Assets and Liabilities

Modelling, Matching and Management

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Introduction

Contents overview

- Basic interest rate theory
- Interest rate risk management
- Stochastic term structure models
- Risk measurement
- Reinsurance and insurance linked securities
- Mean-variance analysis for ALM
- IFRS 17, new accounting standard

Stakeholders

- Shareholders,
- Policyholders,
- Creditors,
- Management,
- Reinsurers,
- Investors in alternative risk capital,
- Supervisors,
- Rating agencies,
- Tax authorities.

Shareholders

- Have invested money in the company,
- Own the net assets of the company,
- Accumulate the after-tax profit or loss,
- Can normally diversify their shareholdings,
- Can "walk away" in case of bankruptcy,
- Can sometimes sell their shareholding,
- Want a competitive return on capital.

Policyholders

- Have paid a premium to the company,
- Have been promised compensation for losses,
- Can normally not diversify their insurance cover,
- Cannot "walk away" in case of bankruptcy,
- Can normally not transfer their policy or claim,
- Are interested in a secure company,
- Do not want to pay a high premium.

Creditors

- Have delivered a service or product to the company,
- Have been promised payment for the service or product,
- Can (normally) not transfer or sell their claim,
- May have an ongoing business relationship,
- Are interested in a secure company.

Management (including employees)

- Have invested their career in the company,
- May or may not be shareholders in the company,
- Can normally not "diversify" their current career,
- Can normally not "walk away" in case of bankruptcy,
- Have a strong influence on the running of the company,
- Focus most on growth and earnings (period accounts),
- Are the guardians of the balance sheet.

Reinsurers

- May have sold protection (non-proportional),
- May participate in the primary business (proportional),
- Cannot easily transfer an existing contract,
- Can refuse to renew a contract,
- Can ask for changed terms at renewal,
- Are interested in profitable underwriting.

Investors in alternative risk capital

- Accept risk from insurers, using financial instruments,
- Compete with and partly replace traditional reinsurance,
- Can move in and out of markets and positions at any time,
- Look for an expected profit commensurate with the risk.

Supervisors

- Monitor compliance with laws and regulations,
- Monitor companies' financial strength,
- Are seen as policyholders' guardian,
- Should they also work for a competitive marketplace?
- Are interested in prudent provisions and adequate capital.

Rating agencies

- Monitor companies' financial strength,
- Focus on adequate capital,
- Focus on prudent provisions,
- Focus on sustainable strategy,
- Focus on profitable underwriting,
- Focus on risk control procedures.

Tax authorities

- Want to collect tax from the company,
- Want realistic provisions and capital,
- Are interested in a profitable business.

Objectives of ALM

- What does ALM stand for?
- Definition of ALM
- Uses of ALM
- Focus of ALM

What does ALM stand for?

- Asset and Liability Matching?
- Asset and Liability Modelling?
- Asset and Liability Measurement?
- Asset and Liability Management?

The answer is, all of the above.

Definition of the Society of Actuaries ALM Principles Task Force:

> *Asset Liability Management is the ongoing process of formulating, implementing, monitoring, and revising strategies related to assets and liabilities to achieve financial objectives, for a given set of risk tolerances and constraints.*

Uses of ALM:

- Risk mitigation: to keep risk exposure within specified limits, for given parameters (strategies),
- Strategic: to formulate asset/liability strategies to achieve financial objectives, subject to acceptable risk.

Focus of ALM:

- Economic: Present value of asset and liability cash flows;
- Market value: Market value of assets and liabilities;
- Accounting results: Book value of assets and liabilities;
- Regulatory: Control of regulatory key figures.
- Long term or short term.

Let's face it. Most financial risk management is ALM:

- Managing liability risk consists of finding offsetting assets;
- Managing asset risk consists of finding offsetting liabilities.

ALM proper consists of modelling and managing the *joint risk* of assets and liabilities.

Sources of risk

- Interest rate risk - main focus of ALM
- Other sources

Interest rate risk is the main focus of ALM.

Interest rate risk manifests itself in many ways:

- Changing market value of assets and/or liabilities,
- Changing present value of asset and/or liability cash flows,
- Changing duration and/or convexity exposure.

Ultimately, interest rate risk is a function of gains and losses on reinvestment and disinvestment of cash flows in the future.

Other risks of an insurance company:

- Market risk,
- Currency risk,
- Liquidity risk,
- Counterparty risk,
- Insurance risk,
- Operational risk.

Managing each of those risks is a form of ALM.

Interest rates

- A continuous model for yield curves.
- Estimating the yield curve.
- Sensitivity of present values.
- Matching ("cash flow matching").
- Immunisation ("duration/convexity matching").
- Stochastic term structure models.
- Simulation of stochastic evolution of the yield curve.

If $P(t)$ is the market price of a *zero-coupon bond* that pays the risk-free amount of €1 at time t, the yield y of the bond is defined by the equation:

$$P(t) = e^{-yt}$$

The yield of the zero-coupon bond is defined as:

$$y = y(t) = -\frac{1}{t}\ln(P(t))$$

$y(t)$ is called the "spot rate" or "zero rate" for maturity t.

Important: The "yield" is just a way of expressing the price.

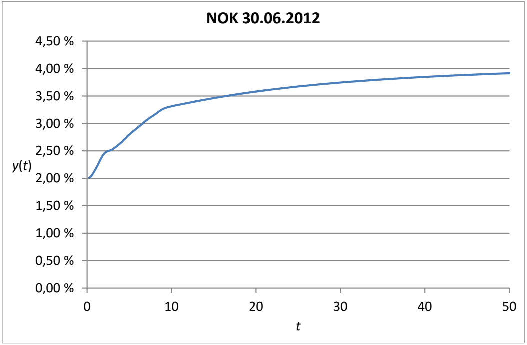
- The current yield curve describes the yields of notional zero-coupon bonds due at different times in the future:

$$y(t) = -\frac{1}{t} \ln(\text{Current market price of €1 payable at time } t)$$

It is...

- *Current*, because the market price changes from day to day.
- *Notional*, because one cannot buy zero-coupon bonds for every maturity.

Interest rates - basic concepts
Yield curve example



Assume that the yield curve $\{y(t) : t > 0\}$ is known.
The arbitrage-free market value of a risk-free, future cashflow $\{c(t_1), c(t_2), \cdots, c(t_n)\}$ is:

$$B = \sum_{i=1}^{n} P(t_i)c(t_i) = \sum_{i=1}^{n} e^{-y(t_i)t_i} c(t_i)$$

Every payment is valued separately as a zero-coupon bond.

The yield curve is a strange animal. Consider this:

- The spot rate $y(t)$ at maturity t is the *constant* yield rate in the interval $(0, t)$ that reproduces the observed price $P(t)$ of €1 payable at time t.
- At the same time we are aware that the yield curve is *not constant*!

The forward rate $y_F(t)$ is the implied yield in the infinitesimal time interval $(t, t + dt)$, defined consistently with the spot rate. Forward rates $y_F(t)$ are defined by spot rates through the equation

$$\int_0^t y_F(s)ds = y(t) \cdot t$$

or, assuming differentiability,

$$y_F(t) = y(t) + t \cdot y'(t).$$

The spot rate is the average of forward rates in the interval $(0, t)$.

Let n be an integer.
If $P(n)$ is the market price of a *zero-coupon bond* that pays the risk-free amount of €1 at time n, the yield i with annual compounding is defined by the equation:

$$P(n) = (1+i)^{-n}$$

The yield of zero-coupon bonds can be calculated explicitly:

$$i = i(n) = P(n)^{-1/n} - 1 = \exp(y(n)) - 1$$

Note the relationship between yield with annual compounding (i) and yield with continuous compounding (y):

$$i = \exp(y) - 1 \text{ and } y = \ln(1+i)$$

Continuous compounding allows a unified and simple notation, e.g.,

$$P(t) = \exp(-y(t)t) = \exp\left(-\int_0^t y_F(s)ds\right)$$

regardless of whether the time t is an integer (whole year) or not.
In this lecture we will use continuous compounding.
In the financial press and many publications, annual and semi-annual compounding is common.

A bond can be defined in general as

> *"a promise to make a series of payments of specified size, at specified times in the future".*

Let us denote by $c(t_i)$ the payment due at time t_i, for $i = 1, \cdots, n$. We assume that the bonds have no credit risk

Let $\{c(t_i) : i = 1, \cdots, n\}$ be the payments stipulated by a bond. Let B be the price being paid for the bond in the market. The average yield $\overline{y}$ of the bond is defined implicitly by the equation

$$B = B(\overline{y}) \stackrel{!}{=} \sum_{i=1}^{n} e^{-\overline{y}t_i} c(t_i) \stackrel{\text{def.}}{=} \int_0^\infty e^{-\overline{y}t} dC(t)$$

The average bond yield is well-defined if all payments are non-negative.

Interest rates - bonds

Bond yield example

		Coupon rate	**5 %**			Average yield y	Average yield y'	
		Face value	**100,00**			2,77455 %	2,81340 %	
						continuous	annual	
Time t	Spot rate $y(t)$	P(t)=Price of €1	$c(t)$	$c(t)\exp(-y(t)t)$		$c(t)\exp(-yt)$	$c(t)(1+y')$^(-t)	IRR of
0,00	0,000 %	1,0000	0,00	0,00				-110,07
1,00	2,169 %	0,9785	5,00	4,89		4,86	4,86	5,00
2,00	2,448 %	0,9522	5,00	4,76		4,73	4,73	5,00
3,00	2,529 %	0,9269	5,00	4,63		4,60	4,60	5,00
4,00	2,648 %	0,8995	5,00	4,50		4,47	4,47	5,00
5,00	2,800 %	0,8693	105,00	91,28		91,40	91,40	105,00
		Total	125,00	110,07	=	110,07	110,07	

Assume that you know the market prices $B_1, \cdots, B_n$ of n different government bonds.

Define the payoff matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} \text{Payments of bond } 1 \\ \vdots \\ \text{Payments of bond } n \end{pmatrix}$$

Some of the c_{ij} may be zero, but all bonds' total payments must be restricted to the time points $t_1, \cdots, t_n$.

We construct a portfolio $(w_1, \cdots, w_n)$ that *replicates* the cash flow of a zero-coupon bond at maturity t_j :

$$(w_1, \cdots, w_n)\, \mathbf{C} \stackrel{!}{=} (0, \cdots, 0, 1, 0, \cdots, 0)$$

The equation is solved by

$$(w_1, \cdots, w_n) = (0, \cdots, 0, 1, 0, \cdots, 0)\, \mathbf{C}^{-1} = \mathrm{row}_j\left(\mathbf{C}^{-1}\right)$$

The price of the zero-coupon bond at maturity t_j must then be

$$P(t_j) = \sum_{i=1}^{n} w_i B_i$$

The implied zero rate $y(t_j)$ is given by solving

$$P(t_j) = e^{-y(t_j)t_j}$$

In principle, finding yield curves is easy matrix algebra. In practice there are a number of problems. For example:

- Not enough traded bonds to cover all time points.
- Payments not at the required time points.
- Lack of long term bonds.

There are a number of techniques and models to deal with that.

In practice you would often use a software or the risk-free rates delivered by EIOPA, Bloomberg or others. They have their models and methods, too.

Market assumption at 31.12.12				
Bond	Maturity 31.12. ...	Face value	Coupon	Average yield_annual
1	2 013	100,00	4,00 %	2,192285 %
2	2 014	100,00	4,00 %	2,473059 %
3	2 015	100,00	4,00 %	2,554222 %
4	2 016	100,00	5,00 %	2,668676 %
5	2 017	100,00	5,00 %	2,813400 %
6	2 018	100,00	5,00 %	2,931906 %
7	2 019	100,00	5,00 %	3,052088 %
8	2 020	100,00	5,00 %	3,149622 %
9	2 021	100,00	5,00 %	3,244703 %
10	2 022	100,00	5,00 %	3,290179 %
11	2 023	100,00	5,00 %	3,322407 %
12	2 024	100,00	5,00 %	3,352730 %
13	2 025	100,00	5,00 %	3,381184 %
14	2 026	100,00	5,00 %	3,407837 %
15	2 027	100,00	5,00 %	3,432781 %

Interest rates - yield curve estimation
Estimating the market yield curve by replication - payment matrix C

Payment matrix C														
2 013	2 014	2 015	2 016	2 017	2 018	2 019	2 020	2 021	2 022	2 023	2 024	2 025	2 026	2 027
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
104,00	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4,00	104,00	-	-	-	-	-	-	-	-	-	-	-	-	-
4,00	4,00	104,00	-	-	-	-	-	-	-	-	-	-	-	-
5,00	5,00	5,00	105,00	-	-	-	-	-	-	-	-	-	-	-
5,00	5,00	5,00	5,00	105,00	-	-	-	-	-	-	-	-	-	-
5,00	5,00	5,00	5,00	5,00	105,00	-	-	-	-	-	-	-	-	-
5,00	5,00	5,00	5,00	5,00	5,00	105,00	-	-	-	-	-	-	-	-
5,00	5,00	5,00	5,00	5,00	5,00	5,00	105,00	-	-	-	-	-	-	-
5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	105,00	-	-	-	-	-	-
5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	105,00	-	-	-	-	-
5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	105,00	-	-	-	-
5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	105,00	-	-	-
5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	105,00	-	-
5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	105,00	-
5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	5,00	105,00

Estimating the market yield curve by replication - clean market price B

The clean market price

Bond	Maturity	Face value	Coupon	Average yield annual	Market price B NPV
1	2 013	100,00	4,00 %	2,192285 %	101,768935
2	2 014	100,00	4,00 %	2,473059 %	102,944219
3	2 015	100,00	4,00 %	2,554222 %	104,124847
4	2 016	100,00	5,00 %	2,668676 %	108,734859
5	2 017	100,00	5,00 %	2,813400 %	110,067565
6	2 018	100,00	5,00 %	2,931906 %	111,228588
7	2 019	100,00	5,00 %	3,052088 %	112,112257
8	2 020	100,00	5,00 %	3,149622 %	112,907489
9	2 021	100,00	5,00 %	3,244703 %	113,512298
10	2 022	100,00	5,00 %	3,290179 %	114,371484
11	2 023	100,00	5,00 %	3,322407 %	115,248582
12	2 024	100,00	5,00 %	3,352730 %	116,056972
13	2 025	100,00	5,00 %	3,381184 %	116,803770
14	2 026	100,00	5,00 %	3,407837 %	117,495240
15	2 027	100,00	5,00 %	3,432781 %	118,136897

Payment time t	P(t)=C^(-1)B	y(t)
1	0,978547	2,1686 %
2	0,952212	2,4484 %
3	0,926940	2,5289 %
4	0,899489	2,6482 %
5	0,869349	2,8002 %
6	0,839009	2,9256 %
7	0,807472	3,0550 %
8	0,776594	3,1605 %
9	0,745374	3,2652 %
10	0,718062	3,3120 %
11	0,692222	3,3441 %
12	0,666958	3,3752 %
13	0,642311	3,4053 %
14	0,618310	3,4340 %
15	0,594978	3,4615 %

Assume you have bonds $i = 1, \cdots, n$. Bond nr. i matures at time t_i and pays coupons c_i, and its current market price is B_i. All bonds have principal 1.

1. Solve for the first bond

$$B_1 = (1 + c_1)P(t_1) \Rightarrow P(t_1) = \frac{B_1}{(1 + c_1)} = e^{-y(t_1)t_1}$$

2. Solve for each subsequent bond

$$B_m = \underbrace{c_m \sum_{i=1}^{m-1} P(t_i)}_{\text{known}} + (1 + c_m)\underbrace{P(t_m)}_{\text{unknown}}$$

$$\Rightarrow P(t_m) = \frac{B_m - c_m \sum_{i=1}^{m-1} P(t_i)}{(1 + c_m)} = e^{-y(t_m)t_m}$$

Interest rates - present value sensitivity
What if the yield curve changes?

Let's assume we have

- a future cash flow $\{C(t) : t > 0\}$ and
- the current yield curve $\{y(t) : t > 0\}$.

The present value of the cash flow is

$$B(y) = \int_0^\infty e^{-y(t)t} dC(t)$$

- Question: How will the present value of change if the yield curve changes?
- The easy answer: Calculate it!
- The traditional answer: Estimate it!

The derivative of the present value with respect to a uniform shift in the entire yield curve is:

$$B'(y) = \lim_{\Delta\overline{y} \to 0} \frac{1}{\Delta\overline{y}} \left(\int_0^\infty e^{-(y(t)+\Delta\overline{y})t} dC(t) - \int_0^\infty e^{-y(t)t} dC(t) \right)$$

And similar for the second derivative.
The first and second derivative of the present value are

$$\begin{aligned} B'(y) &= -\int_0^\infty t e^{-y(t)t} dC(t) \\ B''(y) &= \int_0^\infty t^2 e^{-y(t)t} dC(t) \end{aligned}$$

Using the Taylor expansion we approximate the change in present value if the yield curve shifts:

$$B(y+\Delta\overline{y}) - B(y) \approx B'(y)\Delta\overline{y} + \frac{1}{2}B''(y)\left(\Delta\overline{y}\right)^2$$

Define the *duration* of the cash flow as

$$D = D(y) = -B'(y)/B(y)$$

and its *convexity* as

$$C = C(y) = B''(y)/B(y)$$

Rewrite the Taylor expansion in the following way:

$$\frac{B(y+\Delta\overline{y})-B(y)}{B(y)} \approx -D(y)\Delta\overline{y} + \frac{1}{2}C(y)\left(\Delta\overline{y}\right)^2$$

In words: The relative change in the value of the cash flow when the yield curve is shifted uniformly by a small amount of $\Delta\overline{y}$, can be approximated by:

- To first order: (Minus) the yield change $\Delta\overline{y}$, times duration.
- To second order: Same as above, plus the squared yield change times one-half convexity.

Interest rates - present value sensitivity
Example: Estimating the effect of a yield reduction by D and C

Time t	Spot rate $y(t)$	P(t)=Price of €1	$c(t)$	$c(t)\exp(-y(t)t)$
0,00	0,000 %	1,0000	0,00	0,00
1,00	2,169 %	0,9785	5,00	4,89
2,00	2,448 %	0,9522	5,00	4,76
3,00	2,529 %	0,9269	5,00	4,63
4,00	2,648 %	0,8995	5,00	4,50
5,00	2,800 %	0,8693	105,00	91,28
		Total	125,00	110,07
PV	110,07			
PV x D	502,72	Duration	4,57	
PV x C	2419,65	Convexity	21,98	
Parallel shi	-1,000 %			
Time t	Zero rate $y(t)$	Price of €1	$c(t)$	$c(t)\exp(-y(t)t)$
0,00		1,0000	0,00	0,00
1,00	1,169 %	0,9884	5,00	4,94
2,00	1,448 %	0,9714	5,00	4,86
3,00	1,529 %	0,9552	5,00	4,78
4,00	1,648 %	0,9362	5,00	4,68
5,00	1,800 %	0,9139	105,00	95,96
		Total	125,00	115,22
		Approximations	First order	115,09
			Second order	115,22

Interest rates - present value sensitivity

Example: Estimating the effect of a yield increase by D and C

Time t	Spot rate $y(t)$	P(t)=Price of €1	$c(t)$	$c(t)\exp(-y(t)t)$
0,00	0,000 %	1,0000	0,00	0,00
1,00	2,169 %	0,9785	5,00	4,89
2,00	2,448 %	0,9522	5,00	4,76
3,00	2,529 %	0,9269	5,00	4,63
4,00	2,648 %	0,8995	5,00	4,50
5,00	2,800 %	0,8693	105,00	91,28
		Total	125,00	110,07

PV	110,07		
PV x D	502,72	Duration	4,57
PV x C	2419,65	Convexity	21,98
Parallel shi	**1,000 %**		

Time t	Zero rate $y(t)$	Price of €1	$c(t)$	$c(t)\exp(-y(t)t)$
0,00		1,0000	0,00	0,00
1,00	3,169 %	0,9688	5,00	4,84
2,00	3,448 %	0,9334	5,00	4,67
3,00	3,529 %	0,8995	5,00	4,50
4,00	3,648 %	0,8642	5,00	4,32
5,00	3,800 %	0,8270	105,00	86,83
		Total	125,00	105,16
		Approximations	First order	105,04
			Second order	105,16

Interest rates - present value sensitivity

Example: Five cashflows all adding to €125, with different PV, D and C

Time t	Zero rate $y(t)$	Price of €1	Cashflow 1	Cashflow 2	Cashflow 3	Cashflow 4	Cashflow 5
0,00	0,000 %	1,0000	-	-	-	-	-
1,00	2,169 %	0,9785	5,00	-	125,00	25,00	62,50
2,00	2,448 %	0,9522	5,00	-	-	25,00	-
3,00	2,529 %	0,9269	5,00	-	-	25,00	-
4,00	2,648 %	0,8995	5,00	-	-	25,00	-
5,00	2,800 %	0,8693	105,00	125,00	-	25,00	62,50
		Total	125,00	125,00	125,00	125,00	125,00
		Present value	110,07	108,67	122,32	115,66	115,49
		Duration	4,57	5,00	1,00	2,94	2,88
		Convexity	21,98	25,00	1,00	10,65	12,29
		Dispersion	1,12	0,00	0,00	1,99	3,99

- The duration and convexity of a zero-coupon bond payable at time are t and t^2 independent of the yield.
- Duration and convexity decrease when the yield increases.
- For given duration, convexity increases with the dispersion of the cash flow, because

$$\underbrace{\frac{1}{B(y)} \int_0^\infty (t - D(y))^2 \, e^{-y(t)t} dC(t)}_{\text{Dispersion, similar to variance}} = C(y) - D^2(y)$$

Interest rates - present value sensitivity
Some properties of duration and convexity - 2

- The duration/convexity approximation is an easy way to estimate the sensitivity of a cash flow's present value to (small) changes in the yield curve.
- The average duration/convexity of a portfolio is the present-value-weighted average of the constituent durations/convexities. This makes those quantities easy to use.
- The duration/convexity approximation is valid only when there is a parallel shift in the yield curve.
- The duration/convexity approximation does not tell us what change in the present value to expect, should different parts of the yield curve change by different amounts or even in different directions.

Interest rates - present value sensitivity
Duration - several concepts

Duration comes in several forms:

Macaulay Duration The time weighted present value divided by the present value.

Modified Duration Macaulay Duration by 1 + i(n) / n, where n is the compounding frequency.

Effective Duration Calculated by shocking the yield curve up and down by some change in interest rates, calculating the change in present value, and using a central difference equation.

Dollar Duration Interest rate sensitivity in absolute terms. In our notation, Dollar Duration is $DD(y) = -B'(y) = B(y)D(y)$.

Interest rates - cash flow matching
Asset-Liability Matching - 1

- Assume that the insurance company has an expected liability cash flow of $\{L(t) : t > 0\}$ and valued it in its balance sheet by discounting in accordance with the zero-coupon yield curve $\{y(t) : t > 0\}$:

$$PV_L = \int_0^\infty e^{-y(t)t} dL(t)$$

- Assume also that the insurance company has invested in assets which provide a future cash flow of $\{A(t) : t > 0\}$. The discounted value of the assets is

$$PV_A = \int_0^\infty e^{-y(t)t} dA(t)$$

The only way to protect the surplus against all changes in the yield curve is by matching the asset cash flow to the liability cash flow. In principle, a matching portfolio can be found by solving

$$
\begin{aligned}
& (w_1, \cdots, w_n)\, \mathbf{C} \stackrel{!}{=} (L_1, \cdots, L_n) \\
\Rightarrow \quad & (w_1, \cdots, w_n) = (L_1, \cdots, L_n)\, \mathbf{C}^{-1}
\end{aligned}
$$

where $\mathbf{C}^{nxn}$ is the payoff matrix of n bonds. The market value of the matching portfolio is

$$
(w_1, \cdots, w_n)\, \mathbf{B} = (L_1, \cdots, L_n)\, \mathbf{C}^{-1}\mathbf{B}
$$

This is also known as the discounted value of the liabilities.

Practical problems with matching assets to liabilities

- Not enough bonds available for a long-tailed liability cash flow.
- Insufficient market liquidity at some maturities.
- The matching portfolio may have some $w_i < 0$ (inadmissible).
- Investment manager may consider matching portfolio sub-optimal.
- Liabilities are random and change all the time (need for rebalancing).

Given that full asset-liability matching is not practical, *immunisation* attempts to give approximately the same interest rate sensitivity to the assets and the liabilities.
Let

$$PV = PV_A - PV_L$$

be the surplus or net asset value that we are seeking to protect from changes in the yield curve.

The dollar duration and dollar convexity of the surplus are:

$$
\begin{aligned}
DD &= PV_A D_A - PV_L D_L \\
DC &= PV_A C_A - PV_L C_L
\end{aligned}
$$

A parallel shift of $\Delta\overline{y}$ in the yield curve will change the net asset value by approximately

$$\Delta PV \approx \left(-PV_A \cdot D_A + PV_L \cdot D_L\right) \Delta\overline{y}$$

using the first term of the Taylor expansion.

To protect the value of its surplus a shift in the yield curve, the company could select its assets in such a way that

$$PV_A \cdot D_A = PV_L \cdot D_L$$

This is known as *immunisation* or *duration matching*. It means that the dollar duration of assets equals the dollar duration of liabilities.

If $PV_A = PV_L$, the durations must be equal $(D_A = D_L)$ to achieve immunisation.

One can add a term to the Taylor expansion and write

$$\begin{aligned}\Delta PV &\approx \left(-PV_A \cdot D_A + PV_L \cdot D_L\right) \Delta\overline{y} \\ &\quad + \frac{1}{2}\left(PV_A \cdot C_A - PV_L \cdot C_L\right)\left(\Delta\overline{y}\right)^2\end{aligned}$$

If $PV_A = PV_L$ and $D_A = D_L$, the first term is zero and one should select assets in such a way that

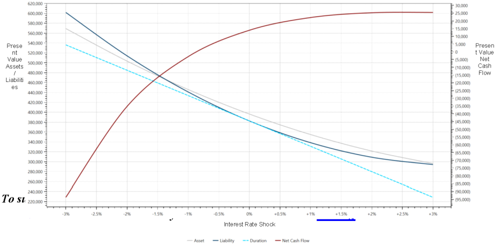
$$C_A \geq C_L$$

The asset cash flow should have the same or a higher convexity than the liability cash flow.

Interest rates - cash flow immunisation

Immunisation - convexity exposure

Figure 7 - Convexity Exposure



- Any liability cash flow can be duration-immunised with two asset cash flows, by solving the set of linear equations that equate
  1. the present value, and
  2. the dollar duration.
- Any liability cash flow can be duration and convexity immunised with three asset cash flows, by solving the set of linear equations that equate
  1. the present value, and
  2. the dollar duration, and
  3. the dollar convexity.

Interest rates - cash flow immunisation

Immunisation example - duration and convexity matching with three bonds

	The bonds you use		
Maturity	2	5	12
Coupon	4,00 %	5,00 %	5,00 %
Face value	100,00	100,00	100,00

Bond investments (number of bonds)		
Maturity: 2	Maturity: 5	Maturity: 12
137 605	64 191	16 509

				Bond payments per face value						Asset cash flow
Time t	Yield curve	Price of €1	Liability cash flow	Maturity: 2	Maturity: 5	Maturity: 12	Maturity: 2	Maturity: 5	Maturity: 12	Total
1	2,168600 %	0,9785	6 527 416	4,00	5,00	5,00	550 419,32	320 954,43	82 546,66	953 920,40
2	2,448388 %	0,9522	4 801 976	104,00	5,00	5,00	14 310 902,30	320 954,43	82 546,66	14 714 403,38
3	2,528864 %	0,9269	3 772 174	-	5,00	5,00	-	320 954,43	82 546,66	403 501,08
4	2,648207 %	0,8995	2 841 440	-	5,00	5,00	-	320 954,43	82 546,66	403 501,08
5	2,800218 %	0,8693	2 224 827	-	105,00	5,00	-	6 740 042,93	82 546,66	6 822 589,58
6	2,925573 %	0,8390	1 724 919	-	-	5,00	-	-	82 546,66	82 546,66
7	3,054961 %	0,8075	1 220 471	-	-	5,00	-	-	82 546,66	82 546,66
8	3,160465 %	0,7766	823 638	-	-	5,00	-	-	82 546,66	82 546,66
9	3,265218 %	0,7454	595 814	-	-	5,00	-	-	82 546,66	82 546,66
10	3,311988 %	0,7181	388 856	-	-	5,00	-	-	82 546,66	82 546,66
11	3,344074 %	0,6922	274 339	-	-	5,00	-	-	82 546,66	82 546,66
12	3,375232 %	0,6670	167 101	-	-	105,00	-	-	1 733 479,77	1 733 479,77
13	3,405255 %	0,6423	96 047	-	-	-	-	-	-	-
14	3,434039 %	0,6183	48 275	-	-	-	-	-	-	-
15	3,461543 %	0,5950	20 081	-	-	-	-	-	-	-
	Total		25 527 374	108,00	125,00	160,00	14 861 321,62	8 023 860,63	2 641 492,98	25 526 675,24

	Liability	Maturity: 2	Maturity: 5	Maturity: 12	Maturity: 2	Maturity: 5	Maturity: 12	Total
Present value	23 146 979	102,94	110,07	116,06	14 165 622	7 065 334	1 916 023	23 146 979
Duration	3,3797	1,96	4,57	9,48	1,9620	4,5673	9,4817	3,3797
Convexity	17,6527	3,89	21,98	103,46	3,8859	21,9833	103,4643	17,6527
Dispersion	6,2303	0,04	1,12	13,56	0,0366	1,1226	13,5608	6,2303

Finding the immunising portfolio

Solve	Liability		Maturity: 2	Maturity: 5	Maturity: 12		Weights w 1--w 3
PV	23 146 979		102,94	110,07	116,06		137 605
PV x D	78 229 713	=	201,97	502,72	1 100,42	x	64 191
PV x C	408 605 988		400,03	2 419,65	12 007,75		16 509

Interest rates - cash flow immunisation

Immunisation example - At the outset, the surplus measured in present value, is zero

Change year 1-2	0,0 %				
Change year 3-15	0,0 %				
Time t	Yield curve	Price of €1	Liability cash flow	Asset cash flow	Surplus
1	2,168600 %	0,9785	6 527 416	953 920	-5 573 496
2	2,448388 %	0,9522	4 801 976	14 714 403	9 912 427
3	2,528864 %	0,9269	3 772 174	403 501	-3 368 673
4	2,648207 %	0,8995	2 841 440	403 501	-2 437 939
5	2,800218 %	0,8693	2 224 827	6 822 590	4 597 763
6	2,925573 %	0,8390	1 724 919	82 547	-1 642 372
7	3,054961 %	0,8075	1 220 471	82 547	-1 137 924
8	3,160465 %	0,7766	823 638	82 547	-741 091
9	3,265218 %	0,7454	595 814	82 547	-513 267
10	3,311988 %	0,7181	388 856	82 547	-306 309
11	3,344074 %	0,6922	274 339	82 547	-191 792
12	3,375232 %	0,6670	167 101	1 733 480	1 566 379
13	3,405255 %	0,6423	96 047	0	-96 047
14	3,434039 %	0,6183	48 275	0	-48 275
15	3,461543 %	0,5950	20 081	0	-20 081
	Total		25 527 374	25 526 675	-699
		Present value	23 146 979	23 146 979	0

Interest rates - cash flow immunisation

Immunisation example - are we protected if the yield curve increases uniformly?

Change year 1-2	1,0 %				
Change year 3-15	1,0 %				
Time t	Yield curve	Price of €1	Liability cash flow	Asset cash flow	Surplus
1	3,168600 %	0,9688	6 527 416	953 920	-5 573 496
2	3,448388 %	0,9334	4 801 976	14 714 403	9 912 427
3	3,528864 %	0,8995	3 772 174	403 501	-3 368 673
4	3,648207 %	0,8642	2 841 440	403 501	-2 437 939
5	3,800218 %	0,8270	2 224 827	6 822 590	4 597 763
6	3,925573 %	0,7901	1 724 919	82 547	-1 642 372
7	4,054961 %	0,7529	1 220 471	82 547	-1 137 924
8	4,160465 %	0,7169	823 638	82 547	-741 091
9	4,265218 %	0,6812	595 814	82 547	-513 267
10	4,311988 %	0,6497	388 856	82 547	-306 309
11	4,344074 %	0,6201	274 339	82 547	-191 792
12	4,375232 %	0,5915	167 101	1 733 480	1 566 379
13	4,405255 %	0,5640	96 047	0	-96 047
14	4,434039 %	0,5375	48 275	0	-48 275
15	4,461543 %	0,5121	20 081	0	-20 081
	Total		25 527 374	25 526 675	-699
		Present value	22 384 651	22 384 602	-49

Interest rates - cash flow immunisation

Immunisation example - are we protected if the yield curve decreases uniformly?

Change year 1-2	-1,0 %				
Change year 3-15	-1,0 %				
Time t	Yield curve	Price of €1	Liability cash flow	Asset cash flow	Surplus
1	1,168600 %	0,9884	6 527 416	953 920	-5 573 496
2	1,448388 %	0,9714	4 801 976	14 714 403	9 912 427
3	1,528864 %	0,9552	3 772 174	403 501	-3 368 673
4	1,648207 %	0,9362	2 841 440	403 501	-2 437 939
5	1,800218 %	0,9139	2 224 827	6 822 590	4 597 763
6	1,925573 %	0,8909	1 724 919	82 547	-1 642 372
7	2,054961 %	0,8660	1 220 471	82 547	-1 137 924
8	2,160465 %	0,8413	823 638	82 547	-741 091
9	2,265218 %	0,8156	595 814	82 547	-513 267
10	2,311988 %	0,7936	388 856	82 547	-306 309
11	2,344074 %	0,7727	274 339	82 547	-191 792
12	2,375232 %	0,7520	167 101	1 733 480	1 566 379
13	2,405255 %	0,7315	96 047	0	-96 047
14	2,434039 %	0,7112	48 275	0	-48 275
15	2,461543 %	0,6913	20 081	0	-20 081
	Total		25 527 374	25 526 675	-699
		Present value	23 950 188	23 950 242	55

Interest rates - cash flow immunisation

Immunisation example - are we protected if the yield curve "tilts"?

Change year 1-2	1,0 %				
Change year 3-15	-1,0 %				
Time t	Yield curve	Price of €1	Liability cash flow	Asset cash flow	Surplus
1	3,168600 %	0,9688	6 527 416	953 920	-5 573 496
2	3,448388 %	0,9334	4 801 976	14 714 403	9 912 427
3	1,528864 %	0,9552	3 772 174	403 501	-3 368 673
4	1,648207 %	0,9362	2 841 440	403 501	-2 437 939
5	1,800218 %	0,9139	2 224 827	6 822 590	4 597 763
6	1,925573 %	0,8909	1 724 919	82 547	-1 642 372
7	2,054961 %	0,8660	1 220 471	82 547	-1 137 924
8	2,160465 %	0,8413	823 638	82 547	-741 091
9	2,265218 %	0,8156	595 814	82 547	-513 267
10	2,311988 %	0,7936	388 856	82 547	-306 309
11	2,344074 %	0,7727	274 339	82 547	-191 792
12	2,375232 %	0,7520	167 101	1 733 480	1 566 379
13	2,405255 %	0,7315	96 047	0	-96 047
14	2,434039 %	0,7112	48 275	0	-48 275
15	2,461543 %	0,6913	20 081	0	-20 081
	Total		25 527 374	25 526 675	-699
		Present value	23 639 526	23 371 086	-268 439

Immunisation example - are we protected if the yield curve steepens?

Change year 1-2	-1,0 %
Change year 3-15	1,0 %

Time t	Yield curve	Price of €1	Liability cash flow	Asset cash flow	Surplus
1	1,168600 %	0,9884	6 527 416	953 920	-5 573 496
2	1,448388 %	0,9714	4 801 976	14 714 403	9 912 427
3	3,528864 %	0,8995	3 772 174	403 501	-3 368 673
4	3,648207 %	0,8642	2 841 440	403 501	-2 437 939
5	3,800218 %	0,8270	2 224 827	6 822 590	4 597 763
6	3,925573 %	0,7901	1 724 919	82 547	-1 642 372
7	4,054961 %	0,7529	1 220 471	82 547	-1 137 924
8	4,160465 %	0,7169	823 638	82 547	-741 091
9	4,265218 %	0,6812	595 814	82 547	-513 267
10	4,311988 %	0,6497	388 856	82 547	-306 309
11	4,344074 %	0,6201	274 339	82 547	-191 792
12	4,375232 %	0,5915	167 101	1 733 480	1 566 379
13	4,405255 %	0,5640	96 047	0	-96 047
14	4,434039 %	0,5375	48 275	0	-48 275
15	4,461543 %	0,5121	20 081	0	-20 081
	Total		25 527 374	25 526 675	-699
		Present value	22 695 313	22 963 758	268 445

Interest rates - cash flow immunisation

Alternatives to immunisation

Alternatives between the extremes of complete matching and total independence, of asset values and liability values.

- Immunising with "maturity buckets" of bonds gives a less spiky asset cash flow.
- Immunising separate maturity sections of the liability cash flow separately gives better protection.
- Not immunising, but limiting that the dollar duration of the surplus: $PV_A D_A - PV_L D_L < \varepsilon \cdot PV_A$.
- The assets should have larger dollar convexity than the liabilities: $PV_A C_A > PV_L C_L$.

The problem: Guaranteed products with long durations are too expensive to immunise with available fixed income assets.

Ways to reduce the problem:

- In an upward-sloping yield curve, exchange shorter maturities for longer maturities, i.e. increase the duration and the yield to maturity. Reduce the interest rate risk.
- Add credit spread. Accept bonds of lower credit quality and exchange interest rate risk for spread risk.
- Add riskier, non-interest-sensitive assets such as equities or real estate, and exchange interest rate risk for market risk.
- Transfer risk to policyholders. Replace guaranteed products with unit-linked products.

(Some) stochastic term structure models

- Cox-Ingersoll.Ross model
- Vasicek model
- Hull-White model
- Ho-Lee model

We have seen that

- Matching protects the surplus against all changes in yield curve, but is not always practical.
- Immunisation protects the surplus against parallel shifts in the yield curve, but not necessarily against other contortions.

To be able to assess what changes one can *reasonably* expect, one must model the stochastic evolution of the yield curve (the "term structure").

The purpose of stochastic term structure models is to represent the evolution of the yield curve as a stochastic process.

- At time t, the market price of €1 payable at time $T > t$ is $P(t, T)$. This price varies randomly from day to day.
- On any given day t, the function $\{P(t, T) : T > t\}$ can be determined, at least in principle, from the observed market prices of bonds and bills.

We write $P(t, T) = \exp(-R(t, T)(T - t))$, so that $R(t, T)$ is the random spot rate (zero rate) at time t for a term of $T - t$.

The short rate at time t is

$$r(t) = \lim_{T \downarrow t} R(t, T).$$

We write

$$P(t, T) = \mathrm{E}\left(\exp\left(-\int_t^T r(s)ds\right)\right)$$

so that

$$R(t, T) = -\frac{1}{T - t} \ln \mathrm{E}\left(\exp\left(-\int_t^T r(s)ds\right)\right)$$

A stochastic model of the evolution of the short rate $r(t)$ together with a no-arbitrage assumption implies the entire yield curve at time t.

Vasicek model

$$dr(t) = a\left(b - r(t)\right)dt + \sigma dz(t)$$

Cox-Ingersoll-Ross model (CIR)

$$dr(t) = a\left(b - r(t)\right)dt + \sigma\sqrt{r(t)}dz(t)$$

The term $z(t)$ is standard Brownian motion (white noise).

- Both the Vasicek and CIR model have mean reversion.
- The parameter b describes the long-term average rate.
- The parameter a describes the strength of the gravitation back to the average rate.
- The Vasicek model allows negative interest rates.
- The CIR model allows only positive interest rates.

The term structure of the Vasicek and CIR model has the affine form

$$P(t,T) = A(t,T)\exp\left(-B(t,T)r(t)\right)$$

or, equivalently,

$$R(t,T) = \frac{1}{T-t}\left(-\ln A(t,T) + B(t,T)r(t)\right)$$

with functions A and B that depend on a, b and σ.
This makes the term structure easy to calculate and simulate once the parameters have been given or estimated.

$$
\begin{aligned}
P(t,T) &= A(t,T)\exp\left(-B(t,T)r(t)\right)\\
B(t,T) &= \frac{1-\exp\left(-a\left(T-t\right)\right)}{a}\\
A(t,T) &= \exp\left(\frac{\left(B(t,T)-\left(T-t\right)\right)\left(a^2b-\frac{1}{2}\sigma^2\right)}{a^2}\right)\\
&\quad\times\exp\left(-\frac{\sigma^2B^2(t,T)}{4a}\right)
\end{aligned}
$$

$$
\begin{aligned}
P(t,T) &= A(t,T)\exp\left(-B(t,T)r(t)\right)\\
B(t,T) &= \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+a)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\\
A(t,T) &= \left(\frac{2\gamma e^{(\gamma+a)(T-t)/2}}{(\gamma+a)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right)^{2ab/\sigma^2}\\
\gamma &= \sqrt{a^2+2\sigma^2}
\end{aligned}
$$

The discrete time version of the CIR and Vasicek model are as follows:

$$r(t+\Delta) = r(t) + a(b - r(t))\Delta + \sigma r^k(t)\left[\sqrt{\Delta}\varepsilon(t)\right]$$

with

$$\begin{aligned}
\Delta &= \text{time step between the observations,} \\
a &= \text{velocity of reverting to mean level,} \\
b &= \text{mean reverting level,} \\
\sigma &= \text{standard deviation,} \\
k &= \text{0 for Vasicek, 0.5 for CIR,} \\
\varepsilon(t) &= \text{standard normal (0,1) random variable.}
\end{aligned}$$

We rewrite this as an ordinary linear regression:

$$\begin{aligned} r(t+\Delta) &= \underbrace{ab\Delta}_{\text{intercept}} + \underbrace{(1-a\Delta)}_{\text{gradient}} r(t) + \underbrace{\sigma r^k(t)\sqrt{\Delta}}_{\text{standard dev.}}\varepsilon(t) \\ &= \alpha + \beta \cdot r(t) + \rho \cdot \varepsilon(t) \end{aligned}$$

We can estimate the intercept, gradient and SD by ordinary least squares and then transform to:

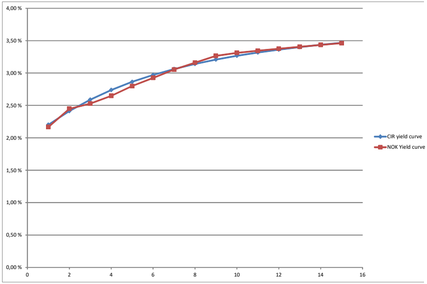
$$\begin{aligned} a^* &= (1-\beta^*)/\Delta, \\ b^* &= \alpha^*/(1-\beta^*), \\ \sigma^* &= \rho^*/\left(\bar{r}^k\sqrt{\Delta}\right) \end{aligned}$$

A practical example

- See "Estimate_CIR_Vasicek_model.xls".
- Schiegl's approach applied to data from the Norwegian National Bank's 3-months interest rates.
- Whether the results are sensible can be discussed.
- The results are sensitive to the time range of observations and the spacing Δ (Schiegl).
- But they are numbers, at least!

Stochastic term structure models - modelling a yield curve

Modelling the NOK yield curve with a CIR model (by eye-fitting)



Stochastic term structure models - modelling a yield curve

Modelling the NOK yield curve with a CIR model - value of a cash flow

Parameters	
a	27,0 %
b	4,0 %
sigma	5,0000 %
r(0)	1,95 %
gamma	27,9 %

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
NOK Yield curve	2,17 %	2,45 %	2,53 %	2,65 %	2,80 %	2,93 %	3,05 %	3,16 %	3,27 %	3,31 %	3,34 %	3,38 %	3,41 %	3,43 %	3,46 %
CIR yield curve	2,20 %	2,41 %	2,59 %	2,74 %	2,86 %	2,97 %	3,06 %	3,14 %	3,21 %	3,27 %	3,32 %	3,36 %	3,40 %	3,44 %	3,47 %
Cash flows															
Liabilities	6 527 416	4 801 976	3 772 174	2 841 440	2 224 827	1 724 919	1 220 471	823 638	595 814	388 856	274 339	167 101	96 047	48 275	20 081
Assets	953 920	14 714 403	403 501	403 501	6 822 590	82 547	82 547	82 547	82 547	82 547	82 547	1 733 480	0	0	0
T-t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A(t,T)	99,51 %	98,20 %	96,30 %	93,99 %	91,39 %	88,63 %	85,77 %	82,87 %	79,97 %	77,10 %	74,29 %	71,54 %	68,86 %	66,27 %	63,75 %
B(t,T)	87,61 %	154,34 %	205,09 %	243,64 %	272,89 %	295,06 %	311,87 %	324,60 %	334,24 %	341,53 %	347,06 %	351,24 %	354,40 %	356,79 %	358,60 %

	r(t)	P(t,t+1)	P(t,t+2)	P(t,t+3)	P(t,t+4)	P(t,t+5)	P(t,t+6)	P(t,t+7)	P(t,t+8)	P(t,t+9)	P(t,t+10)	P(t,t+11)	P(t,t+12)	P(t,t+13)	P(t,t+14)	P(t,t+15)	Liability PV	Asset PV	Surplus PV
Empirical		97,85 %	95,22 %	92,69 %	89,95 %	86,93 %	83,90 %	80,75 %	77,66 %	74,54 %	71,81 %	69,22 %	66,70 %	64,23 %	61,83 %	59,50 %	23 146 979	23 146 979	0
CIR	1,95 %	97,82 %	95,29 %	92,52 %	89,62 %	86,66 %	83,67 %	80,71 %	77,78 %	74,92 %	72,13 %	69,42 %	66,80 %	64,26 %	61,81 %	59,45 %	23 127 064	23 137 939	10 875

The surplus PV is slightly different from 0, because the CIR model is not a perfect replication of the empirical yield curve.

Stochastic term structure models - simulation

One-half year ahead simulations, using CIR and assuming all payments at year end

Parameters

a	27,0 %
b	4,0 %
sigma	5,0000 %
r(0)	1,95 %
gamma	27,9 %
dt	**0,5000**
sigma*sqrt(dt)	0,0354

Projected cash flows

Liabs	6 527 416	4 801 976	3 772 174	…	96 047	48 275	20 081
Assets	953 920	14 714 403	403 501	…	0	0	0

T-t	0,50	1,50	2,50	…	12,50	13,50	14,50
A(t,T)	99,87 %	98,94 %	97,31 %	…	70,19 %	67,55 %	65,00 %
B(t,T)	46,77 %	123,25 %	181,46 %	…	352,93 %	355,68 %	357,76 %

Simulation No.	Previous r(t)	Drift term a(b-r)dt	Uniform (0,1)	dz = Normal (0,1) = NORMINV(Uniform(0,1))	Random term sqrt(r) * sigma*sqrt(dt) * dz	Differential dr	New r(t)	P(t,t+0,5)	P(t,t+1,5)	P(t,t+2,5)	…	P(t,t+12,5)	P(t,t+13,5)	P(t,t+14,5)	Liability PV (left axis)	Asset PV (left axis)	Surplus PV (right axis)
1	1,95 %	0,28 %	0,6956	0,5117	0,25 %	0,53 %	2,48 %	98,72 %	95,96 %	93,03 %	…	64,31 %	61,85 %	59,48 %	23 262 469	23 270 456	7 987
2	1,95 %	0,28 %	0,9148	1,3710	0,68 %	0,95 %	2,90 %	98,52 %	95,46 %	92,32 %	…	63,35 %	60,92 %	58,59 %	23 098 419	23 101 585	3 166
3	1,95 %	0,28 %	0,0125	-2,2414	-1,11 %	-0,83 %	1,12 %	99,35 %	97,58 %	95,35 %	…	67,47 %	64,91 %	62,45 %	23 798 629	23 821 601	22 971
4	1,95 %	0,28 %	0,4786	-0,0538	-0,03 %	0,25 %	2,20 %	98,85 %	96,29 %	93,50 %	…	64,94 %	62,47 %	60,08 %	23 371 274	23 382 397	11 123
5	1,95 %	0,28 %	0,9181	1,3921	0,69 %	0,96 %	2,91 %	98,52 %	95,45 %	92,30 %	…	63,33 %	60,90 %	58,56 %	23 094 403	23 097 450	3 046
6	1,95 %	0,28 %	0,5539	0,1355	0,07 %	0,34 %	2,29 %	98,81 %	96,18 %	93,34 %	…	64,73 %	62,26 %	59,88 %	23 334 787	23 344 863	10 077
7	1,95 %	0,28 %	0,9689	1,8653	0,92 %	1,20 %	3,15 %	98,41 %	95,17 %	91,91 %	…	62,81 %	60,40 %	58,08 %	23 004 742	23 005 105	363
8	1,95 %	0,28 %	0,2343	-0,7247	-0,36 %	-0,08 %	1,87 %	99,00 %	96,69 %	94,07 %	…	65,71 %	63,21 %	60,80 %	23 501 250	23 516 056	14 805
9	1,95 %	0,28 %	0,8431	1,0071	0,50 %	0,77 %	2,72 %	98,61 %	95,67 %	92,62 %	…	63,76 %	61,32 %	58,96 %	23 167 696	23 172 912	5 215
10	1,95 %	0,28 %	0,6595	0,4112	0,20 %	0,48 %	2,43 %	98,74 %	96,02 %	93,11 %	…	64,42 %	61,96 %	59,59 %	23 281 764	23 290 311	8 547
…	…	…	….	…	…	…	…	…	…	…	…	…	…	…	…	…	…

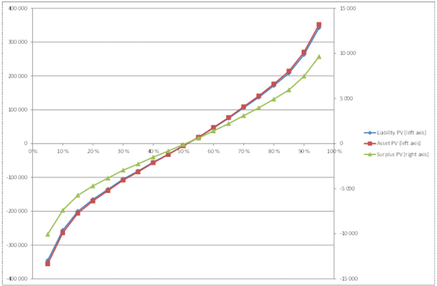
Stochastic term structure models - simulation

Summary of 1000 one-half year ahead simulations

Average	23 364 516	23 375 362	10 847
Percentiles			
5 %	23 018 779	23 019 564	785
10 %	23 108 101	23 111 554	3 454
15 %	23 164 348	23 169 464	5 117
...	...	...	...
50 %	23 358 299	23 369 050	10 751
...	...	...	...
85 %	23 572 749	23 589 550	16 801
90 %	23 627 645	23 645 965	18 320
95 %	23 707 061	23 727 556	20 495
Percentiles corrected for average			
5 %	-345 737	-355 798	-10 062
10 %	-256 415	-263 808	-7 393
15 %	-200 168	-205 898	-5 730
...	...	...	...
50 %	-6 217	-6 312	-95
...	...	...	...
85 %	208 233	214 188	5 955
90 %	263 130	270 603	7 473
95 %	342 545	352 193	9 648
P75-P25	272 021	279 806	7 786
P95-P05	688 282	707 992	19 710

Stochastic term structure models - simulation

Summary of 1000 one-half year ahead simulations



- The term structure of a Vasicek or CIR model can never perfectly replicate the actual (observed) term structure.
- Sometimes one needs a model that starts with today's exact term structure and evolves stochastically from there. Models with that property are called no-arbitrage models.
- Like the Vasicek and CIR models, the Ho-Lee and Hull-White models have an affine term structure of the form $P(t,T) = A(t,T)\exp\left(-B(t,T)r(t)\right)$, with different functions A and B.
- In the Ho-Lee and Hull-White models, the function $A(t,T)$ depends not only on model parameters but also on the initial term structure. The Ho-Lee and Hull-White models permit perfect fit to today's empirical term structure.

Ho-Lee model

$$dr(t) = \theta(t)dt + \sigma dz(t)$$

Hull-White model

$$dr(t) = \left(\theta(t) - ar\left(t\right)\right)dt + \sigma dz(t)$$

The term $z(t)$ is standard Brownian motion (white noise).

$$
\begin{aligned}
P(t,T) &= A(t,T)\exp\left(-B(t,T)r(t)\right) \\
B(t,T) &= T-t \\
A(t,T) &= \frac{P(0,T)}{P(0,t)}\exp\left((T-t)F(0,t)-\frac{1}{2}\sigma^2 t(T-t)^2\right)
\end{aligned}
$$

with

- $P(0,t)$ = today's observed spot rates
- $F(0,t)$ = today's instantaneous (continuous) forward rate

In the Ho-Lee model, the variance of future forward rates around today's forward curve is constant.

$$
\begin{aligned}
P(t,T) &= A(t,T)\exp\left(-B(t,T)r(t)\right) \\
B(t,T) &= \frac{1-\exp\left(-a\left(T-t\right)\right)}{a} \\
A(t,T) &= \frac{P(0,T)}{P(0,t)}\exp\left(B(t,T)F(0,t)\right) \\
&\quad \times \exp\left(-\frac{1}{4a^3}\sigma^2\left(e^{-aT}-e^{-at}\right)\left(e^{-2at}-1\right)\right)
\end{aligned}
$$

with

- $P(0,t) =$ today's observed spot rates
- $F(0,t) =$ today's instantaneous (continuous) forward rate

In the Hull-White model, the variance of future forward rates around today's forward curve is decreasing.

Two main forms of risk measurement in ALM:

- Analysis of sensitivity to changes in financial variables,
- Modelling a probability distribution and simulation.

Risk measures
Quantitative measures of risk exposure

Particularly for interest rate risk sensitivity:

- Duration
- Convexity

Probabilistic risk measures:

- Value at Risk (VaR)
- Tail Value at Risk (TailVar)
- Coherent risk measures
- Spectral risk measures

A random variable X is a "loss" if

- $X > 0$ denotes a loss,
- $X < 0$ denotes a profit.

The probability distribution F of X we call a loss distribution. In most applications

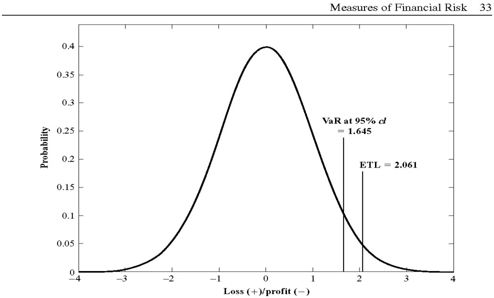
- Loss $X =$ Actual cost – Expected cost, or
- Loss $X =$ Expected income – Actual income

For a loss distribution F, VaR at confidence level α is defined as the α-percentile:

$$VaR_{\alpha}(F) = F^{-1}(\alpha)$$

VaR is the loss level that will not be exceeded with a given probability. For example, α could be 0.90, 0.95, 0.99, 0.995.

Risk measures
Value at Risk (VaR) and Expected Tail Loss (ETL)



Risk measures
Value at Risk (VaR) vs. confidence level

24 Measuring Market Risk

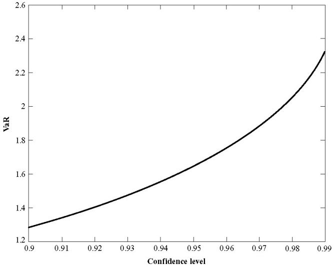


Figure 2.7 VaR and confidence level

Risk measures
Value at Risk (VaR) vs. holding period

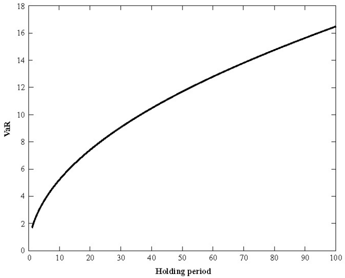


Figure 2.8 VaR and holding period

Some observations about VaR

- VaR involves two arbitrary parameters:
  - the confidence level and
  - the holding period (day, month, year or until final run-off).
- Started by JP Morgan for one-day measurements (RiskMetrics).
- A single measure that applies to all types of losses and aggregates.
- The requirement of a probability implies that the losses are "normal".
- VaR does not tell us the severity of events that surpass the confidence level.

Calculating VaR

- Historical observations:
  - Easy,
  - Realistic, if many observations.
- Historical observations (resampling):
  - Easy,
  - Realistic, if level of confidence realistic,
  - Does not go beyond observed values
- Fitted distributions and simulation:
  - More complicated, possible model error,
  - Allows for a tail beyond the observed,
  - Not necessarily better within the normal range.

Not everyone thinks that VaR is a good idea.

- VaR does not tell us the severity of events beyond the confidence level,
- Two portfolios with the same VaR may have different tail distributions,
- VaR does not necessarily recognise diversification benefits,
- "Perverse incentives", "discourage diversification", and so on and so forth.

Enter Tail Value at Risk (TailVaR)!

For a loss distribution F, TailVaR at confidence level α is defined as

$$\text{TailVaR}_\alpha(F) = \frac{1}{1-\alpha}\int_\alpha^1 \text{VaR}_s(F)ds$$

TailVaR is the average of VaR above the confidence level α.

It is also called Expected Tail Loss (ETL) or Conditional Tail Expectation (CTE).

Risk measures
TailVaR compared with VaR vs confidence level

34 Measuring Market Risk

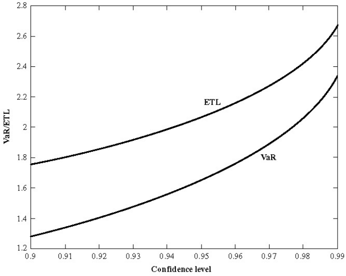


Figure 2.11 ETL and the confidence level

Some observations about TailVaR:

- TailVaR measures the entire tail beyond the VaR percentile.
- TailVaR is indifferent to the size of losses that exceed VaR.
- TailVar is not the only alternative to VaR.
- Its choice of confidence level and holding period is also arbitrary.
- Quantifying extreme tail losses and their probability is difficult.

Risk measures

Coherent risk measures

A risk measure ρ is a functional of a loss distribution F, or alternatively a loss random variable $X \sim F$.

The risk measure is called coherent if it satisfies the following requirements:

- Monotonicity: $X \geq Y \Rightarrow \rho(X) \geq \rho(Y)$
- Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$
- Positive homogeneity: $\rho(aX) = a\rho(X)$ if $a > 0$ is a constant
- Translation invariance: $\rho(X+a) = \rho(X) + a$ for constant a.

TailVaR is coherent, while VaR is not.

A spectral risk measure is defined as a weighted average of VaR at different levels. The weight increases with the severity of the loss:

$$\rho(F) = \int_0^1 \varphi(s) F^{-1}(s) ds = \int_0^1 \varphi(s) \mathrm{VaR}_s(F) ds$$

where $\varphi(s) \geq 0$, $\int_0^1 \varphi(s) ds = 1$ and $\varphi(\cdot)$ is non-decreasing (i.e., larger losses matter more).

One can prove that spectral risk measures are coherent.

TailVaR is a spectral risk measure with $\varphi(s) = \frac{1}{1-\alpha} I(s \geq \alpha)$ (a step function).

The SCR and capital charges are based on 99.5% VaR. Why?

This is the story as it has been told to me:

- CEIOPS originally proposed 99% VaR as the basis of SCR.
- The academic community cried out that VaR was bad and that TailVaR should be used. They are right, of course.
- CEIOPS put out a proposal to use 99% TailVaR.
- The insurance industry cried out that TailVaR was far too complicated to quantify. They are right, of course.
- As a compromise, CEIOPS suggested to use 99.5% VaR as a proxy for 99% TailVaR.

Reinsurance

Proportional reinsurance

- Quota share
- Surplus (variable quota share)

Non-proportional reinsurance

- Excess of loss
- Stop loss

Reinsurance contracts can be compared with Over-The-Counter (OTC) derivatives, the "underlying" being the ceding company's claims.

Reinsurance
Form of recovery

- Proportional reinsurance means that the reinsurer pays a percentage share of every claim from the insured risks. The percentage can be the same for all risks, or vary with the size of the risk.
- Non-proportional reinsurance means that the reinsurer pays that part of every claim, or claim event, that exceeds an agreed threshold. Normally the liability of the reinsurer is limited.

Reinsurance
Properties of reinsurance forms

- Quota share reinsurance is risk sharing, for example if the reinsured does not have enough own capital.
- Surplus reinsurance is sharing only large risks.
- Excess of loss reinsurance covers the part of each claim (or event) that exceeds a threshold. For instance €10" xs €5".
- Stop loss works like XL, but on the year's total claim cost. Often expressed in loss ratios. For instance 100% xs 100%.
- Most insurers combine proportional insurance, with non-proportional reinsurance of the remaining retention and catastrophe risk.

Reinsurance premium =

- The reinsurer's share of the gross premium of the reinsured risks,
- less reinsurance commission,
- plus/minus adjustment of commission, e.g. by a sliding scale,
- plus/minus profit commission.

Reinsurance premium =

- actuarially calculated "minimum & deposit premium",
- plus an exposure-based adjustment at the end of the year,
- plus "reinstatement premium" after reinsurance claims.

- Short-tailed lines are often accounted on a "clean-cut" basis, i.e. with commutation of the reinsurer's liability a few years after the underwriting period.
- Long-tailed lines are always accounted on a "run-off" basis, i.e. with no commutation of the reinsurer's liability before (almost) all claims are settled.
- Valuing the reinsurer's remaining liability is far from easy.

Reinsurance
Commutation considerations

- Commutation means releasing the reinsurer for his remaining liability in return for a cash payment – essentially, commutation amounts to exchanging a paper asset with cash asset.
- For the primary insurer it may be better to receive the cash, rather than to wait for claims to be settled; especially if the solidity of the reinsurer is not assured. The risk is in undervaluing the reinsurer's share of the outstanding liabilities – i.e. receiving less than the true value of the reinsurance asset.
- For the reinsurer it may be an advantage to be finished with the contract. The risk is in overvaluing the reinsurer's share of the outstanding liabilities – i.e. paying too much for the release from liability.

Reinsurance

What complicates the valuation of the reinsurer's remaining liability?

- Have the reported but not settled claims been assessed correctly?
- If reported claims change, will it be on reinsured risks or other risks?
- Will unreported claims be reported on reinsured risks or other risks?
- The tradition of analysing outstanding claim estimates by accident year.
- Complex and opaque reinsurance conditions and wordings.
- Exclusions, limitations, loss corridors, sunset clauses etc.

ART is a collective term for many risk transfer mechanisms:

- Captive insurance companies that one can establish or rent.
- Multi-line, Multi-year cover for several lines or years in one.
- Multi-trigger cover that only is activated by a combination of events.
- Spread-loss cover that is used to spread claim cost over time.
- Risk securitisation.

The term ART has gone out of fashion in recent years.

Finite reinsurance is when the insured finances its own risk over time.
Two main forms:

- Pre-financing: the insured makes a "deposit" with the reinsurer.
- Post-financing: the insured "borrows" from the reinsurer.

There are rules as to when a contract ceases being reinsurance and must be treated as a financial instrument (deposit or debt).

Assets
- Financial assets
- Reinsurers' share of premium provision
- Reinsurers' share of outstanding claim provision
- Receivables from reinsurers
- Other assets

Liabilities
- Gross premium provision
- Gross outstanding claim provision
- Payables to reinsurers
- Other liabilities

Equity = Assets - Liabilities

When accounting for reinsurance transactions it is important to remember that:

- Reinsurance premium is an outlay (liability)
- Reinsurance recoveries are an uncertain income (asset)

Therefore, prudent accounting indicates that

- Reinsurance premium should be recognized immediately,
- Reinsurance recoveries should be recognized only when they are reasonably certain to materialize.

Reinsurance

Risks to be aware of in connection to reinsurance - general

- Counterparty risk
- Exclusions
- Sunset clauses
- Loss corridor
- EML understated
- Adjustment premium
- Limited reinstatements
- Not fully placed contracts
- Misunderstood contracts

Risk: exclusions

- Most reinsurance contracts have exclusions.
- Asbestos liability, acts of terrorism, acts of war etc.
- Beware if the exclusions are for risks that the contract should cover
- Example: pharmaceutical liability reinsurance that excludes "all known side effects".
- Such contracts can be used as window dressing.

Risk: sunset clause

- A sunset clause means that a claim must be reported to the reinsurer within a limited number of years (five or seven) to be covered under the contract.
- The problem is not the IBNR claims. Most claims are reported quickly.
- The problem is the RBNS: open and reopened claims.
- A short sunset clause can severely reduce the value of the reinsurance cover.

Risk: Loss corridor

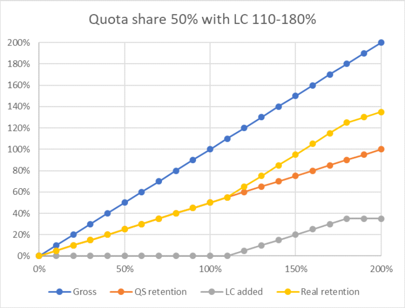
- Normally associated with "quota share" contracts
- Typical loss corridor clause:

"In the case that the combined ratio (the reinsurer's combined ratio is the sum of commission and reinsured losses, divided by the reinsurance premium), is 110% or more, the reinsured shall be liable for all further losses falling under this treaty until a combined ratio of 180% is reached."

- A loss corridor amounts to the reinsured providing stop loss cover for the reinsurer's share.
- Could also be seen as an ex-post premium adjustment or commission adjustment.

Reinsurance

Risks to be aware of in connection to reinsurance - loss corridor



Risk: EML understated

- A company must understand the EML (estimated maximum loss).
- Losses that exceed the limit that most contracts have (also quota share contracts), are normally not covered by reinsurance.
- Unlimited Motor Third Party insurance.
- A company had a stop loss 150% xs 100% for Workers' Compensation insurance and deemed itself perfectly safe - until it turned out that the loss ratio was 1400%. Fortunately they had rich sponsors.

Risk: Adjustment premium

- Premium that will be demanded in the future for the reinsurance cover to be valid.
- Typical case: Adverse Development Cover (ADC) for a run-off, that says that a premium will be due if the development is worse than a certain amount.
- For the accountant, the adjustment premium may be "out of sight and out of mind".
- If a company wants to take credit for the cover, it must take a charge for the adjustment premium.
- Another form of adjustment premium is sliding scale commission.

Retrospective cover

Retrospective cover
Motivation

Sometimes companies are looking to protect a run-off portfolio:

- If they have stopped writing insurance in a LoB
- If they want to reduce capital requirements from a LoB

Retrospective cover can happen in several forms:

- Loss Portfolio Transfer (LPT)
- Adverse development Cover (ADC)
- Part VII transfer in the UK
- Novation
- Sale

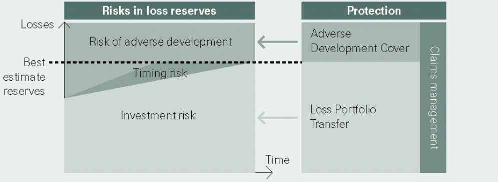
Retrospective cover
General form

Retrospective reinsurance

The economic and regulatory cost of loss reserves can be transferred via a retrospective reinsurance contract. There are two components which are often combined to achieve economic finality on loss reserves:

- **Loss Portfolio Transfer (LPT):** The reinsurer pays off the claims in exchange for the assets covering the loss reserves. This removes the timing and investment risk for the cedent.
- **Adverse Development Cover (ADC):** The reinsurer pays claims in excess of an agreed reserves level in exchange for a risk premium, effectively removing the risk of insufficient reserves for the cedent.

Risks covered by retrospective reinsurance



Source: Swiss Re

Retrospective cover

General form

LPT The reinsurer pays off the claims in exchange for the assets covering the loss reserves. This removes the timing and investment risk for the cedent. The reinsurer does not pay for adverse development of loss reserves.

ADC The reinsurer pays claims in excess of an agreed reserves level in exchange for a risk premium, effectively removing the risk of insufficient reserves for the cedent.

- Insurance risk: A pure LPT that *only covers the value of loss reserves*, reduces net provisions but does not reduce insurance risk. An ADC works like a stop loss contract.
- In order to have real risk transfer and reduce insurance risk, one must combine LPT with ADC.
- Market risk: Reduced by LPT because taken by reinsurer. Not reduced significantly by ADC.
- Counterparty risk: increased, potentially by a significant amount in an LPT. Depends on the rating.
- Operational risk: generally neutral.

Retrospective cover
Example of an Adverse Development Cover

PERIOD "This Contract is in respect of all Loss Settlements paid by the Reassured after 31 December, 2016 but only with respect to policies incepting during the period 1st January, 2011 to 31 December, 2016, both dates inclusive."

LIMITS "EUR 18,000,000 in the aggregate of net Loss Settlements applicable hereunder, excess of EUR 35,000,000 in the aggregate of net Loss Settlements applicable hereunder."

PREMIUM "Premium in full of: EUR 6,000,000.payable at 31 December 2016."

Retrospective cover
Example of an Adverse Development Cover

LOSS ADDITIONAL PREMIUM "In the event that the total paid aggregate of Loss Settlements applicable hereunder is EUR 38,000,000 or greater, the Reassured shall pay to the Reinsurer an Additional Premium of (a) EUR 2,400,000 which will be due and payable immediately and (b) 80% of all further paid losses hereon, subject to a maximum Loss Additional Premium of EUR 5,000,000 in total."

PROFIT COMMISSION "At Commutation the Reassured shall be entitled to a Profit Commission calculated as follows: 25% of a) Income: The paid Premium of EUR 6,000,000, Less b) Outgo: Reinsurer's cost of EUR 2,000,000, plus incurred claims."

Retrospective cover
Example of an Adverse Development Cover

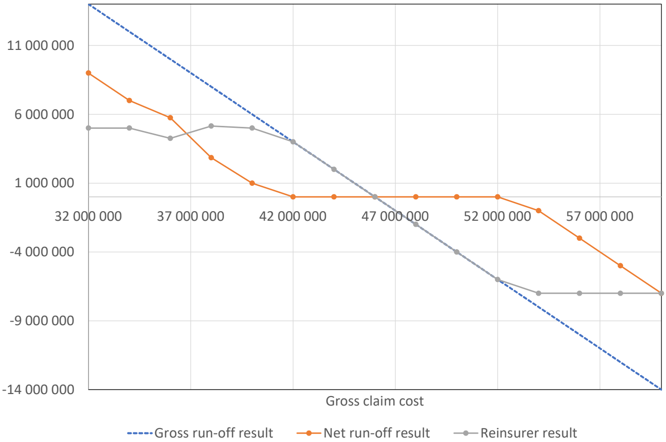
Priority	35 000 000
Limit	18 000 000
M&D premium	6 000 000
LAP threshold	38 000 000
LAP immediate	2 400 000
LAP as a percentage	80 %
LAP maximum	5 000 000
Profit commission	25 %
Margin	2 000 000

Gross claim estimate	46 000 000	Case+RBNS+IBNR+UEP+PDR

Scenarios	Gross claims	Gross run-off result	Reinsured claims	M&D premium	LAP immediate	LAP as a percentage	Profit commission	Reinsurer result	Net run-off result
	32 000 000	14 000 000	0	6 000 000	0	0	1 000 000	5 000 000	9 000 000
	34 000 000	12 000 000	0	6 000 000	0	0	1 000 000	5 000 000	7 000 000
	36 000 000	10 000 000	1 000 000	6 000 000	0	0	750 000	4 250 000	5 750 000
	38 000 000	8 000 000	3 000 000	6 000 000	2 400 000	0	250 000	5 150 000	2 850 000
	40 000 000	6 000 000	5 000 000	6 000 000	2 400 000	1 600 000	0	5 000 000	1 000 000
	42 000 000	4 000 000	7 000 000	6 000 000	2 400 000	2 600 000	0	4 000 000	0
	44 000 000	2 000 000	9 000 000	6 000 000	2 400 000	2 600 000	0	2 000 000	0
Gross claim estimate	46 000 000	0	11 000 000	6 000 000	2 400 000	2 600 000	0	0	0
	48 000 000	-2 000 000	13 000 000	6 000 000	2 400 000	2 600 000	0	-2 000 000	0
	50 000 000	-4 000 000	15 000 000	6 000 000	2 400 000	2 600 000	0	-4 000 000	0
	52 000 000	-6 000 000	17 000 000	6 000 000	2 400 000	2 600 000	0	-6 000 000	0
	54 000 000	-8 000 000	18 000 000	6 000 000	2 400 000	2 600 000	0	-7 000 000	-1 000 000
	56 000 000	-10 000 000	18 000 000	6 000 000	2 400 000	2 600 000	0	-7 000 000	-3 000 000
	58 000 000	-12 000 000	18 000 000	6 000 000	2 400 000	2 600 000	0	-7 000 000	-5 000 000
	60 000 000	-14 000 000	18 000 000	6 000 000	2 400 000	2 600 000	0	-7 000 000	-7 000 000

Retrospective cover

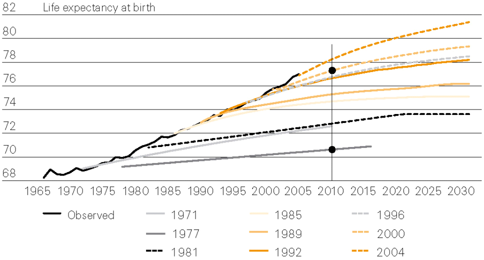
Example of an Adverse Development Cover



Longevity

Longevity
Actual vs, projected life expectancy

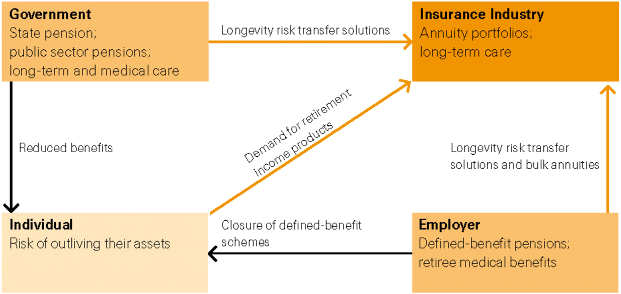
Figure 4: Actual and projected life expectancy at birth, UK males



The chart demonstrates how experts have historically underestimated life expectancy. For example, the life expectancy of a UK male born in 2010 was estimated to be 71 years in 1977. By 2000, this estimate was revised to more than 77 years.

Source: Chris Shaw, "Fifty Years of United Kingdom National Population Projections: How Accurate Have They Been?", Population Trends, 128, Office for National Statistics, 2007

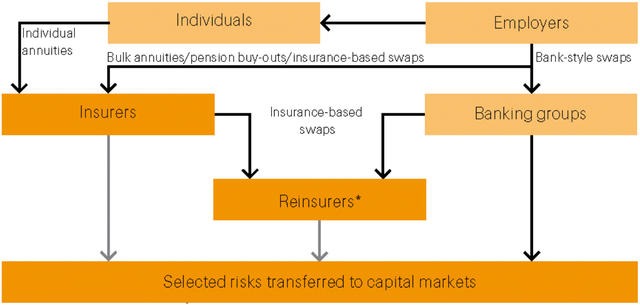
Figure 6: The holders of longevity risk



Source: Swiss Re

Longevity
Spreading longevity risk

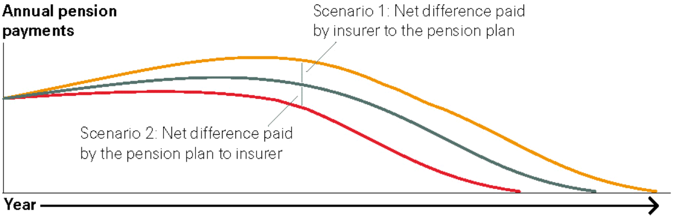
Figure 7: Longevity de-risking strategies



Long-term holders of longevity risk
Insurer/reinsurer retains liabilities to annuitants and pension fund members in a capital market solution
*Reinsurers deal directly with employers by acting as a conventional insurer

Source: Swiss Re

Figure 8: An insurance-based longevity risk transfer solution



– Annual premiums ie "fixed leg"

– Scenario 1*: Illustrative increased annual pension payments if life expectancy improves

– Scenario 2*: Illustrative lower annual pension payments if life expectancy worsens

* ie "floating leg"

Simulated figures – not based on any actual pension plan.

Source: Swiss Re

Alternative risk capital

Alternative risk capital

A short history - 1

- Hurricane Andrew (August 1992).
- RMS projected losses for:
  - Replay of San Francisco 1906 earthquake
  - + major Los Angeles earthquake
  - + major Texas floods
- Realisation that traditional reinsurance capital may be inadequate.
- Major reinsurers found it difficult to find retrocession capacity.
- First private catastrophe bond transaction in 1994 (Hannover Re and Citibank).

- Northridge Earthquake (17 January 1994) led to design of CatEPut contingent capital product.
- Proposal to California Earthquake Authority for Earthquake Risk Bonds by Goldman Sachs et al. created template for public catastrophe bonds.

Alternative risk capital
Types of transactions

- Catastrophe futures
- Catastrophe bonds
- Industry loss warranties
- Collateralised reinsurance
- Sidecars
- Contingent capital

Alternative risk capital
Catastrophe futures

- The "Underlying" is the total claims in one period in a specified area, as reported by all insurers (with an IBNR allowance).
- Subject claims can be limited to catastrophes, like storm, earthquake etc.
- The value of one contract is indexed to a multiple (e.g. \$100,000) of the loss ratio. Loss data is collected from all insurers in the area.
- Alternatively, the value of a contract can be indexed to a multiple of some technical measure, e.g. air pressure in a storm centre.
- The futures price reflects the market's expectation of the index.

Alternative risk capital
Hedging strategy with catastrophe futures

- Hedging strategy for an insurer: buy futures contracts.
- If claims in the quarter are high, the value of the futures contract increases and helps finance the losses of the insurer.
- If claims in the quarter are low, the value of the futures contract decreases. The insurer has a loss on the futures contract but (hopefully) has had a profit in the insurance operation.
- Basis risk! There is no guarantee that the average claim experience will be the same as the claim experience of a specific insurer

Payoff per 100% loss ratio	1 000 000
Cost of one contract	750 000

<u>Insurer</u>

Premium income	100 000 000
Number of contracts bought	100
Cost of contracts bought	75 000 000
Money left	25 000 000

<u>Net claim cost</u>

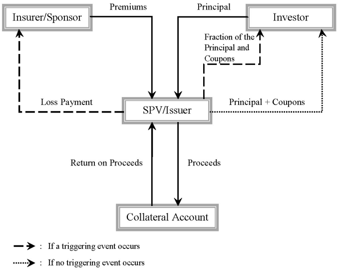
		Contract loss ratio		
		60 %	75 %	90 %
	60 %	0	-15 000 000	-30 000 000
<u>Insurer loss ratio</u>	75 %	15 000 000	0	-15 000 000
	90 %	30 000 000	15 000 000	0

Alternative risk capital
Catastrophe bonds

- A financial instrument that transfers natural catastrophe risk to investors in the capital markets
- Repayment of Investors' funds subject to the outcome of an "insurable" event
- Buyer of cover seeks insurance accounting
- Investors seek to purchase a security
- Simplified structure, using Special Purpose Vehicle / Reinsurer:
- In the event of a pre-defined catastrophe bondholders are forced to forfeit some or all of their interest payments and/or principal repayment

Alternative risk capital
Catastrophe bonds

Figure 1: The structure of a CAT bond transaction.



Alternative risk capital
Properties of catastrophe bonds

- The purpose of the SPV is to segregate the funds received for the catastrophe bond from the insurer's assets.
- The SPV invests in secure bonds with little credit risk.
- Advantage to the insurer: Catastrophe bonds can tap into a huge pool of risk capital (the financial markets), much larger than the risk capital of all reinsurers combined.
- Advantage to the investor: Catastrophe bonds provide an attractive yield because of higher risk, but are uncorrelated with asset market risk. Thus in a portfolio, they have the potential to increase expected return without increasing overall volatility.

- Indemnity: Loss payment based on actual claims incurred by the reinsured
- Parametric: Loss payment based on technical parameters applied to a proxy for the severity of the catastrophe
- Modelled-loss index: Loss payment based on a model applied to a portfolio of underlying exposures
- Industry-loss index: Loss payment based on overall insured losses arising from the loss event

Alternative risk capital
Catastrophe bonds - indemnity triggers

- Eliminates basis risk
- Requires disclosure of underlying risk
- Price dependent on quality of underlying exposure data
- Delay in settlement of losses may delay recovery

Alternative risk capital
Catastrophe bonds - parametric triggers

- Insured retains basis risk
- No disclosure of underlying risk
- Has been used for earthquake and hurricane risk
- Rapid claims payment following trigger event

Alternative risk capital
Catastrophe bonds - modeled loss portfolio triggers

- Basis risk can be minimized
- Limited disclosure of underlying risk
- Flexibility to change model portfolio over time
- Rapid claims payment following trigger event

- Insured retains basis risk
- No disclosure of underlying risk
- Full development of losses may delay recovery

Alternative risk capital

Catastrophe bonds - comparison of triggers

	Indemnity	Parametric	Industry	Model
Basis risk	N	Y	Y	Some
Disclosure	Y	N	N	N
Adjustment	Y	N	Y	Y
Moral hazard	Maybe	N	N	N
Payout	Slow	Rapid	Slow	Rapid
Tradable	Difficult	Y	Y	Y

Alternative risk capital

Catastrophe bonds - pricing

- Probability of loss and expected loss of bond
- Credit rating
- Diversification value
- Data and modeling uncertainty
- Complexity of transaction
- Current bond market pricing
- Reinsurance pricing
- Focus on coupon divided by expected loss

Alternative risk capital

Catastrophe bonds - pros and cons

- Pros
  - No counterparty risk (fully collateralised cover)
  - Risk is transferred to capital markets, less concentration
  - Longer cover periods (up to 5 years or more)
  - Transparent pricing
  - Rapid and undisputed claim settlement
  - Can cover risk that otherwise is uninsurable
- Cons
  - More expensive than traditional reinsurance
  - Basis risk unless indemnity trigger is used
  - More complicated process to set up the transaction

Alternative risk capital
Industry Loss Warranties

- Dual trigger structure – industry and company-specific losses
- Mostly cover property catastrophe risk
- Use of recognised indices:
  - NatCatService (Munich Re, Worldwide)
  - Property Claims Service (US, Canada)
  - Perils (Europe)
  - Sigma (Swiss Re, Worldwide)
- Payoff is usually for the whole limit if the trigger is reached

Alternative risk capital
Collateralised reinsurance

- Traditional reinsurance protection where the investor provides a collateral
- For the insured:
  - + Same as traditional reinsurance
  - + Investor provides collateral $\Longrightarrow$ reduced credit risk
  - - Trust account or letter of credit required
- For the investor:
  - + Another way to load up insurance risk
  - + Wider spectrum of risks available
  - - Not tradable in the secondary market
  - - Higher cost of capital
  - - Not suitable for long-tail lines
- Collateralised reinsurance grown significantly the last 10 years.

Alternative risk capital

Sidecars

- Temporary equity or debt financing to the re/insurance market.
- Provides support where capacity or capital is under stress.
- Typical sidecar profile:
  - Assumes risk through quota share reinsurance.
  - Loss probability $> 5\%$.
  - Expected loss $> 25\%$.
  - Organised as special purpose vehicles
  - Often capitalized by hedge funds.
  - Risk period up to 24 months.

Alternative risk capital
Contingent capital

Basic structure:

- The insurer has the option to call capital under defined events.
- If the event occurs, the investors provide capital to the insurer.
- The capital is in the form of non-voting preference shares.
- The insurer pays annual dividends on the preference shares.
- The insurer converts the preference shares into common equity after 3-4 years or repurchases at original issue price.
- Contingent capital is an umbrella when it's raining.

Alternative risk capital

Main differences between alternative risk capital and reinsurance

- Securities contract – not re/insurance policy,
- No need for insurable interest,
- Not restricted to traditionally ‘insurable’ perils,
- Not restricted to indemnity settlement,
- Basis risk if non-indemnity settlement,
- Fully collateralised,
- Multi-year (usually),
- Single limit – no reinstatement,
- Transparency of pricing,
- Secondary trading possible.

Mean-Variance Analysis

Purpose of this section

The purpose of this section is to show how one can hedge a stochastic liability using correlated assets. We will also see how one can simulate the stochastic development of correlated assets and liabilities.

I. Optimum asset allocation for one period

- Minimum risk portfolio
- Optimal portfolio of risky assets
- Optimal portfolio with a risk-free asset

II. Optimum asset allocation to fund a stochastic liability

- Minimum risk portfolio
- Optimal portfolio of risky assets
- Optimal portfolio with a risk-free asset

Discussion of the mean-variance framework

MVA - optimum asset allocation only
Optimum asset allocation for one period

- Assume that you can invest an amount of $W(0)$ in n different assets numbered $i = 1, \cdots, n$.
- The market value of the assets now is $A_1(0), \cdots, A_n(0)$.
- You will revalue your assets at time $t > 0$.
- The market value of the assets then will be $A_1(t), \cdots, A_n(t)$. This value must include the value of coupons or dividends paid during the period $(0, t]$.
- To all but the insiders, the outcome of $A_1(t), \cdots, A_n(t)$ looks random.

MVA - optimum asset allocation only
Optimum asset allocation for one period

- Define the return of asset no. i by

$$R_i(t) = \left(A_i(t) - A_i(0)\right) / A_i(0).$$

- If you invest $w_i W(0)$ in asset no. i at time 0, your wealth at time t will be

$$W(t) = W(0) \sum_{i=1}^{n} w_i \left(1 + R_i(t)\right) = W(0) \left(1 + \mathbf{w}'\mathbf{R}(t)\right).$$

- Your aggregate return over the period will be $R_{\mathbf{w}}(t) = \mathbf{w}'\mathbf{R}(t)$.
- The asset allocation problem is to find a vector $\mathbf{w} = (w_1, \cdots, w_n)$ with $w_1 + \cdots + w_n = 1$, that provides an adequate expected return with as little as possible uncertainty.

MVA - optimum asset allocation only
Optimum asset allocation for one period

- You decide to measure the uncertainty of $\mathbf{w}'\mathbf{R}(t)$ by its variance.
- In mathematical terms, the asset allocation problem then becomes

$$\text{"minimise } \mathrm{Var}(\mathbf{w}'\mathbf{R}(t)), \text{ subject to certain constraints".}$$

Conceivable investment constraints could be

- No constraints at all, i.e. outright minimisation of the variance;
- An adequate expected return r, i.e., "$E(\mathbf{w}'\mathbf{R}(t)) = r$";
- Exposure limits, e.g., "$w_{\min} \leq w_i \leq w_{\max}$."

We drop the argument t from now on, as we are considering only one period.

- Assume that the return vector $\mathbf{R} = (R_1, \cdots, R_n)'$ is random with a known mean vector

$$\boldsymbol{\mu} = \mathrm{E}(\mathbf{R}) = (\mu_1, \cdots, \mu_n)'$$

- and a known covariance matrix

$$\boldsymbol{\Sigma} = \mathrm{Cov}(\mathbf{R}) = \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \ddots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \cdots & \sigma_n^2 \end{pmatrix}$$

- We assume that there are only risky assets: there exists neither an asset i nor a portfolio (linear combination) of assets, with a secure return. In that case the covariance matrix $\boldsymbol{\Sigma}$ is invertible and positive definite.

- The expected return of a portfolio characterised by the allocation vector $\mathbf{w}$ is

$$\mathsf{E}\left(\mathbf{w}'\mathbf{R}\right) = \mathbf{w}'\mu,$$

- and the variance is

$$\mathsf{Var}\left(\mathbf{w}'\mathbf{R}\right) = \mathbf{w}'\Sigma\mathbf{w}.$$

MVA - optimum asset allocation only
Asset behaviour - assumptions

Asset	Asset class	E(return)	SD(return)
1	Equity	8,84 %	21,39 %
2	Bonds	6,47 %	2,97 %
3	Money market	5,57 %	0,62 %

Correlations	1	2	3
1	100,0 %	16,0 %	15,8 %
2	16,0 %	100,0 %	67,7 %
3	15,8 %	67,7 %	100,0 %

Asset	Asset class	E(return)	Var(return)
1	Equity	8,8 %	4,6E-02
2	Bonds	6,5 %	8,8E-04
3	Money market	5,6 %	3,8E-05

Sigma	1	2	3
1	4,6E-02	1,0E-03	2,1E-04
2	1,0E-03	8,8E-04	1,2E-04
3	2,1E-04	1,2E-04	3,8E-05

A very variance-averse investor could pose the asset allocation problem

$$\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}, \text{ subject to (only) } \mathbf{w}'\mathbf{1} = 1$$

Using Lagrange minimisation, the optimal portfolio can be shown to be

$$\mathbf{w}_{\text{min}} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1} \boldsymbol{\Sigma}^{-1}\mathbf{1}$$

Its expected return is

$$\boldsymbol{\mu}'\mathbf{w}_{\text{min}} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1} \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}$$

and the variance of its return is

$$\mathbf{w}_{\text{min}}'\boldsymbol{\Sigma}\mathbf{w}_{\text{min}} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}$$

MVA - optimum asset allocation only
Minimum variance portfolio - outline of proof

The Lagrangian can be written as

$$L(\mathbf{w},\lambda) = \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \lambda\left(\mathbf{w}'\mathbf{1} - 1\right)$$

To determine $\mathbf{w}_{\text{min}}$ we solve the linear equations

$$\frac{\partial}{\partial \mathbf{w}} L(\mathbf{w},\lambda) = \mathbf{w}'\boldsymbol{\Sigma} - \lambda\mathbf{1}' = \mathbf{0}',$$

$$\frac{\partial}{\partial \lambda} L(\mathbf{w},\lambda) = \mathbf{w}'\mathbf{1} - 1 = 0.$$

MVA - optimum asset allocation only

Minimum variance portfolio - example

Minimum variance portfolio	
Expected return	5,45 %
Variance of return	2,7E-05
Standard deviation of return	0,5 %

Asset class	w_min	Vector	Sigma^-1 *Vector	Sigma * w
Equity	-0,2 %	1	-6,5E+01	2,7E-05
Bonds	-12,6 %	1	-4,7E+03	2,7E-05
Money market	112,8 %	1	4,2E+04	2,7E-05
	100,0 %		3,7E+04	

A more demanding investor could pose the asset allocation problem

$$\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}, \text{ subject to } \mathbf{w}'\boldsymbol{\mu} = r \text{ and (of course) } \mathbf{w}'\mathbf{1} = 1$$

where r is the expected return that an allocation must provide in order to be a candidate.

The optimal portfolio $\mathbf{w}_r$ is now a linear combination of the minimum variance portfolio $\mathbf{w}_{\text{min}}$ and one "reference" risky portfolio $\mathbf{w}_{\text{ref}}$:

$$\mathbf{w}_r = (1 - \upsilon)\,\mathbf{w}_{\text{min}} + \upsilon\mathbf{w}_{\text{ref}}$$

The reference risky portfolio is

$$\mathbf{w}_{\mathsf{ref}} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

or, in special cases, $\mathbf{w}_{\mathsf{ref}} = \mathbf{w}_{\mathsf{min}} + \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$.
The weight of the risky portfolio in the optimal portfolio is

$$v = v(r) = \frac{r - \boldsymbol{\mu}'\mathbf{w}_{\mathsf{min}}}{\boldsymbol{\mu}'\mathbf{w}_{\mathsf{ref}} - \boldsymbol{\mu}'\mathbf{w}_{\mathsf{min}}}$$

Thus the more return you ask for, the more risk you must accept.

The Lagrangian can be written as

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \lambda_1 \left(\mathbf{w}'\mathbf{1} - 1\right) - \lambda_2 \left(\mathbf{w}'\boldsymbol{\mu} - r\right)$$

To determine $\mathbf{w}_r$ we solve the linear equations

$$\frac{\partial}{\partial \mathbf{w}} L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\boldsymbol{\Sigma} - \lambda_1 \mathbf{1}' - \lambda_2 \boldsymbol{\mu}' = \mathbf{0}', \tag{1}$$

$$\frac{\partial}{\partial \lambda_1} L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mathbf{1} - 1 = 0, \tag{2}$$

$$\frac{\partial}{\partial \lambda_2} L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\boldsymbol{\mu} - r = 0. \tag{3}$$

MVA - optimum asset allocation only

Optimal portfolio of risky assets - outline of proof

Using 1 we find that the solution $\mathbf{w}$ is of the general form

$$\mathbf{w} = \lambda_1 \boldsymbol{\Sigma}^{-1}\mathbf{1} + \lambda_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \lambda_1 \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)\mathbf{w}_{\min} + \lambda_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}.$$

Inserting this into 2 we find that

$$\lambda_1 \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right) = 1 - \lambda_2 \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right).$$

If $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \neq 0$, then we can write

$$\mathbf{w} = (1 - v)\,\mathbf{w}_{\min} + v\mathbf{w}_{\text{ref}},$$

with a reference portfolio that is

$$\mathbf{w}_{\text{ref}} = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}.$$

If $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = 0$, we can still write

$$\mathbf{w} = (1 - v)\,\mathbf{w}_{\text{min}} + v\mathbf{w}_{\text{ref}},$$

but the reference portfolio becomes

$$\mathbf{w}_{\text{ref}} = \mathbf{w}_{\text{min}} + \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

(proof as an exercise).
We finally solve 3 to determine the weight to the reference portfolio

$$v = v(r) = \frac{r - \boldsymbol{\mu}'\mathbf{w}_{\text{min}}}{\boldsymbol{\mu}'\mathbf{w}_{\text{ref}} - \boldsymbol{\mu}'\mathbf{w}_{\text{min}}}$$

Reference risky portfolio		Asset class	w_ref	Vector	Sigma^-1 *Vector	Sigma * w
		Equity	-0,2 %	8,8 %	-3,0E+00	4,4E-05
		Bonds	-12,0 %	6,5 %	-2,4E+02	3,2E-05
		Money market	112,1 %	5,6 %	2,3E+03	2,8E-05
Expected return	5,46 %					
Variance of return	2,7E-05					
Standard deviation of return	0,5 %					
			100,0 %		2,0E+03	

MVA - optimum asset allocation only

Optimal portfolio of risky assets - risky portfolio with 6% required return

Optimal portfolio of risky assets	
Required expected return	**6,0 %**
Weighting to w_min (1-nu)	-8267,3 %
Weighting to w_ref (nu)	8367,3 %
Expected return	6,00 %
Variance of return	2,5E-04
Standard deviation of return	1,6 %

Asset class	w_r	Split	w_min	w_ref	Sigma * w
Equity	2,0 %		-0,2 %	-0,2 %	1,4E-03
Bonds	40,6 %		-12,6 %	-12,0 %	4,5E-04
Money market	57,4 %		112,8 %	112,1 %	7,6E-05
	100,0 %	Weight:	-8267,3 %	8367,3 %	

Any required (expected) return r can be generated by the formula

$$\mathbf{w}_r = (1 - \upsilon(r))\,\mathbf{w}_{\min} + \upsilon(r)\mathbf{w}_{\text{ref}},$$

and the variance of the return will be the least possible:

$$\begin{aligned}\sigma^2(r) &= Var\left(\mathbf{w}_r'\mathbf{R}\right) = \\ & (1-\upsilon(r))^2\,\mathbf{w}_{\min}'\boldsymbol{\Sigma}\mathbf{w}_{\min} + \upsilon^2(r)\mathbf{w}_{\text{ref}}'\boldsymbol{\Sigma}\mathbf{w}_{\text{ref}} + \\ & 2\,(1-\upsilon(r))\,\upsilon(r)\mathbf{w}_{\min}'\boldsymbol{\Sigma}\mathbf{w}_{\text{ref}}.\end{aligned}$$

The efficient frontier of risky assets is the curve

$$\left\{(\sigma(r), r) : r \geq \boldsymbol{\mu}'\mathbf{w}_{\min}\right\}$$

in the two-dimensional plane. Proof as an exercise.

Assume now that in addition to the n risky assets, you can invest in a risk-free asset $(i = 0)$ that provides a secure return of $R_0 = \mu_0$.
Your asset allocation problem now becomes

$$\min_{w_0, \mathbf{w}} \mathbf{w}'\mathbf{\Sigma}\mathbf{w}, \text{ subject to } w_0\mu_0 + \mathbf{w}'\boldsymbol{\mu} = r \text{ and } w_0 + \mathbf{w}'\mathbf{1} = 1,$$

where r is the expected return that an allocation must provide in order to be a candidate, and w_0 is the proportion of your wealth to be invested risk-free.

In this case, the optimal portfolio is a combination of

- a risk-free investment of w_0, and
- investment of the remaining $1 - w_0$ in a tangency portfolio $\mathbf{w}_{\tan}$.

The relevant parameters are

$$\mathbf{w}_{\tan} = \mathbf{w}_{\tan}(\mu_0) = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mu_0\mathbf{1})\right)^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mu_0\mathbf{1})$$

$$1 - w_0 = 1 - w_0(r) = \frac{r - \mu_0}{\boldsymbol{\mu}'\mathbf{w}_{\tan} - \mu_0}$$

The Lagrangian can be written as

$$\begin{aligned} L(w_0, \mathbf{w}, \lambda_1, \lambda_2) &= \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \lambda_1 \left(w_0 + \mathbf{w}'\mathbf{1} - 1\right) - \\ & \lambda_2 \left(w_0 \mu_0 + \mathbf{w}'\boldsymbol{\mu} - r\right) \end{aligned}$$

To determine the optimal $(w_0, \mathbf{w})$ we solve the linear equations

$$(\partial/\partial\mathbf{w})\, L(w_0, \mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\,\boldsymbol{\Sigma} - \lambda_1 \mathbf{1}' - \lambda_2 \boldsymbol{\mu}' = \mathbf{0}' \tag{4}$$

$$(\partial/\partial w_0)\, L(w_0, \mathbf{w}, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2 \mu_0 = 0 \tag{5}$$

$$(\partial/\partial \lambda_1)\, L(w_0, \mathbf{w}, \lambda_1, \lambda_2) = w_0 + \mathbf{w}'\mathbf{1} - 1 = 0 \tag{6}$$

$$(\partial/\partial \lambda_2)\, L(w_0, \mathbf{w}, \lambda_1, \lambda_2) = w_0 \mu_0 + \mathbf{w}'\boldsymbol{\mu} - r = 0 \tag{7}$$

Using 4 we find that the solution **w** is of the general form

$$\mathbf{w} = \lambda_1 \mathbf{\Sigma}^{-1} \mathbf{1} + \lambda_2 \mathbf{\Sigma}^{-1} \boldsymbol{\mu}.$$

.

Using 5 we find that $\lambda_1 = -\lambda_2 \mu_0$, so that

$$\mathbf{w} = \lambda_2 \mathbf{\Sigma}^{-1} \left(\boldsymbol{\mu} - \mu_0 \mathbf{1} \right)$$

.

Using 6 we find

$$\lambda_2 = \left(1 - w_0\right) / \left(\mathbf{1}' \mathbf{\Sigma}^{-1} \left(\boldsymbol{\mu} - \mu_0 \mathbf{1} \right) \right)$$

so that $\mathbf{w} = \left(1 - w_0\right) \mathbf{w}_{\text{tan}}$.

Finally, 7 gives us

$$1 - w_0 = 1 - w_0(r) = \frac{r - \mu_0}{\boldsymbol{\mu}' \mathbf{w}_{\text{tan}} - \mu_0}$$

Note that the tangency portfolio is a function of the available risk-free return. The variance of the overall return is

$$\sigma^2(r) = \text{Var}\left(w_0\mu_0 + (1 - w_0)\,\mathbf{w}'_{\text{tan}}\mathbf{R}\right) = (1 - w_0)^2\,\mathbf{w}'_{\text{tan}}\boldsymbol{\Sigma}\mathbf{w}_{\text{tan}}$$

(See exercises for a more detailed proof).

MVA - optimum asset allocation only

Optimal portfolio with a risk-free asset - tangency portfolio

Tangency portfolio	
Expected return	5,53 %
Variance of return	3,2E-05
Standard deviation of return	0,56 %

Asset class	w_tan	Vector	Sigma^-1 *Vector	Sigma * w
Equity	0,1 %	3,8 %	2,2E-01	2,3E-04
Bonds	-4,9 %	1,5 %	-8,2E+00	8,8E-05
Money market	104,8 %	0,6 %	1,7E+02	3,4E-05
	100,0 %		1,7E+02	

MVA - optimum asset allocation only
Optimal portfolio with a risk-free asset - example

Optimal portfolio with risk-free asset		Asset class	(1-w_0) x w_tan		Sigma * w
Required expected return	6,0 %	Equity	0,3 %		4,4E-04
		Bonds	-9,3 %		1,7E-04
Weighting of risk-free asset (w_0)	-89,0 %	Money market	198,0 %		6,4E-05
Weighting of tangency portfolio (1-w_0)	189,0 %				
Expected return	6,00 %				
Variance of return	1,1E-04				
Standard deviation of return	1,06 %				
			189,0 %		

MVA - optimum asset allocation with hedge
Optimal asset allocation to fund a stochastic liability

- Let us briefly reintroduce the time parameter $t > 0$.
- Assume that the assets must support a stochastic liability.
- The value of the liability at time 0 is $L(0)$, and at time t it will be $L(t)$.
- The surplus at time 0 is $S(0) = W(0) - L(0)$. At time t it will be $S(t) = W(t) - L(t)$.
- The funding ratio at time 0 is $F(0) = W(0)/L(0)$.

MVA - optimum asset allocation with hedge
Optimal asset allocation to fund a stochastic liability

Sharpe & Tint (1990) define the surplus return as

$$\frac{S(t)-S(0)}{W(0)} = \left(\frac{W(t)-W(0)}{W(0)}\right) - \frac{L(0)}{W(0)}\left(\frac{L(t)-L(0)}{L(0)}\right) = R_W(t) - \frac{R_L(t)}{F(0)}$$

Here we have defined

$$R_W(t) = \left(W(t) - W(0)\right)/W(0)$$

as asset return and

$$R_L(t) = \left(L(t) - L(0)\right)/L(0)$$

as liability growth.

MVA - optimum asset allocation with hedge
Optimal asset allocation to fund a stochastic liability

Let us assume, as before, that there are n investible assets with a random return characterised by its mean vector and covariance matrix:

$$\mathbf{R}(t) \sim [\boldsymbol{\mu}(t), \boldsymbol{\Sigma}(t)]$$

We now make the additional assumption that liability growth is random, and correlated with asset returns:

$$\begin{aligned} \mathrm{E}\left(R_L(t)\right) &= \mu_L(t) \\ \mathrm{Var}\left(R_L(t)\right) &= \sigma_L^2(t) \\ \mathrm{Cov}\left(R_i(t), R_L(t)\right) &= \gamma_{i,L}(t) = \rho_{i,L}(t)\sigma_i(t)\sigma_L(t) \end{aligned}$$

Denote the vector of covariances by

$$\gamma(t) = \left(\gamma_{1,L}(t), \cdots, \gamma_{n,L}(t)\right)'$$

and assume that you know (have estimated) $\mu_L(t)$, $\sigma_L^2(t)$ and $\gamma(t)$.

MVA - optimum asset allocation with hedge

Optimal asset allocation to fund a stochastic liability

With an arbitrary asset allocation vector **w**, the random surplus return is

$$R_S(t) = \mathbf{w}'\mathbf{R}(t) - \frac{R_L(t)}{F(0)} = \mathbf{w}'\mathbf{R} - \frac{R_L}{F} = R_S$$

It is easy to verify that

$$\begin{aligned} \mathrm{E}(R_S) &= \mathbf{w}'\boldsymbol{\mu} - \frac{\mu_L}{F} \\ \mathrm{Var}(R_S) &= \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F} \end{aligned}$$

Let us minimise the variance, subject to constraints.

If your only aim was to minimise variance, you would solve:

$$\min_{\mathbf{w}} \left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F} \right) \text{ subject to } \mathbf{w}'\mathbf{1} = 1$$

Using Lagrange minimisation, the optimal portfolio can be shown to be

$$\mathbf{w}_{\text{min}}\left(F, \gamma\right) = \left(1 - \upsilon\right)\mathbf{w}_{\text{min}} + \upsilon\mathbf{w}_{\gamma}$$

,

where $\mathbf{w}_{\text{min}}$ is the unconditional minimum variance allocation and $\mathbf{w}_{\gamma}$ is the *liability hedge portfolio*.

MVA - optimum asset allocation with hedge

Optimal asset allocation to fund a stochastic liability, minimum variance portfolio

The liability hedge portfolio is

$$\mathbf{w}_\gamma = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\gamma\right)^{-1}\boldsymbol{\Sigma}^{-1}\gamma$$

The weight of the liability hedge portfolio in the optimal portfolio is

$$v = v\left(F, \gamma\right) = \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\gamma$$

(In the case of $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\gamma = 0$, we can write $\mathbf{w}_\gamma = \mathbf{w}_{\text{min}} + \boldsymbol{\Sigma}^{-1}\gamma$ and $v = 1$).

Outline of proof

The Lagrangian can be written as

$$L(\mathbf{w}, \lambda) = \frac{1}{2}\left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F}\right) - \lambda\left(\mathbf{w}'\mathbf{1} - 1\right).$$

To determine **w** we solve the linear equations

$$\begin{aligned} \frac{\partial}{\partial \mathbf{w}} L(\mathbf{w}, \lambda) &= \mathbf{w}'\boldsymbol{\Sigma} - \frac{1}{F}\gamma' - \lambda\mathbf{1}' = \mathbf{0}', \\ \frac{\partial}{\partial \lambda} L(\mathbf{w}, \lambda) &= \mathbf{w}'\mathbf{1} - 1 = 0. \end{aligned}$$

The first equation gives

$$\mathbf{w} = \lambda \boldsymbol{\Sigma}^{-1}\mathbf{1} + \frac{1}{F}\boldsymbol{\Sigma}^{-1}\gamma',$$

and the second equation gives

$$\lambda = \left(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^{-1}\left(1 - \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\gamma\right).$$

Proceed from there.

Some points to note

- If asset-liability covariance is small relative to the asset variability, then v will be small and the optimal portfolio will be close to $\mathbf{w}_{\min}$.
- In particular, if there is no asset-liability covariance then the optimal portfolio is just $\mathbf{w}_{\min}$.
- The weight given to the liability hedge portfolio is a decreasing function of the initial funding ratio.

MVA - optimum asset allocation with hedge

Optimal asset allocation to fund a stochastic liability, liability hedge portfolio

Liability hedge portfolio	
Expected liability growth	**6,0 %**
Variance of liability growth	**1,6E-03**
Standard deviation of liability growth	4,0 %
Asset return	
Expected return	2,61 %
Variance of asset return	6,5E-03
Standard deviation of return	8,05 %
Surplus return	
Expected return	-1,4 %
Variance of return	9,4E-03
Standard deviation of return	9,7 %

Asset class	w_gamma	gamma	Sigma^-1 *gamma	Assumed correlation	Sigma * w	Var Asset
Equity	-2,1 %	8,6E-04	5,5E-03	**10 %**	-3,3E-03	4,6E-02
Bonds	-319,8 %	5,9E-04	8,2E-01	**50 %**	-2,3E-03	8,8E-04
Money market	422,0 %	6,2E-05	-1,1E+00	**25 %**	-2,4E-04	3,8E-05
	100,0 %		-2,6E-01			

MVA - optimum asset allocation with hedge

Optimal asset allocation to fund a stochastic liability, minimum variance portfolio with a liability hedge

Minimum variance portfolio with liability hedge	
Initial funding ratio	**150,0 %**
Weighting of min.var. Portfolio (1-nu)	117 %
Weighting of liability hedge (nu)	-17 %
Asset return	
Expected return	5,93 %
Variance of return	2,1E-04
Standard deviation of return	1,5 %
Surplus return	
Expected return	1,9 %
Variance of return	5,6E-04
Standard deviation of return	2,4 %

Asset class	w_min (F, gamma)	Split	w_min	w_gamma	Sigma * w
Equity	0,2 %		-0,2 %	-2,1 %	6,0E-04
Bonds	39,8 %		-12,6 %	-319,8 %	4,3E-04
Money market	60,0 %		112,8 %	422,0 %	7,3E-05
	100,0 %	Weight:	117,1 %	-17,1 %	

If you are more interested in beating than in meeting the expected return of the liability hedge portfolio, you would solve:

$$\min_{\mathbf{w}} \left(\mathbf{w}'\mathbf{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F} \right) \text{ subject to } \mathbf{w}'\boldsymbol{\mu} = r \text{ and } \mathbf{w}'\mathbf{1} = 1$$

where r is the expected return that an asset allocation must provide in order to be a candidate for you.
The additional constraint only makes sense if $r \geq \boldsymbol{\mu}'\mathbf{w}_{\min}(F, \gamma)$.

The optimal portfolio can be written in the form

$$\begin{aligned} \mathbf{w}_r(F,\gamma) &= (1-\upsilon-\omega)\,\mathbf{w}_{\text{min}} + \omega\mathbf{w}_{\text{ref}} + \upsilon\mathbf{w}_\gamma \\ &= \mathbf{w}_{\text{min}}(F,\gamma) + \omega\left(\mathbf{w}_{\text{ref}} - \mathbf{w}_{\text{min}}\right) \end{aligned}$$

- $\mathbf{w}_{\text{min}}$ denotes the unconditional minimum variance allocation,
- $\mathbf{w}_{\text{ref}}$ the risky reference portfolio when there is no risk-free asset,
- $\mathbf{w}_\gamma$ the liability hedge portfolio, and
- $\mathbf{w}_{\text{min}}(F,\gamma)$ the minimum surplus variance allocation.
- The weighting parameters are

$$\upsilon = \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\gamma \text{ and } \omega = \frac{r - \boldsymbol{\mu}'\mathbf{w}_{\text{min}}(F,\gamma)}{\boldsymbol{\mu}'\mathbf{w}_{\text{ref}} - \boldsymbol{\mu}'\mathbf{w}_{\text{min}}}.$$

MVA - optimum asset allocation with hedge

Optimal asset allocation to fund a stochastic liability, optimal portfolio of risky assets

Outline of proof

The Lagrangian can be written as

$$\begin{aligned} L(\mathbf{w}, \lambda_1, \lambda_2) &= \frac{1}{2}\left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F}\right) \\ &\quad -\lambda_1\left(\mathbf{w}'\mathbf{1} - 1\right) - \lambda_2\left(\mathbf{w}'\boldsymbol{\mu} - r\right). \end{aligned}$$

To determine **w** we solve the linear equations

$$\begin{aligned} (\partial/\partial\mathbf{w})\, L(\mathbf{w}, \lambda_1, \lambda_2) &= \mathbf{w}'\,\boldsymbol{\Sigma} - \frac{1}{F}\gamma' - \lambda_1\mathbf{1}' - \lambda_2\boldsymbol{\mu}' = \mathbf{0}' \\ (\partial/\partial\lambda_1)\, L(\mathbf{w}, \lambda_1, \lambda_2) &= \mathbf{w}'\mathbf{1} - 1 = 0 \\ (\partial/\partial\lambda_2)\, L(\mathbf{w}, \lambda_1, \lambda_2) &= \mathbf{w}'\boldsymbol{\mu} - r = 0 \end{aligned}$$

and so on ...

MVA - optimum asset allocation with hedge

Optimal asset allocation to fund a stochastic liability, optimal portfolio of risky assets, example

Optimal risky portfolio with liability hedge	
Required expected return	**6,0 %**
Weighting of liability hedge (nu)	-17 %
Weighting to reference portfolio (omega)	1006,1 %
Weighting of min.var. Portfolio (1-nu-omega)	-889 %
Asset return	
Expected return	6,00 %
Variance of return	2,7E-04
Standard deviation of return	1,6 %
Surplus return	
Expected return	2,0 %
Variance of return	5,6E-04
Standard deviation of return	2,4 %

Asset class	w_r (F, gamma)	Split	w_min	w_ref	w_gamma	Sigma * w
Equity	0,4 %		-0,2 %	-0,2 %	-2,1 %	7,7E-04
Bonds	46,2 %		-12,6 %	-12,0 %	-319,8 %	4,8E-04
Money market	53,4 %		112,8 %	112,1 %	422,0 %	7,9E-05
	100,0 %	Weight:	-889,0 %	1006,1 %	-17,1 %	

Let us finally develop the case where the investor has access to a risk-free asset with secure return μ_0. The problem is then to

$$\min_{w_0,\mathbf{w}} \left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F} \right) \text{ subject to } \begin{aligned} w_0\mu_0 + \mathbf{w}'\boldsymbol{\mu} &= r \\ \text{and } w_0 + \mathbf{w}'\mathbf{1} &= 1 \end{aligned}$$

The parameter w_0 denotes the proportion of assets invested risk-free.

The optimal portfolio consists of

- a risk-free investment of w_0,
- investment of $1 - w_0 - \upsilon$ in the tangency portfolio $\mathbf{w}_{\tan}$,
- investment of υ in the liability hedge portfolio $\mathbf{w}_\gamma$.

The weightings are

$$\upsilon = \frac{1}{F}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\gamma \text{ and } 1 - w_0 = \frac{r - \upsilon\boldsymbol{\mu}'\left(\mathbf{w}_\gamma - \mathbf{w}_{\tan}\right) - \mu_0}{\boldsymbol{\mu}'\mathbf{w}_{\tan} - \mu_0}.$$

The Lagrangian arguments are as before, therefore the proof is omitted.

MVA - optimum asset allocation with hedge

Optimal asset allocation to fund a stochastic liability, optimal portfolio with a risk-free asset, example

Optimal portfolio with risk-free asset and liability hedge		Asset class	(1-w_0) x w_risky	Split	w_tan	w_gamma	Sigma * w
Required expected return	6,0 %	Equity	0,5 %		0,1 %	-2,1 %	8,3E-04
Weighting of liability hedge (nu)	-17 %	Bonds	49,1 %		-4,9 %	-319,8 %	4,9E-04
Weighting of tangency portfolio (1-w_0-nu)	111,9 %	Money market	45,2 %		104,8 %	422,0 %	7,9E-05
Weighting of risk-free asset (w_0)	5,1 %						
Asset return							
Expected return	6,0 %						
Variance of return	2,8E-04						
Standard deviation of return	1,68 %						
			94,9 %	Weight:	111,9 %	-17,1 %	
Surplus return							
Expected return	2,0 %						
Variance of return	5,6E-04						
Standard deviation of return	2,4 %						

- Using the mean-variance framework provides insight into the effect of correlation between asset classes, and between asset classes and liabilities.
- Estimating the covariances is easy in principle. Having to rely on estimated covariances in allocating your portfolio may be more problematic. It requires a great deal of confidence in the estimates.
- The mean-variance framework may return allocations that are not feasible, because they are outside the investment mandate. There exists software to do the minimisation with arbitrary constraints.

Mean-Variance Analysis

Discussion of mean-variance framework

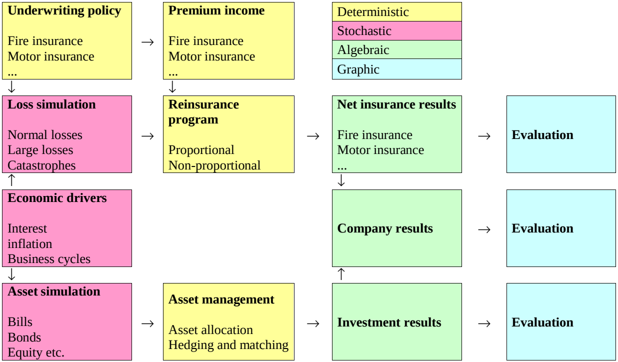
- Even if your asset allocation is subject to constraints, you should calculate the cost of those constraints in terms of lost return or increased volatility, relative to what an unconstrained allocation could achieve.
- Given the framework (means and covariances), the method returns an optimal asset allocation. "Optimal" does not necessarily mean "very good" - it just means the best that could be achieved under the given assumptions.
- Asset returns are not normally distributed! However, relying on means and covariances does not imply that you subscribe to the normality assumption. It only means that you select two readily available distribution characteristics and ignore the rest.

DFA: Modelling and simulation of an insurance company's results for:

- Asset allocation,
- Capital allocation,
- Market strategies,
- Business mix,
- Pricing decisions,
- Product design,
- Reinsurance design.

For an example, see Kaufmann et al. (2001).

Dynamic Financial Analysis (DFA)
Overall structure



Modelling

- Processes (insurance, investments, ...),
- Dependence (correlations, copulas, ...),
- Strategies (products, assets, reinsurance, ...),
- Economic drivers (inflation, cycle, ...).

Analysis

- Choose strategies,
- Stochastic simulation,
- Compute overall results,
- Evaluate overall results,
- Go back & improve.

Cost of capital pricing of a stochastic liability
The cost of capital method

The cost-of-capital method for calculating a risk margin comprises the following essential steps:

- Calculate the supporting capital required (SCR) to support the liability on the balance date $(t = 0)$.
- Calculate the supporting capital required to support the remaining outstanding liability on every balance date in the future (SCR_t at time $t = 1, 2, \cdots$).
- The supporting capital required could come from a regulatory requirement, a risk measure (VaR or TailVaR), or be a percentage of the liability.
- For each year $t = 0, 1, 2, \cdots$, charge CoC% of the supporting capital required. It is the reward that shareholders require for putting their capital at risk.
- The risk margin is the discounted value of the charges, $\text{RM} = \text{CoC\%} \sum_{t=0}^{\infty} (1+i)^{-t} \text{SCR}_t$.

You can use as an approximation

$$\text{SCR}_t^* = \text{SCR}_0 \times \frac{\text{OS}_t}{\text{OS}_0}$$

where OS_t is the expected outstanding liability at balance date t.Then

$$\begin{aligned} \text{RM}^* &= \text{CoC\%} \times \text{SCR}_0 \times \sum_{t=0}^{\infty} (1+i)^{-t} \frac{\text{OS}_t}{\text{OS}_0} \\ &= \text{CoC\%} \times \frac{\text{SCR}_0}{\text{OS}_0} \times \sum_{t=0}^{\infty} (1+i)^{-t} \text{OS}_t. \end{aligned}$$

See "Example_CoC_Pricing.xlsx".

See a separate set of slides

Equitable Life is the world's oldest mutual life insurer. The company's problems were revealed when it emerged it had insufficient funds to honour Guaranteed Annuity Rate policies (GARs), which gave investors a guaranteed minimum income when they retired. It suffered a near collapse in 2000 after it lost a House of Lords court case brought by GAR policyholders, leaving the mutual society with a liability of £1.5 billion and forcing its closure to new business.

ALM gone wrong
HIH/FAI

On 1 March 2001, the Australian Prudential Regulation Authority (APRA) served notices calling on HIH to show cause why an inspector should not be appointed under s52 of the Insurance Act 1973. On 15 March, the date of expiry of the "show cause" letter, HIH applied to the courts to be placed into provisional liquidation.

The ensuing investigation revealed that HIH had an estimated deficit of assets over liabilities on the order of A$5 billion in a total balance sheet of about A$7 billion. The Australian government appointed a Royal Commission to look into the events surrounding HIH's collapse.

ALM gone wrong
HIH/FAI

Particular issues that Justice Owen assessed as contributing to HIH's failure included the following:

- Poor corporate governance
- Underprovisioning
- Abuses of reinsurance
- Lack of integrity of information provided to the HIH board, to the auditors, and to the regulator
- Conglomerate complexity
- Inadequate and inappropriate asset valuations.

With respect to the regulator, APRA, Justice Owen made the following comments:

- “APRA did not cause or contribute to the collapse of HIH; nor could it have taken steps to prevent the failure of the company. A regulator cannot be expected to provide a guarantee that no company under it supervision will ever fail.”
- “However, the manner in which APRA exercised its powers and discharged its responsibilities under the Insurance Act fell short of that which the community was entitled to expect from the prudential regulator of the insurance industry.”

ALM gone wrong

HIH/FAI - abuse of reinsurance

- FAI bought a reinsurance contract and booked a large expected recovery for a modest premium.
- To repay the "loan", the company bought six other contracts from the same reinsurer, to come into force in the year after.
- The company promised the reinsurer that it would not make any claim against the six "repayment" contracts.
- No provision was made for the reinsurance premium of the six "repayment" contracts, because they were seen as "not in force" or "unearned".
- FAI was sold to HIH on the basis of its official accounts. HIH went bankrupt, and both buyer and seller were sent to prison for fraud.
- The reinsurance counterparties were two of the world's most respected companies.

ALM gone wrong
Silver Pensjonsforsikring

- Many private sector employees in Norway used to have generous DB pension plans. On changing jobs, the accued pension benefits were converted to "free policies".
- Most insurers never credit more to free policies than the guaranteed rate.
- Silver came into the market and promised better benefits with more aggressive investment.
- In the low interest rate environment and with Solvency II, Silver ran into trouble.
- Silver was put under public administration in February 2017 and stopped making payments.
- A resolution plan was made in 2017-18.

ALM gone wrong
Silver Pensjonsforsikring

The resolution plan involved the sale of Silver to another insurer, with:

- Free policies with guaranteed rate to be converted to unit linked policies with no guarantee.
- Age pension entitlements to be reduced 0,25% to 1,25%.
- Disability and dependents pension entitlements to remain unchanged.
- Existing unit linked pensions to continue unchanged.
- Missed pension payments in 2017 to be reimbursed

There were threats of legal action against the Finance Department by the owners and some policyholders.

ALM gone wrong
Operational risk

- The company had a reinsurance contract.
- On the premium side it looked like a proportional contract. On the claim side it was a non-proportional contract.
- The programmers of the insurance system thought that in a proportional contract, every claim payment must generate a proportional receivable from the reinsurer.
- The insurance system sent "receivables" to the accounting system. Not individually auditable, but as a monthly sum.
- This went on for several years. The annual amounts were not big enough to alert the auditor, but in the end there was about €9" of "receivables" in the balance sheet.
- The reinsurance manager saw that something was wrong but was not listened to.
- The company was in sales negotiations when the mistake was discovered. The price dropped considerably.

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