Models for Nonnegative Outcomes

Continuous Outcomes and Count Data

Log-Linear and Exponential Regression Models

Poisson and Negative Binomial Models

Panel Data Models

- Nonnegative outcomes can be:
  - Continuous: $Y \in [0, +\infty[$
    - Examples: prices, wages,...
  - Discrete (counts): $Y \in \{0,1,2,3, \dots\}$
    - Examples: patents applied for by a firm in a year, times someone is arrested in a year,...

- Linear regression models are not the most suitable option because:
  - May generate negative predictions for the dependent variable
  - At least close to the lower bound of Y, it does not make sense to assume constant partial effects

Log-linear regression model:

$$ln(Y_i) = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_k x_{ki} + u_i$$

- Assumption: $E(u_i|x) = 0$
- With this transformation, the dependent variable becomes unbounded: $Y \in]0, +\infty[\Longrightarrow \ln(Y) \in]-\infty, +\infty[$
- However, two new problems arise:
  - The log-linear model is not defined for $Y = 0$; adding a small constant value to Y or dropping zeros are not in general good solutions
  - Prediction is more interesting in the original scale, $\widehat{Y}_i$, and not in the logarithmic scale, $\widehat{ln(Y_i)}$; the log-linear model gives the latter directly but retransforming it to the original scale requires additional assumptions and calculations and/or the application of relatively complex methods

Exponential regression model:

$$Y = exp(x'\beta + u)$$
$$E(Y|X) = exp(x'\beta)$$

- Assumption: $E(e^u|x) = 1$
- Advantages:
  - $\widehat{Y}_i$ is always nonnegative
  - Predictions are obtained directly in the original scale, without requiring any retransformations
- Partial effects:

$$\Delta X_j = 1 \Longrightarrow \Delta E(Y|X) = \beta_j exp(x'\beta)$$

  - The sign of the effect is given by the sign of β_j
  - β_j can be interpreted as a semi-elasticity, since:

$$100\beta_j = 100\frac{\Delta E(Y|X)}{E(Y|X)}, \text{i.e. } \Delta X_j = 1 \Longrightarrow \%\Delta E(Y|X) = 100\beta_j\%$$

- Assumptions and estimation methods according to the type of nonnegative outcome:
  - Continuous response:
    - Assumption: only $E(Y|X)$; estimation: QML
  - Count data - two alternatives:
    - Assumption: only $E(Y|X)$; estimation: QML
    - Assumption: $E(Y|X)$ and $Pr(Y = j|X)$; estimation: ML

- Three main distribution functions are used as basis for QML and/or ML estimation:
  - Poisson
  - Negative Binomial 1
  - Negative Binomial 2

Poisson regression model:

$$Y_i \sim Poisson(\lambda_i) \Longrightarrow Pr(Y_i = y|x_i) = \frac{e^{-\lambda_i}\lambda_i{}^y}{y!}$$

where $\lambda_i = E(Y|X) = exp(x'\beta)$

- Estimation methods: ML (only count data) or QML, since the Poisson distribution belongs to the linear exponential family

- By definition, $E(Y|X) = Var(Y|X)$ (equidispersion), which may be a strong assumption is some empirical applications

Stata
ML: poisson $YX_1 \dots X_k$
QML: poisson $YX_1 \dots X_k$, robust

Negative binomial regression models:

- Two variants, both allowing for overdispersion ($\delta > 0$):
  - NEGBIN1: $Var(Y|X) = (1 + \delta)E(Y|X)$ - ML estimation
  - NEGBIN2: $Var(Y|X) = [1 + \delta E(Y|X)]E(Y|X)$ - it belongs to the linear exponential family, enabling estimation by both ML (only count data) and QML

Stata

NEGBIN1: nbreg $Y X_1 \ldots X_k$, dispersion(constant)
NEGBIN2 (ML): nbreg $Y X_1 \ldots X_k$, dispersion(mean)
NEGBIN2 (QML): nbreg $Y X_1 \ldots X_k$, dispersion(mean) robust

- Overdispersion test:

$H_0: \delta = 0$ (Poisson model)

$H_1: \delta \neq 0$ (Negative Binomial 1 or 2 model)

Base model:

- Continuous / count data:

$$E(Y_{it}|x_{it},\alpha_i) = exp(\gamma_i + x_{it}'\beta) = \alpha_i exp(x_{it}'\beta)$$

- Count data:

$$Pr(Y_{it} = y|x_{it},\alpha_i) = \frac{e^{-\lambda_{it}}{\lambda_{it}}^{y}}{y!}$$

$$\lambda_i = E(Y_{it}|x_{it},\alpha_i) = \alpha_i exp(x_{it}'\beta)$$

Pooled estimator:

- Based on the cross-sectional assumption $E(Y_{it}|x_{it}) = exp(x_{it}'\beta)$
- Produces consistent estimators only if $E(\alpha_i|x_{it}) = 1$
- Does not require the Poisson distributional assumption
- Using a robust vce controls for both overdispersion and time dependence

Stata
poisson $Y X_1 \ldots X_k$, vce(cluster *clustvar*)

Random Effects Poisson Estimator:

- Assumptions:
  - $Y_{it} \sim Poisson(\lambda_{it})$
  - $\lambda_i = E(Y_{it}|x_{it}, \alpha_i) = \alpha_i exp(x_{it}'\beta)$
  - $log(\alpha_i) = \gamma_i \sim Gamma(1, \eta)$

- Resulting model:
  - NEGBIN2-type model
  - Estimation method: ML
  - $E(Y_{it}|x_{it}) = exp(x_{it}'\beta)$, which implies that the Pooled estimator is consistent under random effects of this type

Stata
xtpoisson $Y X_1 \ldots X_k$, re

- Alternative model: assumes $log(\alpha_i) = \gamma_i \sim N(0, \sigma^2)$ and produces $(Y_{it}|x_{it}) = exp(x_{it}'\beta)$ but has no close form solution

Fixed Effects Estimators:

- Fixed effects Poisson estimator (three equivalent versions):
  - Pooled estimator with individual effects
  - Estimator conditional on $\sum_{t=1}^{T} Y_{it}$, with $\sum_{t=1}^{T} Y_{it} \neq 0$
  - Quasi mean-differenced GMM estimator (Hausman, Hall and Griliches, 1984)

- Quasi-differences GMM estimator:
  - Chamberlain (1992)
  - Wooldridge (1997)

Do not require the Poisson distributional assumption

Fixed effects Poisson estimator:

- May be derived using the three equivalent versions
- Pooled estimator with individual effects:
  - Adds individual dummies, associated to the $\gamma_i' s$
  - As in linear models, β is consistently estimated even in short panels (no incidental parameters problem)
- The quasi mean-differenced GMM estimator is based on the following moment condition:

$$E\left(Y_{it} - \frac{\lambda_{it}}{\bar{\lambda}_i}\bar{Y}_i \middle| x_{it}\right) = 0,$$

where $\lambda_{it} = exp(x_{it}'\beta)$ and $\bar{\lambda}_i = \frac{1}{T}\sum_{t=1}^{T}\lambda_{it}$

- Requires strictly exogenous explanatory variables

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xtpoisson $Y X_1 \dots X_k$, fe

Quasi-differences GMM estimator :

- Chamberlain (1992):

$$E\left(\frac{\lambda_{i,t-1}}{\lambda_{i,t}}Y_{it} - Y_{i,t-1}\middle| x_{it}\right) = 0$$

- Wooldridge (1997):

$$E\left(\frac{Y_{it}}{\lambda_{it}} - \frac{Y_{i,t-1}}{\lambda_{i,t-1}}\middle| x_{it}\right) = 0$$

- In both cases the explanatory variables do not need to be strictly exogenous, so these estimators are particularly useful in dynamic models