

# Probability Theory and Stochastic Processes

## LIST 1<sup>1</sup>

### Measure and probability

- (1) Decide if  $\mathcal{F}$  is  $\sigma$ -algebra of  $\Omega$ , where:
- (a)  $\mathcal{F} = \mathcal{P}(\Omega)$ ,  $\Omega = \mathbb{R}^n$ .
  - (b)  $\mathcal{F} = \{\emptyset, \{1, 2\}, \{3, 4, 5, 6\}, \Omega\}$ ,  $\Omega = \{1, 2, 3, 4, 5, 6\}$ .
  - (c)  $\mathcal{F} = \{\emptyset, \{0\}, \mathbb{R}^-, \mathbb{R}_0^-, \mathbb{R}^+, \mathbb{R}_0^+, \mathbb{R} \setminus \{0\}, \mathbb{R}\}$ ,  $\Omega = \mathbb{R}$ .
- (2) Let  $(\Omega, \mathcal{F})$  be a measurable space and  $A_1, A_2, \dots \in \mathcal{F}$ . Prove:
- (a)  $\bigcap_{i=1}^{+\infty} A_i \in \mathcal{F}$
  - (b)  $A_1 \setminus A_2 \in \mathcal{F}$
- (3) Let  $\Omega$  be a finite set with  $\#\Omega = n$ . Compute  $\#\mathcal{P}(\Omega)$ . *Hint:* Find a bijection between  $\mathcal{P}(\Omega)$  and the space  $\{v \in \mathbb{R}^n : v_i \in \{0, 1\}\}$ .
- (4) Let  $\Omega = [-1, 1] \subset \mathbb{R}$ . Determine if the following collection of sets is a  $\sigma$ -algebra:
- $$\mathcal{F} = \{A \in \mathcal{B}(\Omega) : x \in A \Rightarrow -x \in A\}.$$
- (5) Determine if the intersection and the union of  $\sigma$ -algebras are still  $\sigma$ -algebras.
- (6) Show that
- (a) if  $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{P}$ , then  $\sigma(\mathcal{I}_1) \subset \sigma(\mathcal{I}_2)$ .
  - (b)  $\sigma(\sigma(\mathcal{I})) = \sigma(\mathcal{I})$  for any  $\mathcal{I} \subset \mathcal{P}$ .
  - (c)  $\mathcal{I} = \{[a, +\infty[ : a \in \mathbb{R}\}$  generates the Borel  $\sigma$ -algebra of  $\mathbb{R}$ .
- (7) Let  $(\Omega, \mathcal{F})$  be a measurable space and  $A, B \in \mathcal{F} \setminus \{\emptyset, \Omega\}$  such that  $A \neq B$ . Compute  $\#\sigma(\{A, B\})$  if
- (a)  $A \cap B \neq \emptyset$  and  $A \cup B \neq \Omega$ .
  - (b)  $A \cap B = \emptyset$  and  $A \cup B \neq \Omega$ .
  - (c)  $A \cap B \neq \emptyset$  and  $A \cup B = \Omega$ .
  - (d)  $A \cap B = \emptyset$  and  $A \cup B = \Omega$ .

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<sup>1</sup>Send comments and/or corrections to [jldias@iseg.ulisboa.pt](mailto:jldias@iseg.ulisboa.pt). Harder questions are marked with \*. Collaboration among colleagues is encouraged, but each student should write his/her own solutions, understand them and give credit to the collaborators.

(8) Let  $\mu: \mathcal{P}(\mathbb{R}) \rightarrow [0, +\infty]$  be given by  $\mu(\emptyset) = 0$ ,  $\mu(\mathbb{R}) = 2$  and  $\mu(X) = 1$  if  $X \in \mathcal{P}(\mathbb{R}) \setminus \{\emptyset, \mathbb{R}\}$ . Determine if  $\mu$  is  $\sigma$ -subadditive and  $\sigma$ -additive.

(9) Let  $\mu_n$  be a measure and  $a_n \geq 0$  for all  $n \in \mathbb{N}$ . Prove that

$$\mu = \sum_{n=1}^{+\infty} a_n \mu_n,$$

is also a measure. Furthermore, show that if  $\mu_n$  is a probability measure for all  $n$  and  $\sum_n a_n = 1$ , then  $\mu$  is also a probability measure.

(10) Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $A_1, A_2, \dots \in \mathcal{F}$ . Prove that:

- (a) In the definition of measure, the condition  $\mu(\emptyset) = 0$  can be replaced by the existence of a set  $E \in \mathcal{F}$  with finite measure,  $\mu(E) < +\infty$ .
- (b) If  $A_i \subset A_{i+1}$ , then  $\mu(\bigcup_i A_i) = \lim_{i \rightarrow +\infty} \mu(A_i)$ .
- (c) If  $A_{i+1} \subset A_i$  and  $\mu(A_1) < +\infty$ , then  $\mu(\bigcap_i A_i) = \lim_{i \rightarrow +\infty} \mu(A_i)$ .

(11) Let  $(\Omega, \mathcal{F}, P)$  probability space,  $A_1, A_2, \dots \in \mathcal{F}$  and  $B$  is the set of points in  $\Omega$  that belong to an infinite number of  $A_n$ 's:

$$B = \bigcap_{n=1}^{+\infty} \bigcup_{k=n}^{+\infty} A_k.$$

Show that:

(a) (First Borel-Cantelli lemma) If

$$\sum_{n=1}^{+\infty} P(A_n) < +\infty,$$

then  $P(B) = 0$ .

(b) \*(Second Borel-Cantelli lemma) If

$$\sum_{n=1}^{+\infty} P(A_n) = +\infty$$

and

$$P\left(\bigcap_{i=1}^n A_i\right) = \prod_{i=1}^n P(A_i),$$

for every  $n \in \mathbb{N}$  (i.e. the events are mutually independent), then  $P(B) = 1$ .