Stochastic Calculus - Part 9

ISEG

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Stochastic Differential Equations

- Deterministic ordinary differential equations (ODE's)

$$f\left(t, x(t), x'(t), x''(t), \ldots\right) = 0, \qquad 0 \leq t \leq T.$$

- ODE of order 1:

$$\frac{dx(t)}{dt} = b(t, x(t))$$

or

$$dx(t) = b(t, x(t))\, dt$$

- Discrete version

$$\Delta x(t) = x(t + \Delta t) - x(t) \approx b(t, x(t))\, \Delta t$$

- Example

$$\frac{dx(t)}{dt} = cx(t)$$

has solution

$$x(t) = x(0)\,e^{ct}.$$

Stochastic differential equation

- SDE in differential form:

$$\begin{aligned} dX_t &= b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t, \qquad (1) \\ X_0 &= X_0 \end{aligned}$$

- $b(t, X_t)$ is called the drift coeffcient, $\sigma(t, X_t)$ is called the diffusion coefficient.
- SDE in integral form

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s. \qquad (2)$$

- "Naif" interpretation of SDE solution:
  $\Delta X_t \approx b(t, X_t)\,\Delta t + \sigma(t, X_t)\,\Delta B_t.$ and
  $\Delta X_t \approx N\left(b(t, X_t)\,\Delta t, (\sigma(t, X_t))^2\,\Delta t\right).$

Definition

A solution of SDE (1) or (2) is a stochastic process $\{X_t\}$ that satisfies:

1. $\{X_t\}$ is an adapted process with continuous trajectories
2. $\mathbb{E}\left[\int_0^T (\sigma(s, X_s))^2 \, ds\right] < \infty$.
3. $\{X_t\}$ satisfies SDE (1) or (2)

- The solutions of SDE's are also called "diffusions" or "diffusion processes".

Use of Itô formula to solve an SDE

- **Example**: Geometric Brownian motion (gBm). SDE:

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{3}$$

or

$$X_t = X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t X_s dB_s \tag{4}$$

- How to solve this SDE?
- Assume that $X_t = f(t, B_t)$ with $f \in C^{1,2}$. By Itô formula:

$$X_t = f(t, B_t) = X_0 + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, B_s) \right) ds + \tag{5}$$

$$+ \int_0^t \frac{\partial f}{\partial x}(s, B_s) \, dB_s.$$

- Comparing (4) and (5) twe have that (there is uniqueness of representation as an Itô process)

$$\frac{\partial f}{\partial s}(s, B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, B_s) = \mu f(s, B_s), \qquad (6)$$

$$\frac{\partial f}{\partial x}(s, B_s) = \sigma f(s, B_s). \qquad (7)$$

- Differentiating (7), we obtain

$$\frac{\partial^2 f}{\partial x^2}(s,x) = \sigma \frac{\partial f}{\partial x}(s,x) = \sigma^2 f(s,x)$$

and replacing in (6), we get

$$\left(\mu - \frac{1}{2}\sigma^2\right) f(s,x) = \frac{\partial f}{\partial s}(s,x)$$

- Separating the variables: $f(s,x) = g(s)h(x)$, we have

$$\frac{\partial f}{\partial s}(s,x) = g'(s)h(x)$$

and

$$g'(s) = \left(\mu - \frac{1}{2}\sigma^2\right) g(s)$$

which is a linear EDO with solution

$$g(s) = g(0)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) s\right]$$

- Using (7), we have $h'(x) = \sigma h(x)$. Hence

$$f(s,x) = f(0,0)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) s + \sigma x\right].$$

- Conclusion:

$$X_t = f(t, B_t) = X_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right] \tag{8}$$

which is a geometric Brownian motion (gBm).

- Remark: Note that we have obtained the solution of an SDE by solving a deterministic PDE (partial differential equation).

- Let us confirm that (8) satisfies the SDE (3) or (4).
- Applying the Itô formula to $X_t = f(t, B_t)$, with

$$f(t, x) = X_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma x\right],$$

we obtain

$$\begin{aligned} X_t &= X_0 + \int_0^t \left[\left(\mu - \frac{1}{2}\sigma^2\right)X_s + \frac{1}{2}\sigma^2 X_s\right]ds + \int_0^t \sigma X_s dB_s \\ &= X_0 + \mu\int_0^t X_s ds + \sigma\int_0^t X_s dB_s \end{aligned}$$

- or

$$dX_t = \mu X_t dt + \sigma X_t dB_t.$$

- **Example:** Ornstein-Uhlenbeck process (or Langevin equation):

$$dX_t = \mu X_t dt + \sigma dB_t.$$

or

$$X_t = X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t dB_s.$$

- Remark: in the discret form we would have

$$X_{t+1} = (1+\mu) X_t + \sigma (B_{t+1} - B_t)$$

or

$$X_{t+1} = \phi X_t + Z_t,$$

with $\phi = 1+\mu$ and $Z_t \sim N(0, \sigma^2)$. This is an autoregressive time series of order 1.

- **Example:** Ornstein-Uhlenbeck process (or Langevin equation) - cont.
- Let

$$Y_t = e^{-\mu t} X_t$$

or $Y_t = f(t, X_t)$, with $f(t,x) = e^{-\mu t} x$. By Itô formula,

$$Y_t = Y_0 + \int_0^t \left(-ce^{-\mu s} X_s + ce^{-\mu s} X_s + \frac{1}{2}\sigma^2 \times 0 \right) ds + \int_0^t \sigma e^{-\mu s} dB_s.$$

- Hence,

$$X_t = e^{\mu t} X_0 + e^{\mu t} \int_0^t \sigma e^{-\mu s} dB_s.$$

- If X_0 =cte., this process is called an Ornstein-Uhlenbeck process.

- **Example:** The geometric Brownian motion (again)
- Let

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{9}$$

or

$$X_t = X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t X_s dB_s. \tag{10}$$

- Assumption

$$X_t = e^{Z_t}.$$

or

$$Z_t = \ln(X_t).$$

- Applying the Itô formula to $f(X_t) = \ln(X_t)$, we get

$$\begin{aligned} dZ_t &= \frac{1}{X_t} dX_t + \frac{1}{2}\left(\frac{-1}{X_t^2}\right)(dX_t)^2 \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t. \end{aligned}$$

- That is

$$Z_t = Z_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t$$

- and

$$X_t = X_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right].$$

- By the same procedure, one can show that the linear homogeneous equation

$$dX_t = b(t) X_t dt + \sigma(t) X_t dB_t$$

has the solution

$$X_t = X_0 \exp\left[\int_0^t \left(b(s) - \frac{1}{2}\sigma(s)^2\right) ds + \int_0^t \sigma(s)\, dB_s\right].$$

Exercise

- Exercise: Solve the SDE

$$\begin{aligned} dX_t &= a(m - X_t)\, dt + \sigma dB_t, \\ X_0 &= x, \end{aligned}$$

where $a, \sigma > 0$ and $m \in \mathbb{R}$. Calculate also the mean and variance of X_t when $t \to \infty$ and find the distribution of X_t when $t \to \infty$ (invariant or stationary distribution).