considering a Poisson($\lambda = 0.1$) distribution for the claim counts build the associated transition rule and transition probability matrices.

(c) Refer to the system above and Table 1. Determine the percentage of the basic premium to be paid by a driver that originally entered the scale at level 100%, drove with no claims for seven years, then filed two claims during the eighth year, and has been driving claim-free for three years since then.

(d) Suppose that for some 3-state BMS and some given θ, the steady state premium distribution is given by vector $\big(\theta, (1-\theta)^2, \theta(1-\theta)\big)$, where θ is the probability of getting at least one claim in one year. Number of claims is mixed Poisson distributed with parameter taking values according to a mean one exponential distribution.

Compute the resulting limiting distribution.

4. Suppose you work as an actuary for an insurer and you need to define a new tariff for the third party liability in motor insurance to put in place for the coming year 2020. As done in the past we model separately the claim frequency (denoted as N) and the expected cost per unit (denoted as Y). For the expected cost we decided to consider five rating factors: **[30]**

- Driver's age (Age 1: ≤ 20 years of age, 2: 21-25, 3: 26-35, 4: 36-50 and 5: > 50)
- Experience (1: ≤ 2 license years , 2: > 2)
- Region (1: Greater Lisbon, 2: Greater Oporto, 3: South, 4: Centre and 5: North)
- Capital insured (Cap 1: Minimum legal, 2: Medium, 3: Insurer's maximum).
- Horse power (HP 1: ≤ 75, 2: $75 - 120$ and 3: > 120)

Consider the model output shown in the Annex.

(a) For the estimation purpose, the chief actuary is advising you to use annual data from 2017 instead of that of 2018. What do you think? Comment briefly.

(b) In this example, for the severity component, what sort of model do you think is being proposed for the estimation: Additive or Multiplicative? Write down the model that is being proposed.

(c) Argue the use of the model and refer possible changes to be made. Use existing information when available.

(d) Assume the proposed estimation is valid (or is last). Suppose you select a driver with 19 years of age, inexperienced, from greater Lisbon area, driving a 70HP vehicle, with minimum capital insured. His policy is part of the BMS described above, and he has had no claims at all in the current year. What would be the corresponding pure premium for next year? Does it correspond to the minimum premium charged?

Annex

```
Call:
glm(formula = Y ~ Age + Experience + Region + Cap +
    HP, family = Gamma(link = "log"), data = dados)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.0741  -0.4282  -0.0868   0.2428   4.8993

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   7.287927   0.018583 392.181  < 2e-16 ***
Age2         -0.072685   0.015306  -4.749 2.07e-06 ***
Age3         -0.045600   0.016796  -2.715  0.00664 **
Age4         -0.234246   0.018794 -12.464  < 2e-16 ***
Age5         -0.254591   0.022173 -11.482  < 2e-16 ***
Experience2   0.087757   0.013683   6.414 1.46e-10 ***
Region2       0.063669   0.015509   4.105 4.06e-05 ***
Region3       0.030505   0.017394   1.754  0.07949 .
Region4       0.128912   0.019063   6.763 1.40e-11 ***
Region5       0.104366   0.021465   4.862 1.17e-06 ***
Cap2          0.003539   0.009874   0.358  0.72006
Cap3          0.006322   0.011188   0.565  0.57205
HP2           0.050938   0.010551   4.828 1.39e-06 ***
HP3           0.148721   0.013310  11.174  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for Gamma family taken to be 0.302994)

    Null deviance: 4590.4  on 16353  degrees of freedom
Residual deviance: 4422.0  on 16340  degrees of freedom
AIC: 260880

Number of Fisher Scoring iterations: 5
```

Solutions:

1. (a)

$$E[X] = E[E[X|\Theta]] = E[1/\Theta] = \int_0^\infty \theta^3 e^{-500\theta} 500^5/4! d\theta = \frac{500}{4}$$

$$\begin{aligned} V[X] &= E[v(\Theta)] + V[\mu(\Theta)] = v + a \\ &= V[1/\Theta] + E[1/\Theta^2] = 2\,E[1/\Theta^2] + E[1/\Theta]^2 = \frac{5}{2^4 \times 3} 500^2 \\ E[1/\Theta^2] &= \int_0^\infty \theta^2 e^{-500\theta} 500^5/4! d\theta = \frac{500^2}{2^2 \times 3} \end{aligned}$$

.

(b)

$$f(100, 250, 150) = \int_0^\infty f(100|\theta) f(250|\theta) f(150|\theta) \pi(\theta) d\theta = \frac{500^5}{4!} \frac{7!}{1\,000^8}.$$

(c) Similarly,

$$f(100, 250, 150, x_4) = \frac{500^5}{4!} \frac{8!}{(1000 + x_4)^9}$$

.

(d)

$$f(x_4|1000, 250, 150) = \frac{f(100, 250, 150, x_4)}{f(100, 250, 150)} = \frac{8 \times 1\,000^8}{(1\,000 + x_4)^9} \to Pareto(8, 1\,000)$$

(e) Collective premium: $\mu = 500/4 = 125$, already calculated. Bayesian premium: Mean of $Pareto(8, 1\,000)$: $1\,000/7 \simeq 142.86$. Sample mean: $500/3 \simeq 166.(6)$.

$$\mu < E[X_4|100, 250, 150] < \bar{X}.$$

Obs.: The order is not necessarily this one.

(f) We have exact credibility: $k = v/a = 4$, $z = 3/(3 + 4) = 3/7 \Rightarrow$

$$\tilde{\mu}(\theta) = \frac{3}{7}\bar{x} + \frac{4}{7}\mu = \frac{1\,000}{7}$$

(g) Posterior is $Gamma(8, 0.001)$, mean 8/1000. This is an example of conjugate distributions, and exact credibility.

2.

Risk \ **Year**	1	2	3	4	5
1	80	67	65	93	118
2	112	136	60	93	104
3	80	75	90	111	116
4	135	121	155	123	77
5	112	136	93	71	104

(a) $\tilde{\mu}(\theta) = \hat{z}\,\bar{x} + (1 - \hat{z})\hat{\mu}$, $\hat{z} = n/(n + \hat{k})$, $\hat{k} = \hat{v}/\hat{a}$.

$$\hat{\mu} = 101.08\,; \quad \hat{v} = 596.3\,; \quad \hat{a} = 72.432\,; \quad \hat{k} = 8.23254915\,; \quad \hat{z} = 0.377856144$$

Credibility estimates

1	2	3	4	5
94.85293074	101.0497715	98.55592096	109.0603218	101.881055

(b) As expected, the mean of the credibility estimates equals <u>101.08</u>, the collective premium estimate. "The risks paying less are compensated by those who pay more", balancing the payments.

(c) $E[\hat{\upsilon}] = E[E[\hat{\upsilon}|\Theta]]$. $E[\hat{\upsilon}|\Theta]$ and $E[\hat{\upsilon}]$ come:

$$\begin{aligned} E[\hat{\upsilon}|\Theta=\theta] &= \frac{1}{5}\sum_{j=1}^{5} E[S_j'^2|\theta] = \frac{1}{5}\sum_{j=1}^{5} V[X|\theta] = \frac{1}{5} n\, \upsilon(\theta) = \upsilon(\theta)\,. \\ E[\hat{\upsilon}] &= E[\upsilon(\Theta)] = \upsilon\,. \end{aligned}$$

In the 2nd step we assumed that, given that $\Theta = \theta$, we have a random sample (Bühlmann's H_1) and so it is well known that the sample variance $S_j'^2$ is an unbiased estimator of the respective population variance.

3.

(a) It is an optimal system, fair (in the sense that it results from the application of Bayes theorem) and it is financially balanced (the average income of the insurer stays at 100, year after year)

(b) The transition rule matrix is:

$$T = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \left(\begin{array}{cccccccccccc}
\{1+\} & \{0\} & & & & & & & & & & \\
\{1+\} & & \{0\} & & & & & & & & & \\
\{1+\} & & & \{0\} & & & & & & & & \\
\{2+\} & \{1\} & & & \{0\} & & & & & & & \\
\{2+\} & & \{1\} & & & \{0\} & & & & & & \\
\{3+\} & \{2\} & & \{1\} & & & \{0\} & & & & & \\
\{3+\} & & \{2\} & & \{1\} & & & \{0\} & & & & \\
\{4+\} & \{3\} & & \{2\} & & \{1\} & & & \{0\} & & & \\
\{4+\} & & \{3\} & & \{2\} & & \{1\} & & & \{0\} & & \\
\{5+\} & \{4\} & & \{3\} & & \{2\} & & \{1\} & & & \{0\} & \\
\{5+\} & & \{4\} & & \{3\} & & \{2\} & & \{1\} & & & \{0\} \\
\{6+\} & \{5\} & & \{4\} & & \{3\} & & \{2\} & & \{1\} & & \{0\}
\end{array}\right) \end{array}$$

For the transition probability matrix, for each entry in T consider $Pr(\{i\}) = e^{-\lambda}\lambda^i/i!$ and $Pr(\{(i+1)+\}) = 1 - \sum_{k=0}^{i} e^{-\lambda}\lambda^k/k!$, for $i = 0, 1, 2...$, and $\lambda = 0.1$.

(c) R: 40%.

(d) We look for vector $\big(E[\Theta], E[(1-\Theta)^2], E[\Theta(1-\Theta)]\big)$, with

$$\begin{aligned} E[\Theta] &= E[1-e^{-\Lambda}] = 1 - M_\Lambda(-1) = 1/2 \\ E[(1-\Theta)^2] &= E[e^{-2\Lambda}] = M_\Lambda(-2) = 1/3 \\ E[\Theta(1-\Theta)] &= E[(1-e^{-\Lambda})e^{-\Lambda}] = M_\Lambda(-1) - M_\Lambda(-2) = 1/6\,, \end{aligned}$$

where $M_\Lambda(\cdot)$ is the m.g.f. of r.v Λ.

4.

(a) Those of 2018 may not yet be stabilized yet, not yet *up to date*, not be complete or fully reported data... Still, we need to be sure it is a stable year for the business.

(b) The proposed model is a Multiplicative one, for the Severity expectation, where Premium = Frequency × Severity:

$$E[Y] = \gamma_0 \times \text{Age} \times \text{Experience} \times \text{Region} \times \text{Cap} \times \text{HP}$$

(c) At least, 'Cap' is not significant because of a high p-level, it looks like we should cut both levels, I mean the factor, however we always should be carefull, like changing stepwise, level after level. In theory, We should do the jointly null test for 'Cap', not the separate null tests.

Also, level Region3 is questionable... However, we could first re-run without the 'Cap' factor, and decide after.

(d)

$$0.1 \exp\{7.287927\}$$

No it does not, the same sort of individual but with 'Age level 5 is charged less.