Models in Finance - Class 4

Master in Actuarial Science

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ISEG

One-dimensional Itô's formula or Itô's lemma

- Itô's formula or Itô's lemma is a stochastic version of the chain rule.
- Suppose we have a function of a function $f(b_t)$ and we consider f is a C^2 class function. We want to find $\frac{d}{dt}f(b_t)$. Then by Taylor's theorem (2nd order expansion):

$$\delta f(b_t) = f'(b_t)\,\delta b_t + \frac{1}{2}f''(b_t)\,(\delta b_t)^2 + \cdots$$

Dividing by δt and letting $\delta t \to 0$, we obtain the classical chain rule:

$$\frac{d}{dt}f(b_t) = f'(b_t)\,\frac{db_t}{dt} + \frac{1}{2}f''(b_t)\,\frac{db_t}{dt}\lim_{\delta t \to 0}(\delta b_t) = f'(b_t)\,\frac{db_t}{dt}$$

or

$$df(b_t) = f'(b_t)\,db_t.$$

One-dimensional Itô's formula or Itô's lemma

- What if we replace b_t (deterministic) by the sBm B_t?Then, the 2nd order term $\frac{1}{2}f''\left(B_t\right)\left(\delta B_t\right)^2$ cannot be ignored because $\left(\delta B_t\right)^2 \approx \left(dB_t\right)^2 \approx dt$ is not of the order $\left(dt\right)^2$, that is (Itô formula):

$$df\left(B_t\right) = f'\left(B_t\right)dB_t + \frac{1}{2}f''\left(B_t\right)dt. \tag{1}$$

- Example: Compute the stochastic differential of B_t^2 and represent this process using a stochastic integral.
- We have $B_t^2 = f\left(B_t\right)$ with $f\left(x\right) = x^2$. Therefore, by (1)

$$\begin{aligned} d\left(B_t^2\right) &= 2B_t dB_t + \frac{1}{2}2\left(dB_t\right)^2 \\ &= 2B_t dB_t + dt. \end{aligned}$$

(Taylor expansion of B_t^2 as a function of B_t and assuming that $\left(dB_t\right)^2 = dt$). Note that in integral form the result is equivalent to $\int_0^t B_s dB_s = \frac{1}{2}\left(B_t^2 - t\right)$.

One-dimensional Itô's formula or Itô's lemma

- If f is a C^2 function then

$$\begin{aligned} f(B_t) &= \text{stochastic integral}+\text{process with differentiable paths} \\ &= \text{Itô process} \end{aligned}$$

- We can replace condition 2) $E\left[\int_0^T u_t^2 dt\right] < \infty$ in the definition of $L^2_{a,T}$ by the (weaker condition):
  2') $P\left[\int_0^T u_t^2 dt < \infty\right] = 1.$
- Let $L_{a,T}$ be the space of processes that satisfy condition 1 of the definition of $L^2_{a,T}$ and condition 2'). The Itô integral can be defined for $u \in L_{a,T}$ but, in this case, the stochastic integral may fail to have zero expected value and the Itô isometry may fail to be verified.

- Define $L^1_{a,T}$ as the space of processes v such that:
  1. v is an adapted and measurable process.
  2. $P\left[\int_0^T |v_t|\, dt < \infty\right] = 1.$
- An adapted and continuous process $X = \{X_t, 0 \leq t \leq T\}$ is called an Itô process if it satisfies the decomposition:

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds, \tag{2}$$

where $u \in L_{a,T}$ and $v \in L^1_{a,T}$.

One-dimensional Itô's formula or Itô's lemma

Theorem

(One-dimensional Itô's formula or Itô's lemma): Let $X = \{X_t, 0 \leq t \leq T\}$ *a Itô process of type (2). Let* $f(t,x)$ *be a* $C^{1,2}$ *function. Then* $Y_t = f(t, X_t)$ *is an Itô process and we have:*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, u_s dB_s + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, v_s ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\, u_s^2 ds.$$

- In the differential form, the Itô formula is:

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)\, dt + \frac{\partial f}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\,(dX_t)^2 .$$

where $(dX_t)^2$ can be computed using (2) and the table of products

$\times$	dB_t	dt
dB_t	dt	0
dt	0	0

- Itô's formula for $f(t,x)$ and $X_t = B_t$, or $Y_t = f(t, B_t)$.

$$
\begin{aligned}
f(t, B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s \\
+ \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\, ds.
\end{aligned}
$$

$$
\begin{aligned}
df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t)\, dt + \frac{\partial f}{\partial x}(t, B_t)\, dB_t \\
+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t)\, dt.
\end{aligned}
$$

- Itô's formula for $f(x)$ and $X_t = B_t$, or $Y_t = f(B_t)$.

$$df(B_t) = \frac{\partial f}{\partial x}(B_t)\, dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(B_t)\, dt.$$

Multidimensional Itô's formula or Itô's lemma

- Assume that $B_t := \left(B_t^1, B_t^2, \ldots, B_t^m\right)$ is an m-dimensional standard Brownian motion, that is, components B_t^k, $k = 1, \ldots, m$ are one-dimensional independent sBm.
- Consider a Itô process of dimension n, defined by

$$
\begin{aligned}
X_t^1 &= X_0^1 + \int_0^t u_s^{11} dB_s^1 + \cdots + \int_0^t u_s^{1m} dB_s^m + \int_0^t v_s^1 ds, \\
X_t^2 &= X_0^2 + \int_0^t u_s^{21} dB_s^1 + \cdots + \int_0^t u_s^{2m} dB_s^m + \int_0^t v_s^2 ds, \\
&\vdots \\
X_t^n &= X_0^n + \int_0^t u_s^{n1} dB_s^1 + \cdots + \int_0^t u_s^{nm} dB_s^m + \int_0^t v_s^n ds.
\end{aligned}
$$

Multidimensional Itô's formula

- In differential form:

$$dX_t^i = \sum_{j=1}^{m} u_t^{ij} dB_t^j + v_t^i dt,$$

with $i = 1, 2, \ldots, n$.

- Or, in compact form:

$$dX_t = u_t dB_t + v_t dt,$$

where v_t is n-dimensional, u_t is a $n \times m$ matrix of processes.

- We assume that the components of u belong to $L_{a,T}$ and the components of v belong to $L_{a,T}^1$.

Multidimensional Itô's formula

- If $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^p$ is a $C^{1,2}$ function, then $Y_t = f(t, X_t)$ is a Itô process and we have the Itô formula or Itô lemma:

$$dY_t^k = \frac{\partial f_k}{\partial t}(t, X_t)\, dt + \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i}(t, X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f_k}{\partial x_i \partial x_j}(t, X_t)\, dX_t^i dX_t^j.$$

Multidimensional Itô's formula

- The product of the differentials $dX_t^i dX_t^j$ is computed following the product rules:

$$dB_t^i dB_t^j = \begin{cases} 0 & \text{se } i \neq j \\ dt & \text{se } i = j \end{cases},$$
$$dB_t^i dt = 0,$$
$$(dt)^2 = 0.$$

Multidimensional Itô's formula

- If B_t is a n-dimensional sBm and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function with $Y_t = f(B_t)$ then:

$$f(B_t) = f(B_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(B_t)\, dB_s^i + \frac{1}{2} \int_0^t \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(B_t) \right) ds$$

Integration by parts formula

- Integration by parts formula: If X_t^1 and X_t^2 are Itô processes and $Y_t = X_t^1 X_t^2$, then by Itô's formula applied to $f(x) = f(x_1, x_2) = x_1 x_2$, we get

$$d\left(X_t^1 X_t^2\right) = X_t^2 dX_t^1 + X_t^1 dX_t^2 + dX_t^1 dX_t^2.$$

That is:

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_s^2 dX_s^1 + \int_0^t X_s^1 dX_s^2 + \int_0^t dX_s^1 dX_s^2.$$

Example

- Consider the process

$$Y_t = \left(B_t^1\right)^2 + \left(B_t^2\right)^2 + \cdots + \left(B_t^n\right)^2.$$

Represent this process in terms of Itô stochastic integrals with respect to n-dimensional sBm.

- By n-dimens. Itô formula applied to $f(x) = f(x_1, x_2, \ldots, x_n) = x_1^2 + \cdots + x_n^2$, we obtain

$$\begin{aligned} dY_t &= 2B_t^1 dB_t^1 + \cdots + 2B_t^n dB_t^n \\ &\quad + ndt. \end{aligned}$$

That is:

$$Y_t = 2\int_0^t B_s^1 dB_s^1 + \cdots + 2\int_0^t B_s^n dB_s^n + nt.$$

Exercise

- Exercise: Let $B_t := \left(B_t^1, B_t^2\right)$ be a two dimensional Bm Represent the process
$$Y_t = \left(B_t^1 t, \left(B_t^2\right)^2 - B_t^1 B_t^2\right)$$
as an Itô process.
- By the multidimensional Itô's formula applied to $f(t,x) = f(t, x_1, x_2) = \left(x_1 t, x_2^2 - x_1 x_2\right)$, we obtain: (Details: homework)
$$\begin{aligned} dY_t^1 &= B_t^1 dt + t dB_t^1, \\ dY_t^2 &= -B_t^2 dB_t^1 + \left(2B_t^2 - B_t^1\right) dB_t^2 + dt \end{aligned}$$
that is
$$\begin{aligned} Y_t^1 &= \int_0^t B_s^1 ds + \int_0^t s dB_s^1, \\ Y_t^2 &= -\int_0^t B_s^2 dB_s^1 + \int_0^t \left(2B_s^2 - B_s^1\right) dB_s^2 + t. \end{aligned}$$

- Exercise: Assume that a process X_t satisfies the SDE

$$dX_t = \sigma(X_t)\, dB_t + \mu(X_t)\, dt.$$

Compute the stochastic differential of the process $Y_t = X_t^3$ and represent this process as an Itô process.

Basic Ideas of the proof of Itô's formula

- The process

$$Y_t = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, u_s dB_s$$
$$+ \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, v_s ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\, u_s^2 ds.$$

  is an Itô process.
- We assume that f and its partial derivatives are bounded (the general case can be proved approximating f by bounded functions with bounded derivatives).
- The Itô stoch. integral can be approximated by a sequence of stochastic integrals of simple processes and so we can assume that u and v are simple processes.

- Consider a partition of $[0,t]$ into n equal sub-intervals:

$$f\left(t,X_t\right)=f\left(0,X_0\right)+\sum_{k=0}^{n-1}\left(f\left(t_{k+1},X_{t_{k+1}}\right)-f\left(t_k,X_{t_k}\right)\right).$$

- By Taylor formula:

$$f\left(t_{k+1},X_{t_{k+1}}\right)-f\left(t_k,X_{t_k}\right)=\frac{\partial f}{\partial t}\left(t_k,X_{t_k}\right)\Delta t+\frac{\partial f}{\partial x}\left(t_k,X_{t_k}\right)\Delta X_k$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(t_k,X_{t_k}\right)\left(\Delta X_k\right)^2+Q_k,$$

where Q_k is the remainder or error of the Taylor formula.

- We also have that

$$
\begin{aligned}
\Delta X_k = X_{t_{k+1}} - X_{t_k} &= \int_{t_k}^{t_{k+1}} v_s ds + \int_{t_k}^{t_{k+1}} u_s dB_s \\
&= v(t_k)\,\Delta t + u(t_k)\,\Delta B_k + S_k,
\end{aligned}
$$

where S_k is the remainder or error.

- Therefore:

$$
\begin{aligned}
(\Delta X_k)^2 = {} & (v(t_k))^2 (\Delta t)^2 + (u(t_k))^2 (\Delta B_k)^2 \\
& + 2v(t_k)\,u(t_k)\,\Delta t \Delta B_k + P_k,
\end{aligned}
$$

where P_k is the remainder or error term

- If we replace all this terms, we obtain:

$$f(t, X_t) - f(0, X_0) = I_1 + I_2 + I_3 + \frac{1}{2} I_4 + \frac{1}{2} K_1 + K_2 + R,$$

where

$$\begin{aligned}
I_1 &= \sum_k \frac{\partial f}{\partial t}(t_k, X_{t_k}) \Delta t, \\
I_2 &= \sum_k \frac{\partial f}{\partial t}(t_k, X_{t_k}) v(t_k) \Delta t, \\
I_3 &= \sum_k \frac{\partial f}{\partial x}(t_k, X_{t_k}) u(t_k) \Delta B_k, \\
I_4 &= \sum_k \frac{\partial^2 f}{\partial x^2}(t_k, X_{t_k}) (u(t_k))^2 (\Delta B_k)^2 .
\end{aligned}$$

- $$K_1 = \sum_k \frac{\partial^2 f}{\partial x^2}\left(t_k, X_{t_k}\right)\left(v\left(t_k\right)\right)^2\left(\Delta t\right)^2,$$
$$K_2 = \sum_k \frac{\partial^2 f}{\partial x^2}\left(t_k, X_{t_k}\right) v\left(t_k\right) u\left(t_k\right) \Delta t \Delta B_k,$$
$$R = \sum_k \left(Q_k + S_k + P_k\right).$$

- When $n \to \infty$, it is easy to show that

$$I_1 \to \int_0^t \frac{\partial f}{\partial t}(s, X_s)\, ds,$$
$$I_2 \to \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, v_s ds,$$
$$I_3 \to \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, u_s dB_s.$$

- As we have seeen before (quadratic variation of sBm), we have that

$$\sum_k (\Delta B_k)^2 \to t,$$

hence

$$I_4 \to \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\, u_s^2 ds.$$

- On the other hand, we also have

$$K_1 \to 0,$$
$$K_2 \to 0.$$

- It is also possible to show (but more technical and hard) that

$$R \to 0.$$

- Conclusion: In the limit, when $n \to \infty$, we obtain the one-dimensional Itô's formula.