Stochastic Calculus - part 5

ISEG

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Stochastic integrals - motivation

- Consider a stochastic differential equation

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)'' \frac{dB_t}{dt}''.$$

- $''\frac{dB_t}{dt}''$ is a "stochastic noise". Does not exist in a "classical sense" since B is not differentiable.
- Stochastic differential eq. in integral form:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + " \int_0^t \sigma(s, X_s)\, dB_s "$$

- How to define stochastic integrals of type:

$$\int_0^T u_s dB_s \quad ?$$

where B is a Brownian motion and u is an appropriate stochastic process.

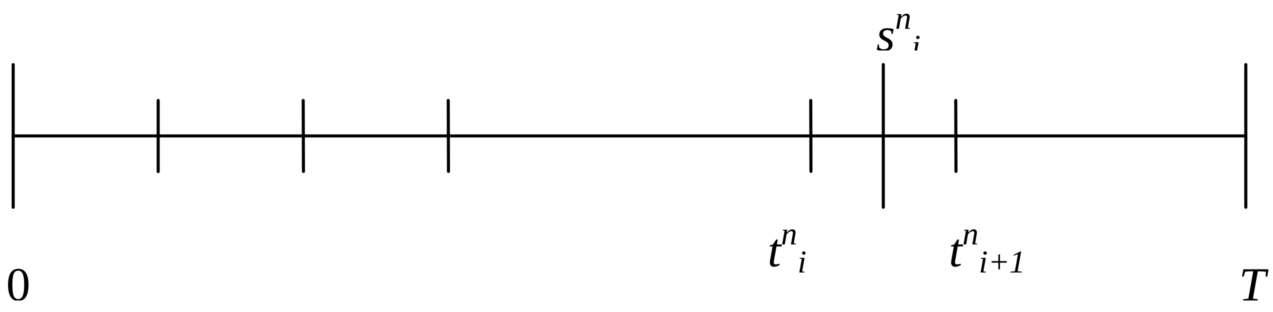
Riemann-Stieltjes integral

- Alternative 1: Consider the integral as a Riemann-Stieltjes integral.
- Consider a sequence of partitions of $[0, T]$ and a sequence of points in each interval of the partition:

$$\tau_n: \ 0 = t_0^n < t_1^n < t_2^n < \cdots < t_{k(n)}^n = T$$
$$s_n: \ t_i^n \leq s_i^n \leq t_{i+1}^n, \quad i = 0, \ldots, k(n) - 1,$$

such that $\lim_{n\to\infty} |\tau_n| := \lim_{n\to\infty} \left[\sup_i \left(t_{i+1}^n - t_i^n\right)\right] = 0.$

Riemann-Stieltjes integral



Riemann-Stieltjes integral:

$$\int_0^T fdg := \lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(s_i^n\right) \Delta g_i, \Delta g_i := g(t_{i+1}^n) - g(t_i^n),$$

if the limit exists and does not depend on the sequences τ_n and s_n.

- The Riemann Stieltjes (R-S) integral $\int_0^T fdg$ exists is f is continuous and g bounded total variation, i.e.

$$\sup_{\tau_n} \sum_i |\Delta g_i| < \infty.$$

- If f is continuous and $g \in C^1(0, T)$ then the (R-S) integral $\int_0^T fdg$ exists and

$$\int_0^T fdg := \int_0^T f(t) g'(t) dt,$$

- In the Brownian motion case B, $B'(t)$ does not exist, and therefore one cannot define the pathwise integral

$$\int_0^T u_t(\omega) dB_t(\omega) \overset{\times}{\neq} \int_0^T u_t(\omega) B_t'(\omega) dt$$

- We know that the Brownian motion as unbounded total variation. Therefore, if u is continuous (but not differentiable) one cannot define the (R-S) integral $\int_0^T u_t(\omega) dB_t(\omega)$.
- If u has trajectories of class C^1, integration by parts can be applied, and the (R-S) exists and

$$\int_0^T u_t(\omega) dB_t(\omega) = u_T(\omega) B_T(\omega) - \int_0^T u_t'(\omega) B_t(\omega) dt.$$

- Problem: For example, $\int_0^T B_t(\omega) dB_t(\omega)$ does not exist as a R-S integral. We need to consider processes more irregular than C^1 processes. How to define the stochastic integral for these processes?

- We will "construct" the stochastic integral $\int_0^T u_t dB_t$ using a probabilistic approach. We will show that we can define stochastic integrals for adapted stochastic processes u that belong to a certain space: the space $L^2_{a,T}$.

Definition

The space $L^2_{a,T}$ is the set of stochastic processes $u = \{u_t,\ t \in [0, T]\}$, such that:

1. u is adapted and measurable: i.e. u_t is $\mathcal{F}_t$-measurable and the map $(s, \omega) \to u_s(\omega)$, defined on $[0, T] \times \Omega$ is measurable with respect to the σ-algebra $\mathcal{B}_{[0,T]} \times \mathcal{F}_T$.
2. $E\left[\int_0^T u_t^2 dt\right] < \infty$.

- We need condition 1 in order to show that the r.v. $\int_0^t u_s ds$ is $\mathcal{F}_t$-measurable.
- Condition 2 allows us to show that u, as a map of t and ω, belongs to $L^2([0, T] \times \Omega)$ and that (by Fubini Theorem):

$$E\left[\int_0^T u_t^2 dt\right] = \int_0^T E\left[u_t^2\right] dt = \int_{[0,T]\times\Omega} u_t^2(\omega)\, dt P(d\omega).$$

- Strategy: We will define $\int_0^T u_t dB_t$ for $u \in L^2_{a,T}$ as the mean-square limit (limit in $L^2(\Omega)$) of integrals of simple processes.

Stochastic integral for simple processes

Definition

A stochastic process u is a simple process if

$$u_t = \sum_{j=1}^{n} \phi_j \mathbf{1}_{(t_{j-1}, t_j]}(t), \tag{1}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$, the r.v. ϕ_j are $\mathcal{F}_{t_{j-1}}$-measurable and $E\left[\phi_j^2\right] < \infty$. The set of simple processes is denoted by $\mathcal{S}$.

Definition

If u is a simple process of form (1) ($u \in \mathcal{S}$), then we define the stochastic integral of u with respect to the Brownian motion B by

$$\int_0^T u_t dB_t := \sum_{j=1}^{n} \phi_j \left(B_{t_j} - B_{t_{j-1}}\right).$$

Example

Consider the simple process

$$u_t = \sum_{j=1}^{n} B_{t_{j-1}} \mathbf{1}_{(t_{j-1}, t_j]}(t).$$

Then

$$\int_0^T u_t dB_t = \sum_{j=1}^{n} B_{t_{j-1}} \left(B_{t_j} - B_{t_{j-1}}\right).$$

By the independence ot the increments of B, it is clear that

$$\begin{aligned} E\left[\int_0^T u_t dB_t\right] &= \sum_{j=1}^{n} E\left[B_{t_{j-1}}\left(B_{t_j} - B_{t_{j-1}}\right)\right] \\ &= \sum_{j=1}^{n} E\left[B_{t_{j-1}}\right] E\left[B_{t_j} - B_{t_{j-1}}\right] = 0. \end{aligned}$$

Isometry property

Proposition

(Isometry property). Let $u \in \mathcal{S}$. The, the following isometry property is satisfied:

$$E\left[\left(\int_0^T u_t dB_t\right)^2\right] = E\left[\int_0^T u_t^2 dt\right]. \tag{2}$$

Proof.

With $\Delta B_j := B_{t_j} - B_{t_{j-1}}$, we have

$$\begin{aligned} E\left[\left(\int_0^T u_t dB_t\right)^2\right] &= E\left[\left(\sum_{j=1}^n \phi_j \Delta B_j\right)^2\right] \\ &= \sum_{j=1}^n E\left[\phi_j^2 (\Delta B_j)^2\right] + 2\sum_{i<j}^n E\left[\phi_i \phi_j \Delta B_i \Delta B_j\right]. \end{aligned}$$

□

Proof.

(cont.) Note that $\phi_i \phi_j \Delta B_i$ is $\mathcal{F}_{t_{j-1}}$-measurable and ΔB_j is independent of $\mathcal{F}_{t_{j-1}}$. Therefore

$$\sum_{i<j}^n E\left[\phi_i \phi_j \Delta B_i \Delta B_j\right] = \sum_{i<j}^n E\left[\phi_i \phi_j \Delta B_i\right] E\left[\Delta B_j\right] = 0.$$

On the other hand, since ϕ_j^2 i $\mathcal{F}_{t_{j-1}}$-measurable and ΔB_j is independent of $\mathcal{F}_{t_{j-1}}$,

$$\begin{aligned} \sum_{j=1}^n E\left[\phi_j^2 (\Delta B_j)^2\right] &= \sum_{j=1}^n E\left[\phi_j^2\right] E\left[(\Delta B_j)^2\right] \\ &= \sum_{j=1}^n E\left[\phi_j^2\right] (t_j - t_{j-1}) = \\ &= E\left[\int_0^T u_t^2 dt\right]. \end{aligned}$$

□

Linearity and zero mean properties

- Other properties of $\int_0^T u_t dB_t$ for $u \in \mathcal{S}$:
  1. Linearity: If $u, v \in \mathcal{S}$:
  $$\int_0^T (au_t + bv_t)\, dB_t = a \int_0^T u_t dB_t + b \int_0^T v_t dB_t. \qquad (3)$$
  2. Zero mean:
  $$E\left[\int_0^T u_t dB_t\right] = 0. \qquad (4)$$

Proof.

Exercise (you only need to use the definition of stochastic integral for simple processes) □

The Itô integral for adapted processes

Lemma

If $u \in L^2_{a,T}$ then exists a sequence of simple processes $\left\{u^{(n)}\right\}$ such that

$$\lim_{n\to\infty} E\left[\int_0^T \left|u_t - u_t^{(n)}\right|^2 dt\right] = 0. \qquad (5)$$

Proof.

Step 1. Assume that u is continuous in quadratic mean:

$$\lim_{s\to t} E\left[\left|u_t - u_s\right|^2\right] = 0.$$

Define $t_j^n := \frac{j}{n} T$ and

$$u_t^n = \sum_{j=1}^n u_{t_{j-1}^n} \mathbf{1}_{\left(t_{j-1}^n, t_j^n\right]}(t). \qquad (6)$$

Proof.

(cont.) By the Fubini theorem

$$E\left[\int_0^T \left|u_t - u_t^{(n)}\right|^2 dt\right] = \left[\int_0^T E\left[\left|u_t - u_t^{(n)}\right|^2\right] dt\right]$$
$$= \sum_{j=1}^{n} \int_{t_{j-1}^n}^{t_j^n} E\left[\left|u_{t_{j-1}^n} - u_t\right|^2 dt\right]$$
$$\leq T \sup_{|t-s|\leq \frac{T}{n}} E\left[\left|u_s - u_t\right|^2\right] \underset{n\to\infty}{\longrightarrow} 0.$$

□

Proof.

(cont.) Step 2. Assume that $u \in L^2_{a,T}$ and consider the sequence of processes $\left\{v^{(n)}\right\}$

$$v_t^n = n \int_{t-\frac{1}{n}}^{t} u_s ds.$$

These processes are continuous in quadratic mean (even the paths are continuous) and belong to $L^2_{a,T}$. On the other hand, we have that

$$\lim_{n\to\infty} E\left[\int_0^T \left|u_t - v_t^{(n)}\right|^2 dt\right] = 0$$

since

$$\lim_{n\to\infty} \int_0^T \left|u_t(\omega) - v_t^{(n)}(\omega)\right|^2 dt = 0.$$

□

Proof.

(cont.) and we apply the dominated convergence theorem in $[0, T] \times \Omega$, since by the Cauchy-Schwarz inequality (and changing the integration order):

$$
\begin{aligned}
E\left[\int_0^T \left|v_t^{(n)}\right|^2 dt\right] &= E\left[n^2 \int_0^T \left|\int_{t-\frac{1}{n}}^t u_s ds\right|^2 dt\right] \\
&\leq nE\left[\int_0^T \left(\int_{t-\frac{1}{n}}^t u_s^2 ds\right) dt\right] \\
&= nE\left[\int_0^T u_s^2 \left(\int_s^{s+1/n} dt\right) ds\right] \\
&= E\left[\int_0^T u_s^2 ds\right].
\end{aligned}
$$

□

Definition of stochastic integral of an adapted process

Definition

The stochastic integral or Itô integral of $u \in L^2_{a,T}$ is defined as the mean-square limit (in $L^2(\Omega)$):

$$\int_0^T u_t dB_t = \lim_{n\to\infty} (L^2) \int_0^T u_t^{(n)} dB_t, \tag{7}$$

where $\left\{u^{(n)}\right\}$ is a sequence of simple processes that satisfies (5).

- The limit exists, since by the isometry property the sequence $\left\{ \int_0^T u_t^{(n)} dB_t \right\}$ is a Cauchy sequence in $L^2(\Omega)$ and therefore it is convergent.

Proof.

$$E\left[\left(\int_0^T u_t^{(n)} dB_t - \int_0^T u_t^{(m)} dB_t\right)^2\right] = E\left[\int_0^T \left(u_t^{(n)} - u_t^{(m)}\right)^2 dt\right]$$
$$\leq 2E\left[\int_0^T \left(u_t^{(n)} - u_t\right)^2 dt\right] + 2E\left[\int_0^T \left(u_t - u_t^{(m)}\right)^2 dt\right] \overset{n,m\to\infty}{\to} 0.$$

□

Properties of the Itô integral

- Properties of the Itô integral $\int_0^T u_t dB_t$ for $u \in L^2_{a,T}$.
  1. Isometry:
  $$E\left[\left(\int_0^T u_t dB_t\right)^2\right] = E\left[\int_0^T u_t^2 dt\right]. \quad (8)$$
  2. Zero mean:
  $$E\left[\int_0^T u_t dB_t\right] = 0 \quad (9)$$
  3. Linearity:
  $$\int_0^T (au_t + bv_t) dB_t = a\int_0^T u_t dB_t + b\int_0^T v_t dB_t. \quad (10)$$

Proof.

These properties can be easily proved for processes $u \in \mathcal{S}$ (psimple processes). Then, passing to the limit, they are also satisfied for processes $u \in L^2_{a,T}$. □

Example

Let us show that

$$\int_0^T B_t dB_t = \frac{1}{2}B_T^2 - \frac{1}{2}T.$$

Since $u_t = B_t$ is continuous, let us consider the approximating sequence of simple processes (6)

$$u_t^n = \sum_{j=1}^{n} B_{t_{j-1}^n} \mathbf{1}_{\left(t_{j-1}^n, t_j^n\right]}(t),$$

with $t_j^n := \frac{j}{n}T$.

The stochastic integral for adapted processes

Example

(cont.)

$$\begin{aligned}\int_0^T B_t dB_t &= \lim_{n\to\infty}(L^2) \int_0^T u_t^{(n)} dB_t = \\ &= \lim_{n\to\infty}(L^2) \sum_{j=1}^{n} B_{t_{j-1}^n}\left(B_{t_j^n} - B_{t_{j-1}^n}\right) \\ &= \lim_{n\to\infty}(L^2)\ \frac{1}{2}\sum_{j=1}^{n}\left[\left(B_{t_j^n}^2 - B_{t_{j-1}^n}^2\right) - \left(B_{t_j^n} - B_{t_{j-1}^n}\right)^2\right] \\ &= \frac{1}{2}\left(B_T^2 - T\right),\end{aligned}$$

where we have used the fact that (quadratic variation of B.m.)
$E\left[\left(\sum_{j=1}^n \left(\Delta B_{t_j^n}\right)^2 - T\right)^2\right] \to 0$ and $\frac{1}{2}\sum_{j=1}^n \left(B_{t_j^n}^2 - B_{t_{j-1}^n}^2\right) = \frac{1}{2}B_T^2$.

- Let us prove that $E\left[\left(\sum_{j=1}^{n}\left(\Delta B_{t_j^n}\right)^2 - T\right)^2\right] \rightarrow 0$. By the independence of increments and $E\left[\left(\Delta B_{t_j^n}\right)^2\right] = \Delta t_j^n$,

$$E\left[\left(\sum_{j=1}^{n}\left(\Delta B_{t_j^n}\right)^2 - T\right)^2\right] = E\left[\left(\sum_{j=1}^{n}\left[\left(\Delta B_{t_j^n}\right)^2 - \Delta t_j^n\right]\right)^2\right]$$
$$= \sum_{j=1}^{n} E\left[\left(\Delta B_{t_j^n}\right)^2 - \Delta t_j^n\right]^2.$$

Using formula $E\left[(B_t - B_s)^{2k}\right] = \frac{(2k)!}{2^k \cdot k!}(t-s)^k$, we have

$$E\left[\left(\sum_{j=1}^{n}\left(\Delta B_{t_j^n}\right)^2 - T\right)^2\right] = \sum_{j=1}^{n}\left[3\Delta t_j^n - 2\left(\Delta t_j^n\right)^2 + \left(\Delta t_j^n\right)^2\right]$$
$$= 2\sum_{j=1}^{n}\left(\Delta t_j^n\right)^2 = 2T\sup_j\left|\Delta t_j^n\right| \underset{n\rightarrow\infty}{\longrightarrow} 0.$$

Let us see that $(dB_t)^2 = dt$.

- By formula $E\left[(B_t - B_s)^{2k}\right] = \frac{(2k)!}{2^k \cdot k!}(t-s)^k$, we have that

$$Var\left[(\Delta B)^2\right] = E\left[(\Delta B)^4\right] - \left(E\left[(\Delta B)^2\right]\right)^2$$
$$= 3(\Delta t)^2 - (\Delta t)^2 = 2(\Delta t)^2.$$

We also know that
$$E\left[(\Delta B)^2\right] = \Delta t.$$

Therefore, if Δt is small, the variance $(\Delta B)^2$ is negligible when compared with its mean value $\Longrightarrow$ when $\Delta t \rightarrow 0$ or "$\Delta t = dt$", we have that
$$(dB_t)^2 = dt. \tag{11}$$