LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Mathematics Department
Undergraduate Degrees in Economics and Management
Mathematics 2

Mid Term Exam - April 11th, 2019
Duration: 1h15m

Name:
Number:

1. [2,0 pts] Classify the quadratic form

$$q(x, y, z) = 5ax^2 + 2ay^2 - z^2 - 6axy$$

for all possible values of the parameter $a \in \mathbb{R}$.

The associated symmetric matrix is $A = \begin{bmatrix} 5a & -3a & 0 \\ -3a & 2a & 0 \\ 0 & 0 & -1 \end{bmatrix}$,

with minors $\Delta_1 = 5a$, $\Delta_2 = 10a^2 - 9a^2 = a^2$, $\Delta_3 = -a^2$

. If $a>0$, $\Delta_1>0$, $\Delta_2>0$, $\Delta_3<0$: q is underdetermined

If $a<0$, $\Delta_1<0$, $\Delta_2>0$, $\Delta_3<0$: q is negative defined

If $a=0$, the minor's method does not apply:

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a diagonal matrix. The eigenvalues are $\lambda = 0$ and $\lambda = -1$

q is negative semi-defined

2. a. [1,0 pts] Consider a square matrix $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$ such that

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \text{ and } A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}.$$

Write down the characteristic polynomial of A and deduce the value of $\det(A)$.

The eigenvalues (the roots of the characteristic polynomial p) are 2 and -3.

Hence $p(\lambda) = \det(A - \lambda Id) = (\lambda-2)(\lambda+3) = \lambda^2 + \lambda - 6$

$\det(A) = p(0) = -6$.

b. [0,5 pts] Give an example of a 2 by 2 square matrix with no eigenvalues.

For example $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: $p(\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

has no roots

3. [1,5 pts] Compute the equation of the tangent straight line to the curve

$$e^{xy} + y\cos(x) + x^2 y = 2$$

at point $(0, 1)$.

The curve C is the level curve of $f(x,y) = e^{xy} + y\cos(x) + x^2y$ associated to the value $k = 2$ (it goes through $(0,1)$ since $f(0,1) = 2$).

$\vec{\nabla} f(0,1)$ is orthogonal to C at $(0,1)$

$\vec{\nabla} f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = \left(ye^{xy} - y\sin x + 2xy, xe^{xy} + \cos(x) + x^2\right)$

$\vec{\nabla} f(0,1) = (1,1)$

A director vector of the tangent line is then, for example, $\vec{n}(1,-1)$ and the slope is $m = -1$

Hence the tangent has equation

$y = -(x-0) + 1 = -x + 1$

4. [2,0 pts] Consider the two variable function defined by the expression $f(x,y) = \dfrac{\ln\left(1 - x^2 - \frac{1}{4}y^2\right)}{\sqrt{xy}}$.

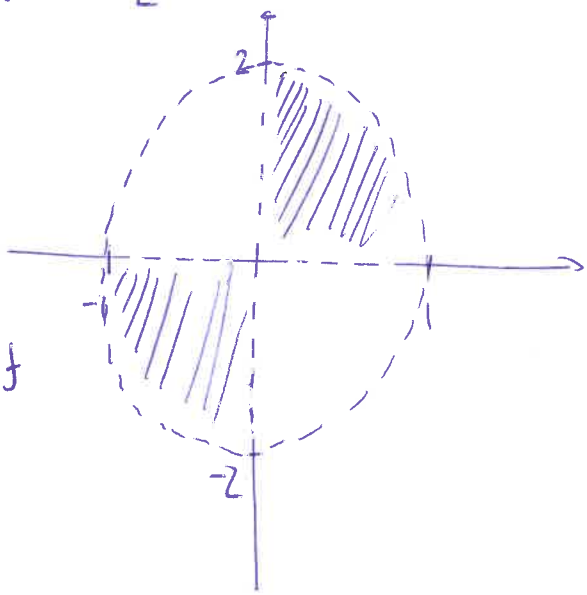
Determine the domain D_f of f and represent it graphically. Sketch also the boundary of D_f.

$$(x,y) \in D_f \iff xy > 0 \wedge 1 - x^2 - \frac{y^2}{4} > 0$$

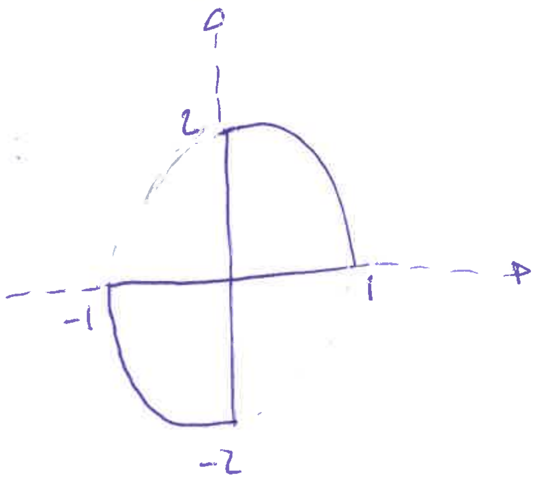
$$\iff xy > 0 \wedge \frac{x^2}{1^2} + \frac{y^2}{2^2} < 1$$

$$D_f = \left\{(x,y) \,\middle|\, \left[(x>0 \wedge y>0) \vee (x<0 \wedge y<0)\right] \wedge \frac{x^2}{1^2} + \frac{y^2}{2^2} < 1\right\}$$

D_f



Bdy D_f:



5. Consider the function f defined in $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \dfrac{x^5 + x^3 + 3yx^2 + xy^2 + 3y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

a. [2,0 pts] Show that f is differentiable at point $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h^5+h^3}{h^3} = \lim_{h\to 0} 1+h^2 = 1.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{3h^3}{h^3} = 3$$

We now study the limit $\displaystyle\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-h-3k}{\sqrt{h^2+k^2}}$

$$\frac{f(h,k)-f(0,0)-h-3k}{\sqrt{h^2+k^2}} = \frac{h^5+h^3+3kh^2+hk^2+3k^3-(h+3k)(h^2+k^2)}{\sqrt{h^2+k^2}\cdot h^2+k^2} = \frac{h^5}{(h^2+k^2)^{3/2}}$$

$$\left|\frac{h^5}{(h^2+k^2)^{3/2}}\right| \leq \frac{|h|^5}{(h^2+k^2)^{3/2}} \leq \frac{(h^2+k^2)^{5/2}}{(h^2+k^2)^{3/2}} = h^2+k^2 \longrightarrow 0,$$ hence

f is differentiable

b. [1,0 pts] Using the fact that f is differentiable at $(0,0)$, give an approximation of

$f(0.05; 0.01)$.

$$f(0.05;0.01) \approx f(0,0) + D^1_{(0,0)}f(0.05;0.01)$$

$$= 0 + 0.05\times 1 + 3\times 0.01 = 0.08.$$