Master in Mathematical Finance, ISEG, University of Lisbon

Stochastic Calculus

Final Exam; Exam duration: 2 hours; June, 11, 2018

Justify your answers and calculations

1. Consider a standard Brownian motion $B = \{B_t, t \geq 0\}$.

(a) Define the process X by

$$X_t = e^{-\alpha t} \sin(\gamma B_t) + B_t^2 - t,$$

where α and γ are real parameters. Deduce how α and γ should be related for the process X to be a martingale. Give also an example of numerical values for α and γ such that X is a martingale.

(b) Let $\lambda > 0$ be a constant. For what values of λ, is the process

$$Y_t = \begin{cases} t^\lambda B_{\frac{1}{t}} & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

a standard Brownian motion? Show that for these values of λ, the process Y is indeed a standard Brownian motion.

2. Consider the n-dimensional Brownian motion $B = \{B_t, t \in [0,T]\}$ with $B_t = \left(B_t^{(1)}, B_t^{(2)}, \ldots, B_t^{(n)}\right) \in \mathbb{R}^n$. Define the process

$$Z_t = Z_0 \exp\left(\sum_{i=1}^{n} \alpha_i t + \delta_i B_t^{(i)}\right),$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\delta_1, \delta_2, \ldots, \delta_n$ are real parameters. Show that $Z_t = C + K \int_0^t Z_s ds + \int_0^t Z_s (\mathbf{v} \cdot dB)$, where C, K and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are constants and $\cdot$ denotes the scalar (inner) product and express the constants C, K and $\mathbf{v}$ in terms of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\delta_1, \delta_2, \ldots, \delta_n$ and Z_0.

3. Let $B = \{B_t, t \in [0,T]\}$ be a Brownian motion. Consider a SDE of the type

$$\begin{aligned} dX_t &= f(t, X_t)dt + g(t, X_t)dB_t, \\ X_0 &= x > 0, \end{aligned}$$

where $f(t,x)$ and $g(t,x)$ are continuous and deterministic functions.

(a) If $f(t, X_t) = 5X_t^2$ and $g(t, X_t) \equiv 0$, what can you say about the existence and uniqueness of solutions in the interval $[0,T]$? Explain your answer and, if possible, find the solution or solutions.

(b) Considering $f(t, X_t) = \sin(t) X_t$ and $g(t, X_t) = e^{-t} X_t$, with $X_0 = 2$, comment on the existence and uniqueness of solutions and solve the SDE.

4. Consider the boundary value problem with domain $[0,T] \times \mathbb{R}$:

$$\begin{gathered} \frac{\partial F}{\partial t} + 6x \frac{\partial F}{\partial x} + 8 \frac{\partial^2 F}{\partial x^2} + x = 0, \quad t > 0, \ x \in \mathbb{R} \\ F(T, x) = x^2. \end{gathered}$$

(a) Show that the solution of the problem can be represented by the stochastic representation (Feynman-Kac) formula

$$F(t,x) = \mathbb{E}_{t,x}\left[X_T^2\right] + \int_t^T \mathbb{E}_{t,x}\left[X_s\right] ds,$$

where X is a diffusion process that satisfies

$$\begin{aligned} dX_s &= 6X_s ds + 4X_s dB_s, \quad \text{if } s > t, \\ X_t &= x. \end{aligned}$$

Hint: Define X as the solution of the SDE above, assume that F actually solves the PDE, consider the process $Y_s = F(s, X_s)$ and apply the Itô formula.

(b) Deduce an explicit expression for the process X and for the solution $F(t,x)$ of the boundary value problem.

5. Consider the Black-Scholes model with a risky asset with price S_t and a riskless asset with price B_t. The assets follow the SDE's

$$dS_t = 0.1\, S_t\, dt + 0.25\, S_t\, d\overline{W}_t \quad \text{ e } \quad dB_t = 0.05 B_t\, dt, \text{ with } S_0 = 2, B_0 = 1.$$

where $\overline{W}$ is a Brownian motion. Consider also a contingent claim (financial derivative) with payoff (at maturity T) given by $\chi = \Phi(S_T) = S_T^4 + 30\ln(S_T)$. Write the SDE for S_t under que equivalent martingale measure (or risk neutral measure) $\mathbb{Q}$, specify also what is the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ and calculate the price of the contingent claim (financial derivative) at time 0 with maturity 1 year.

6. Let $B = \{B_t, t \geq 0\}$ be a Brownian motion. Define the operator $\mathcal{K} : L^2_{a,T} \to L^2_{a,T}$ by

$$(\mathcal{K}X)_t = \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s,$$

where the functions b and σ satisfy the Lipschitz and linear growth properties (as in the existence and uniqueness theorem for SDE's). In the space $L^2_{a,T}$ consider the norm

$$\|X\| = \left(\int_0^T e^{-\lambda s} E\left[|X_s|^2\right] ds\right)^{\frac{1}{2}},$$

where λ is a positive number. Show that if λ is large enough, then

$$\|\mathcal{K}X - \mathcal{K}Y\| \leq C\,\|X - Y\|,$$

for a positive constant $C < 1$ (and therefore the operator $\mathcal{K}$ is a contraction in $L^2_{a,T}$).

Hint: You can start by showing that $E\left[|(\mathcal{K}X)_t - (\mathcal{K}Y)_t|^2\right] \leq C_1 E\left[\int_0^t (X_s - Y_s)^2\, ds\right]$. Then you multiply the previous inequality by $e^{-\lambda t}$ and integrate in $[0,T]$.

Marks: 1(a):2.5, (b):2.0, 2:2.0, 3(a):2.25, (b):2.25, 4(a):2.5, (b):2.0, 5:2.5, 6:2.0