Master in Mathematical Finance, ISEG, University of Lisbon

Stochastic Calculus

Final Exam; Exam duration: 2 hours; July, 3, 2018

Justify your answers and calculations

1. Consider a standard Brownian motion $B = \{B_t, t \geq 0\}$ and the filtration $\{\mathcal{F}_t\}$ generated by the Brownian motion.

(a) Define the process Y by

$$Y_t = \exp\left(\int_0^t e^{-\alpha s} dB_s + k\int_0^t e^{-\beta s} ds\right),$$

where α, β and k are real parameters. Deduce how α, β and k should be related for the process Y to be a martingale.

(b) Consider the process

$$X_t = B_t - tB_1, \quad t \in [0,1].$$

Verify if this process is: (i) Gaussian; (ii) adapted to $\{\mathcal{F}_t\}$; (iii) a martingale with respect to $\{\mathcal{F}_t\}$; and (iv) deduce its covariance function.

2. Consider the standard Brownian motion $B = \{B_t, t \in [0,T]\}$ and define the process

$$Z_t = \int_0^t (t-s)^{-\alpha} dB_s.$$

Deduce the values of α for which the stochastic Itô integral Z_t is well defined and calculate the covariance function for the process Z.

3. Let $B = \{B_t, t \in [0,T]\}$ be a Brownian motion. Consider a SDE of the type

$$\begin{aligned} dX_t &= f(t,X_t)dt + g(t,X_t)dB_t, \\ X_0 &= x > 0, \end{aligned}$$

where $f(t,x)$ and $g(t,x)$ are continuous and deterministic functions.

(a) If $f(t,X_t) = t^2 \ln\left(1 + X_t^4\right)$ and $g(t,X_t) \equiv t\exp\left(\cos\left(X_t\right)\right)$, what can you say about the existence and uniqueness of solutions in the interval $[0,T]$? Explain your answer.

(b) Considering $f(t,X_t) = X_t^\delta$ and $g(t,X_t) = X_t$, solve the SDE when $\delta = 1$ and when $\delta = -1$ (Hint: perhaps it is useful to consider an appropriate exponential integrating factor when $\delta = -1$).

4. Consider the boundary value problem with domain $[0,T] \times \mathbb{R}$:

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + 32x^2\frac{\partial^2 F}{\partial x^2}(t,x) + 10x\frac{\partial F}{\partial x}(t,x) = 4F(t,x), &\quad 0 \leq t < T,\ x \in \mathbb{R}, \\ F(T,x) = 1 + x^4. & \end{aligned}$$

Specify the infinitesimal generator of the associated diffusion, deduce explicitly the form of this diffusion process, present the stochastic representation formula for the solution of the boundary value problem and an explicit expression (as explicit as you can) for the final solution of the problem.

5. Consider the Black-Scholes model with a risky asset with price S_t and a riskless asset with price B_t. The assets follow the SDE's

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, d\overline{W}_t \quad \text{and} \quad dB_t = r\, B_t\, dt,$$

where $\overline{W}$ is a Brownian motion under the measure $\mathbb{P}$.

(a) Write the SDE satisfied by the process $Y_t = S_t^n$, with $n \geq 2$, under the equivalent martingale measure $\mathbb{Q}$ and solve this stochastic differential equation.

(b) Deduce a formula for the price of the contingent claim (financial derivative) with payoff $\chi = \Phi(S_T) = \frac{1}{K} S_T^6 + K \mathbf{1}_{\{S_T < S_0\}}$, at time 0, where K is a positive constant, T is the expiry date and S_0 is the price of the underlying risky asset at time 0.

6.Let $B = \{B_t, t \geq 0\}$ be a Brownian motion. Consider $\varepsilon > 0$ and the function

$$f_\varepsilon(x) = \begin{cases} |x| & \text{if } |x| \geq \varepsilon \\ \frac{1}{2}\left(\varepsilon + \frac{x^2}{\varepsilon}\right) & \text{if } |x| < \varepsilon \end{cases}.$$

Assume that you can apply the classical Itô formula. Show that

$$f_\varepsilon(B_t) = \frac{\varepsilon}{2} + \int_0^t f_\varepsilon'(B_s)\, dB_s + \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{]-\varepsilon,\varepsilon[}(B_s)\, ds,$$

and that when $\varepsilon \to 0$ (consider the mean-square limit), we obtain

$$\int_0^t f_\varepsilon'(B_s)\, dB_s \to \int_0^t \operatorname{sign}(B_s) dB_s.$$

Moreover, explain why we need to assume that we can apply the classical Itô formula for a function of class C^2.

Marks: 1(a):2.25, (b):2.25, 2:2.5, 3(a):2.0, (b):2.5, 4:2.5, 5(a):2.0, (b):2.0, 6:2.0