Stochastic Calculus - part 17

ISEG

The Black-Scholes model

- The Black-Scholes model: 2 assets with dynamics

$$dB(t) = rB(t)\,dt, \tag{1}$$

$$dS_t = \alpha S_t dt + \sigma S_t d\overline{W}_t, \tag{2}$$

where r, α and σ are parameters.

- $B(t)$ represents the deterministic price of a riskless asset (a bond or a bank deposit).
- S_t is the (stochastic) price process of a risky asset (a stock or an index).
- $\overline{W}_t$ is a standard Brownian motion with respect to the original probability measure P.

The Black-Scholes model

- r: risk-free interest rate (or short rate of interest).
- α: mean rate of return of the risky asset
- σ: Volatility of the risky asset
- The solution of (2) is the geometric Brownian motion:

$$S_t = S_0 \exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma \overline{W}_t\right).$$

Financial Derivatives

- Consider a contingent claim (a financial derivative), with payoff given by

$$\chi = \Phi\left(S\left(T\right)\right). \tag{3}$$

Assume that this derivative may be traded in the market and that its price process is

$$\Pi\left(t\right) = F\left(t, S_t\right), \quad t \in \left[0, T\right], \tag{4}$$

where $F \in C^{1,2}$.

Financial Derivatives

- Applying It's formula to (4) and considering (2), we get

$$dF(t,S_t) = \left(\frac{\partial F}{\partial t}(t,S_t) + \alpha S_t \frac{\partial F}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t)\right) dt + \left(\sigma S_t \frac{\partial F}{\partial x}(t,S_t)\right) d\overline{W}_t.$$

Financial Derivatives

That is,

$$F(t,S_t) = F(0,S_0) + \int_0^t \left(\frac{\partial F}{\partial t}(r,S_r) + AF(r,S_r)\right) dr + \int_0^t \left(\sigma S_r \frac{\partial F}{\partial x}(r,S_r)\right) d\overline{W}_r,$$

where

$$Af(t,x) = \alpha x \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x)$$

is the infinitesimal generator associated to the diffusion S_t that has the dynamics (2).

Financial Derivatives

- We may also write

$$d\Pi(t) = \alpha_\Pi(t)\,\Pi_t dt + \sigma_\Pi(t)\,\Pi_t d\overline{W}_t, \tag{5}$$

where

$$\alpha_\Pi(t) = \frac{\left(\frac{\partial F}{\partial t}(t,S_t) + \alpha S_t \frac{\partial F}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t)\right)}{F(t,S_t)}, \tag{6}$$

$$\sigma_\Pi(t) = \frac{\sigma S_t \frac{\partial F}{\partial x}(t,S_t)}{F(t,S_t)}. \tag{7}$$

Portfolios

- Portfolio (a_t, b_t)
- a_t is the number of stocks (or units of the risky asset) in the portfolio at instant t.
- b_t is the number of bonds (or units of the riskless asset) in the portfolio at instant t.
- If a_t is negative, we have a short position in the risky asset (for example, "short selling" of stocks)
- If b_t is negative, we have a short position in the riskless asset.

Portfolios

- The value of the portfolio at instant t is given by

$$V\left(t\right) = a_t S_t + b_t B_t.$$

- We assume that the portfolio is self-financed, that is

$$dV_t = a_t dS_t + b_t dB_t.$$

- In a self-financed portfolio, any variation on the value of the portfolio is only due to price changes of the assets, so cash infusion or withdrawal is not allowed.

Pricing by the no arbitrage principle

- We can also consider a portfolio with two other assets: the risky asset and the derivative with the same underlying asset. Let $u_S\left(t\right)$ and $u_\Pi\left(t\right)$ be the relative quantities of each of these assets in the portfolio, so that $u_S\left(t\right) + u_\Pi\left(t\right) = 1$. The dynamics for the value of the portfolio (which is also assumed self-financed) are described by

$$dV_t = u_S\left(t\right) V_t \frac{dS_t}{S_t} + u_\Pi\left(t\right) V_t \frac{d\Pi_t}{\Pi_t}.$$

Substituting (2) and (5), we obtain

$$\begin{aligned} dV_t = {} & V_t\left[u_S\left(t\right)\alpha + u_\Pi\left(t\right)\alpha_\Pi\left(t\right)\right]dt \\ & + V\left[u_S\left(t\right)\sigma + u_\Pi\left(t\right)\sigma_\Pi\left(t\right)\right]d\overline{W}_t. \end{aligned}$$

Pricing by the no arbitrage principle

- We construct the portfolio $(u_S(t), u_\Pi(t))$ in such a way that the stochastic part of dV_t is zero.
- Let $u_S(t), u_\Pi(t)$ be solutions of the system of linear equations

$$\begin{cases} u_S(t) + u_\Pi(t) = 1, \\ u_S(t)\sigma + u_\Pi(t)\sigma_\Pi(t) = 0. \end{cases}$$

- This system has solution

$$u_S(t) = \frac{\sigma_\Pi(t)}{\sigma_\Pi(t) - \sigma},$$
$$u_\Pi(t) = \frac{-\sigma}{\sigma_\Pi(t) - \sigma}.$$

Pricing by the no arbitrage principle

- Substituting (7) on the expressions above, we get

$$u_S(t) = \frac{S_t \frac{\partial F}{\partial x}(t, S_t)}{S_t \frac{\partial F}{\partial x}(t, S_t) - F(t, S_t)}, \tag{8}$$
$$u_\Pi(t) = \frac{-F(t, S_t)}{S_t \frac{\partial F}{\partial x}(t, S_t) - F(t, S_t)}. \tag{9}$$

- With this portfolio we have (value of the portfolio without a stochastic differential):

$$dV_t = V_t \left[u_S(t)\alpha + u_\Pi(t)\alpha_\Pi(t) \right] dt. \tag{10}$$

- By the no-arbitrage principle we have, from (10), that

$$u_S(t)\alpha + u_\Pi(t)\alpha_\Pi(t) = r \tag{11}$$

Pricing by the no arbitrage principle - Black-Scholes model

- Replacing (6), (8) and (9) in the no arbitrage condition (11), we get

$$\frac{\partial F}{\partial t}(t, S_t) + rS_t \frac{\partial F}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t) - rF(t, S_t) = 0.$$

- Furthermore, it is clear that in the maturity date of the derivative we have

$$\Pi(T) = F(T, S_T) = \Phi(S(T)) \qquad (12)$$

Pricing by the no arbitrage principle - Black-Scholes model

Theorem

(Black-Scholes eq.) Assume that the market is specified by eqs. (1)-(2) and we want to price a derivative with payoff given by (3). Then, the only price function of the form (4) that is consistent with the principle of no arbitrage is the solution F of the following boundary values problem, defined in the domain $[0, T] \times \mathbb{R}^+$*:*

$$\frac{\partial F}{\partial t}(t, x) + rx \frac{\partial F}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) = 0, \qquad (13)$$

$$F(T, x) = \Phi(x).$$

Pricing by the no arbitrage principle - Black-Scholes model

- In order to determine the Black-Scholes equation (13), we need to assume that the derivative price takes the form $\Pi(t) = F(t, S_t)$ and that there exists a market for the derivative to be traded. However, it is not unusual for derivatives to be traded "over the counter" (OTC), so it is not always the case.
- To solve this problem, we shall see how to obtain the same equation (13) without those hypothesis.

Some notes on arbitrage

- An arbitrage opportunity on a financial market is defined as a self-financed portfolio h such that:

$$V^h(0) = 0,$$
$$V^h(T) > 0 \quad a.s.$$

- an arbitrage opportunity is the possibility of obtaining a positive profit from no investment, with probability 1, i.e., with no risk involved.
- The no-arbitrage principle simply states that, given a derivative with price $\Pi(t)$, we consider that $\Pi(t)$ is such that there are no arbitrage opportunities in the market.

Proposition

If a self-financed portfolio h is such that the portfolio value has the dynamics

$$dV^h(t) = k(t)\,V^h(t)\,dt,$$

where $k(t)$ is an adapted process, then we must have $k(t) = r$ for all t, or otherwise arbitrage opportunities exist.