INTEREST RATE & CREDIT RISK MODELS

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Oct 2019

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Objectives

Main Goal

- Be able to evaluate and manage the risks embedded in debt securities:
  - credit
  - interest rate

Interest Rate Theory

- Understand the distinction between different interest rates in the market: spot rates, forward rates, yield-to-maturity, etc.
- Identify under which conditions a stochastic interest rate model is needed and when a deterministic interest rate models is sufficient.
- Identify and apply the stochastic spot rate models, deriving the basic term structure properties from the spot rate.
- Handle forward rate models and understanding the fundamental difference the spot and the forward rate modeling approach.

OBJECTIVES

Credit Risk Theory

- Understand the approach of the structural models, in particular the Merton Model.
- Be familiar with the main reduced form models.
- Understand the limitations of structural and reduced form models when used to model portfolio credit derivatives.
- Be able to incorporate correlated defaults.

PART I

FIXED INCOME MARKETS AND INTEREST RATE RISK

1. INTRODUCTION
1.1. INTEREST RATE RISK

- Fixed-income market - the global financial market on which various medium to long term fixed interest rate instruments, such as bonds, swaps, FRAs, swaptions and caps are traded.

- Several fixed-income markets operate => many concepts of interest rates have been developed.

- Interest Rate Risk - changes in the net present value (the price) of a stream of future cash flows resulting from changes in interest rates.

- Management of interest rate risk - pricing and hedging of interest rate products and balance sheets.

INTEREST RATE RISK IN BANKS BALANCE SHEETS

- In banks, the goal is to measure the sensitivity of the balance sheet and the P&L to interest rate shifts.
- 2 types of interest rate risk:
  - Risk of Net Interest Income fluctuation
  - Risk of optionality embedded in assets and liabilities, e.g. prepayment of loans and early redemption of deposits, impacting on cash-flows.
- Target variables:
  - Net Interest Income (NII) - captures cash-flows in a given period (e.g. 1 year)
  - Economic Value (EV) : NPV of assets minus liabilities - allows to consider all balance sheet cash-flows, usually by using duration.
- Earnings-at-risk (EaR):
  - Impact on earnings (NII or EV) from several very unfavorable scenarios for interest rates.

INTEREST RATE RISK IN BANKS BALANCE SHEETS

- Sources of interest rate risk:
  - liquidity flows:
    - Direct – new loans, debt issued or deposits received
    - Indirect - prepayments, early redemptions
  - repricing of existing assets and liabilities
- Measurement:
- Interest rate or repricing gaps - corresponding to the differences between the assets and the liabilities to be repriced in different time buckets (usually up to 1 year), excluding non-interest rate bearing balance sheet items (e.g. fixed assets and capital, even though capital may be considered as a fixed rate liability). As in liquidity risk, these gaps may be static or dynamic.
- Difference between average repricing term of assets and liabilities (with fixed rates, corresponds to the differences between residual maturities).

Interest Rate Risk in Banks Balance Sheets

- EBA guidelines:

- FIs must measure their exposure to IRR in the banking book, in terms of both potential changes to EV, and changes to expected NII or earnings, considering:
  - different scenarios for potential changes in the level and shape of the yield curve, and to changes in the relationship between different market rates (i.e. basis risk);
  - assumptions made on non-interest bearing assets and liabilities of the banking book (including capital and reserves);
  - assumptions made on customer behaviour for 'non-maturity deposits' (i.e. the maturity assumed for liabilities with short contractual maturity but long behavioural maturity);
  - behavioural and automatic optionality embedded in assets or liabilities, considering:

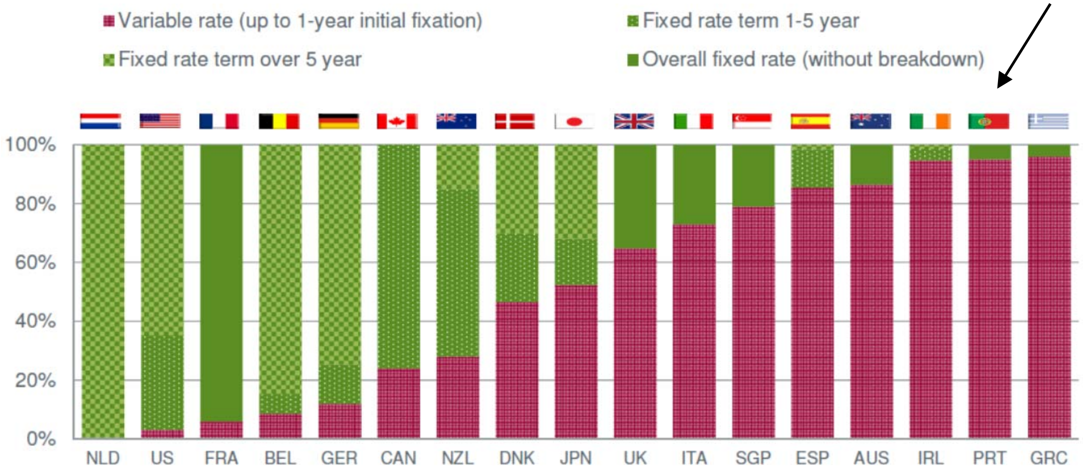
    (a) impacts on current and future loan prepayment speeds from the underlying economic environment, interest rates and competitor activity;

    (b) the speed/elasticity of adjustment of product rates to changes in market interest rates; and

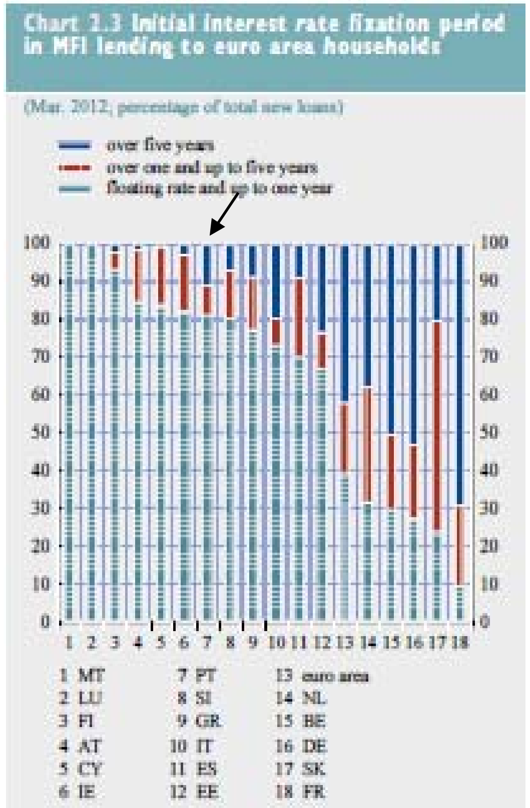
    (c) the migration of balances between product types, due to changes in their features.

INTEREST RATE RISK IN BANKS BALANCE SHEETS

- Banking assets in Portugal are mostly indexed to money market reference rates:



Source: Fitch (2016), 2016 Fitch Credit Outlook Conference, Lisbon, 28 Jan.



Source: European Central Bank (2012), "Financial Stability Review", June.

INTEREST RATE RISK IN BANKS BALANCE SHEETS

- Portuguese banks usually have positive interest rate gaps, as credit rates are mostly indexed to money market rates (e.g. Euribor), while among liabilities only bonds issued are usually indexed, given that term deposits are typically short term liabilities (though may be renewed) and their interest rates are typically fixed by the bank.
- Therefore, short term interest rate decreases are, *ceteris paribus*, unfavorable to banks (as long as repricing gaps are shorter for assets than for liabilities).
- However, we must also bear in mind that higher interest rates may reduce credit risk.
- Hedging of gaps is done through the spot market, forward/futures, options or swaps, as well as by changing the pricing structure of balance sheet.

Interest Rate Risk in Banks Balance Sheets

- Interest rate risk management in FIs creates a demand for mathematical models:

  - Calculation of durations/modified durations for bonds
  - Pricing of exotic options
  - Prepayment models for loans and deposits in banks
  - Behavior models for key balance sheet items, namely as a function of interest rates
  - Estimation of the Term Structure of Interest Rates

1.2. From Bonds to Interest Rates

- **Price of a coupon-paying bond with discrete compounding*:***

(1) $$P = \sum_{t=1}^{N} \frac{C_i}{(1+y)^t} + \frac{M}{(1+y)^N}$$

or

(1a) $$P = C \left[\frac{1 - \frac{1}{(1+y)^N}}{y} \right] + \frac{M}{(1+y)^N}$$

where:

P = price;

Ct = coupons paid on the bond;

M = repayment value of the bond;

y = yield to maturity (with discrete compounding of interest);

t = periods in which the cash-flows are generated.

- **Price of a coupon-paying bond with continuous compounding:**

(2) $$P = \sum_{t=1}^{N} (C_t + M) \cdot e^{-yt}$$

- **Price of a coupon-paying bond with continuous compounding and payments:**

(3) $$P = \int_{t=0}^{N} (C_t - M) \cdot e^{-yt} \cdot dt$$

1.2. FROM BONDS TO INTEREST RATES

- Yield curve – graphical representation of the relationship between YTM and maturities.

- Problems with using YTM to characterise the term structure of interest rates:

  - interest earned on different bonds but paid in the same periods are discounted at different rates
  - interest earned on a bond paid at different points in time is discounted at the same rate (flat yield curve).
  - does not provide any information on interest rates for maturities that do not coincide with the maturities of the existing securities.

  - yield curves designed from linear interpolations of YTM usually exhibit a very irregular shape and therefore hardly plausible.

1.2. From Bonds to Interest Rates

- Term structure of interest rates (TSIR) – relationship between interest rates and maturities.
- 3 different ways to represent TSIR:
  - Spot rates
  - Forward rates
  - Discount rates

- In order to overcome the drawbacks of using YTM, the spot curve must be constructed on the basis of zero-coupon yields or spot rates.
- However, in most countries, zero-coupon government bonds are limited or restricted to the shortest maturities (up to 1y; in Portugal, the only zero-coupon Government bonds are Treasury bills).

- The TSIR will have to be estimated.

1.2. FROM BONDS TO INTEREST RATES

- Replacing the YTM by the spot rates in (1), one gets:

(4) $$P = \sum_{t=1}^{N} \frac{C_t}{(1+s_t)^t} + \frac{M}{(1+s_N)^N}$$

- The spot rates are related to the discount factors as it results from (4):

(5) $$d_t = \frac{1}{(1+s_t)^t}$$

where:

dt = discount factor;

st = spot rate.

- Rewriting (4) by using (5), one obtains:

(6) $$P = \sum_{t=1}^{N} C_t \cdot d_t + M \cdot d_N$$

1.2. FROM BONDS TO INTEREST RATES

- With continuous interest compounding, the discount function becomes:

(7) $d_t = e^{-s_t \cdot t}$

- Spot rate can be deducted from the discount function:

- Discrete interest compounding:

(8) $s_t = \left(\dfrac{1}{d_t}\right)^{1/t} - 1$

- Continuous interest compounding:

(9) $s_t = -\dfrac{\ln(d_t)}{t}$

1.2. From Bonds to Interest Rates

- The forward rate may be calculated from the spot rates of different maturities, assuming that, due to arbitrage, the return of a spot investment for a longer maturity is equal to the return in a shorter maturity followed by an investment whose return is given by the forward rate:

(10) $${}_{n}f_{t} = \left[\frac{(1+s_{n-t})^{n-t}}{(1+s_{n})^{n}}\right]^{-\frac{1}{t}} - 1$$

- (8)+(10) =>

(10a) $${}_{n}f_{t} = \left[\frac{d_{n}}{d_{n-t}}\right]^{-\frac{1}{t}} - 1$$

- n=1 and (10a) =>

(10b) $${}_{n}f_{1} = \frac{d_{n} - d_{n-1}}{d_{n-1}} = -\frac{\Delta d_{n-1}}{d_{n-1}}$$

1.2. FROM BONDS TO INTEREST RATES

- Given that the value of an amount invested at rate *s*, continuously compounded, during the term t, is equivalent to

(10c) $$\lim_{n \to \infty}\left(1+\frac{s_t}{n}\right)^{n \cdot t} = e^{s_t \cdot t}$$

- With continuous compounding, the amount invested at rate *s* for a maturity m+n is equivalent to:

(10d) $$e^{s_{m+n}(m+n)} = e^{s_m \cdot m} \cdot e^{{}_m f_n \cdot n}$$

(10e) $$s_{m+n} \cdot (m+n) = s_m \cdot m + {}_m f_n \cdot n$$

(11) $${}_m f_n = \frac{s_{m+n} \cdot (m+n) - s_m \cdot m}{n}$$

1.2. From Bonds to Interest Rates

- Using (9), (11) becomes:

(11a) $${}_m f_n = \frac{-\ln(d_{m+n}) + \ln(d_m)}{n}$$

- n->0 =>

(12) $${}_m f_0 = \lim_{n \to 0}\left(\frac{-\ln(d_{m+n}) + \ln(d_m)}{n}\right) = -\frac{\partial[\ln(d_m)]}{\partial m} = \frac{\partial(m \cdot s_m)}{\partial m}$$

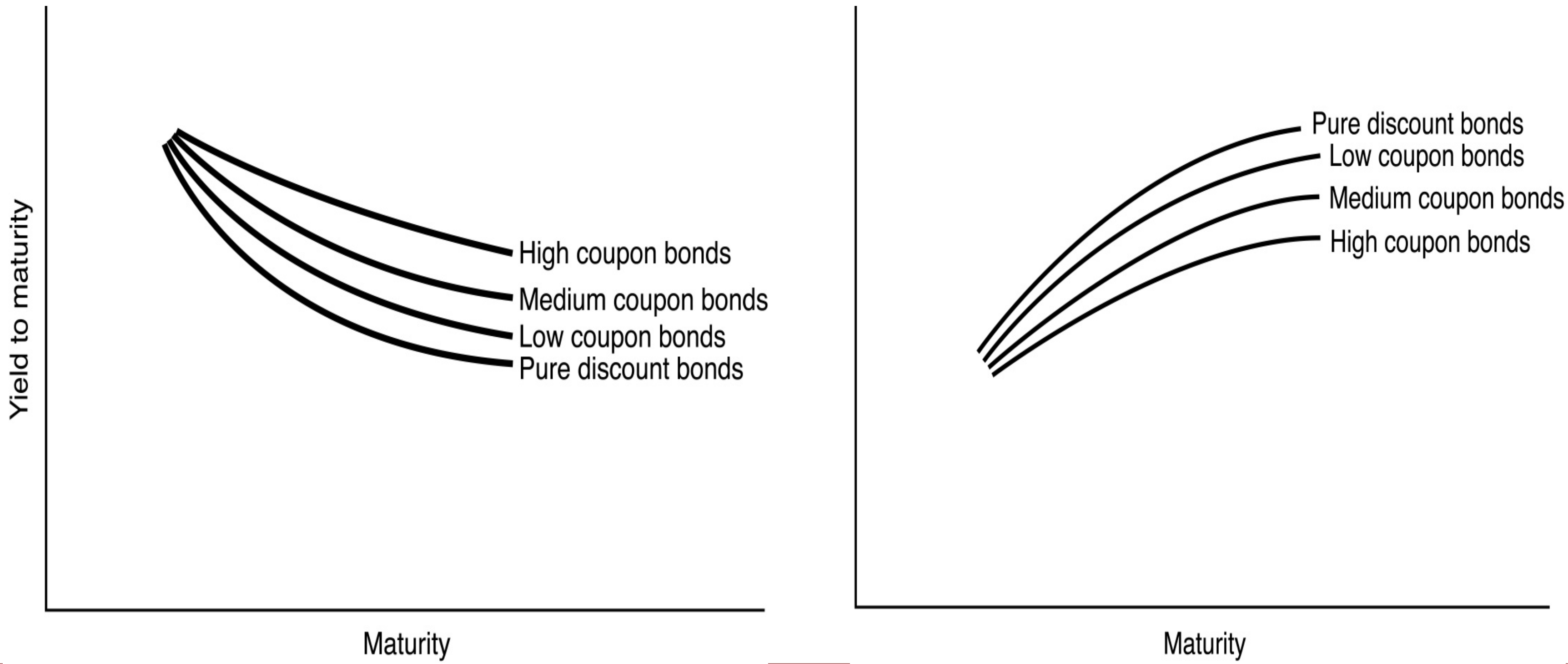
or

(13) $${}_m f_0 = s_m + m \cdot \frac{\partial(s_m)}{\partial m}$$

(14) $$s_m = \frac{1}{m} \cdot \int_0^m {}_\mu f_0 \, d\mu$$ Spot rate=simple average of the instantaneous forward rates

1.3. YIELD-TO-MATURITY

- The YTM depends on:
  - The bond’s maturity
  - The bond’s coupon rate
- The yield curve has a less pronounced shape and curvature when the coupons are higher, as the higher weight of intermediate cash-flows reduces the effect of higher maturities in the discount process.



2 - TERM STRUCTURES (TS)

2.1. Introduction

2.2. Stylized Facts

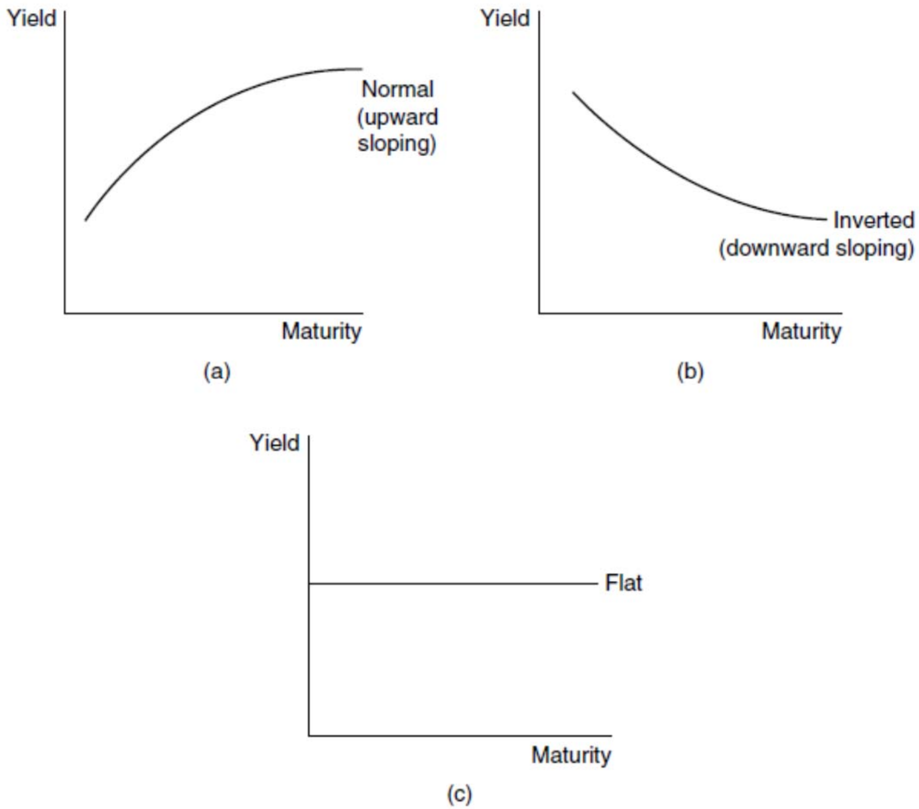
2.3. Theories of the TS

2.1. INTRODUCTION

- The yield curve may assume different monotonic or non-monotonic shapes.

- Monotonic shapes:

Three Hypothetical Yield Curves



Fabozzi, Frank J., (2012), "The Handbook of Fixed Income Securities", 8th Edition, McGraw-Hill Education

- The term structure of interest rates changes in response to:

  - Economic shocks
  - Market-specific events
  - Policy decisions

- Example
  - The announcement in 2018 by the Italian Government that next year's budget would involve a higher deficit => sell-off Italian Govt. bonds => increase in Italian yields and also some contagion on Spanish and Portuguese Govt. bonds.

2.2. Stylized Facts

- Volatility
- Correlation
- Standard Movements
  - Shift Movements
  - Twist Movements
  - Butterfly Movements

STYLIZED FACTS (1) : VOLATILITY

- Yields and bond prices are typically much less volatile than prices in other asset classes.
- In countries where the credibility of monetary policy is lower, or, correspondingly, the currency is weaker, the volatility of short term interest rates is usually higher than long term interest rates.
- In countries where credit risk issues arise, long term interest rates typically become more volatile.
- Even though volatility of short-term rates is usually higher, volatility of bond prices in longer maturities is higher, due to the higher impact of interest rate shifts on the net present value of the cash-flows of bonds with higher maturities (durations).
- Volatility of bond prices changes along time, as volatility is influenced by duration and convexity (related to the first and the second derivative of the bond price in order to the yield).
- Moreover, duration and convexity, as well as yields, change along time.

STYLIZED FACTS (1) : VOLATILITY

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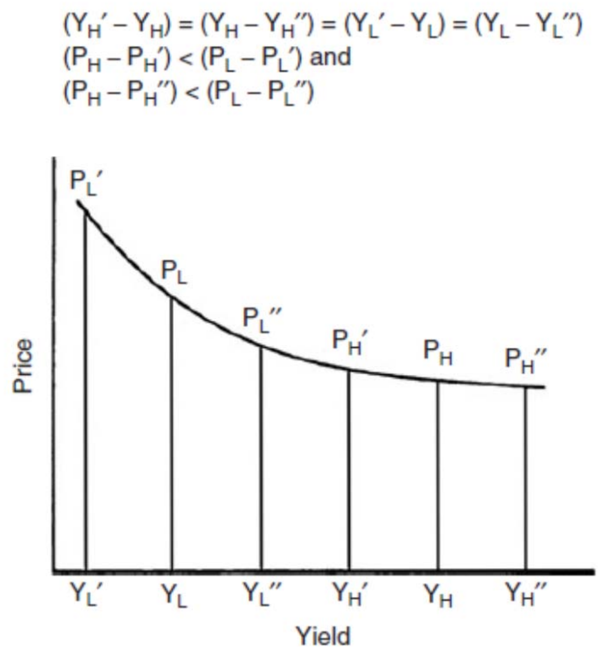
- Moreover, duration and convexity, as well as yields, change along time.

STYLIZED FACTS (1) : VOLATILITY

- With higher yields, the volatility of bond prices (P_H) due to yield changes is lower.
- As it is illustrated below, $P_H' - P_H'' < P_L' - P_L''$

The Effect of Yield Level on Price Volatility



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STYLIZED FACTS (1) : VOLATILITY

- Changes in bond prices are close to symmetric for small yield changes.
- For larger symmetric yield changes, price increases are higher than price decreases.
- Prices of bonds with higher coupon rates are less volatile, given the higher weight of intermediate cash-flows => zero-coupon bonds are the most volatile, for a given maturity.

Price/Yield Relationship for Four Hypothetical Option-Free Bonds

	Price ($)			
Yield (%)	6%/5 Year	6%/20 Year	9%/5 Year	9%/20 Year
4.00	108.9826	127.3555	122.4565	168.3887
5.00	104.3760	112.5514	117.5041	150.2056
5.50	102.1600	106.0195	115.1201	142.1367
5.90	100.4276	101.1651	113.2556	136.1193
5.99	100.0427	100.1157	112.8412	134.8159
6.00	100.0000	100.0000	112.7953	134.6722
6.01	99.9574	99.8845	112.7494	134.5287
6.10	99.5746	98.8535	112.3373	133.2472
6.50	97.8944	94.4479	110.5280	127.7605
7.00	95.8417	89.3225	108.3166	121.3551
8.00	91.8891	80.2072	104.0554	109.8964

Instantaneous Percentage Price Change for Four Hypothetical Bonds (Initial yield for all four bonds is 6%)

	Percentage Price Change			
New Yield	6%/5 Year	6%/20 Year	9%/5 Year	9%/20 Year
4.00	8.98	27.36	8.57	25.04
5.00	4.38	12.55	4.17	11.53
5.50	2.16	6.02	2.06	5.54
5.90	0.43	1.17	0.41	1.07
5.99	0.04	0.12	0.04	0.11
6.01	−0.04	−0.12	−0.04	−0.11
6.10	−0.43	−1.15	−0.41	−1.06
6.50	−2.11	−5.55	−2.01	−5.13
7.00	−4.16	−10.68	−3.97	−9.89
8.00	−8.11	−19.79	−7.75	−18.40

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STYLIZED FACTS (2) : CORRELATION

- Rates with different maturities are
  - Positively but not perfectly correlated, meaning that there is more than one factor behind the yield curve dynamics
  - Correlation decreases with differences in maturity

- Example:

	1M	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y
1M	1									
3M	0.999	1								
6M	0.908	0.914	1							
1Y	0.546	0.539	0.672	1						
2Y	0.235	0.224	0.31	0.88	1					
3Y	0.246	0.239	0.384	0.808	0.929	1				
4Y	0.209	0.202	0.337	0.742	0.881	0.981	1			
5Y	0.163	0.154	0.255	0.7	0.859	0.936	0.981	1		
7Y	0.107	0.097	0.182	0.617	0.792	0.867	0.927	0.97	1	
10Y	0.073	0.063	0.134	0.549	0.735	0.811	0.871	0.917	0.966	1

STYLIZED FACTS (3): STANDARD MOVEMENTS

- The evolution of the interest rate curve can be split into 3 standard movements, regardless the time period or the market:
  - **Shift movements** (changes in level), which account for 70 to 80% of observed movements on average.
  - **Twist movements** (changes in slope), which accounts for 15 to 30% of observed movements on average.
  - **Butterfly movements** (changes in curvature), which accounts for 1 to 5% of observed movements on average.

=> 1 or 2-factor models tend to be enough to explain the behavior of the yield curve in most occasions.

2.3. THEORIES OF THE TS

- Explanatory theories of the TSIR depend mostly on:
  - the preferences of market participants for maturities, namely their credit, liquidity and interest rate risk aversion.
  - the expectations on the future behavior of short-term interest rates, i.e. monetary policy.
- Term structure theories attempt to explain the relationship between "risk-free" interest rates and the corresponding maturities.
- Explanatory theories:
  - Expectations
  - Preferred habitat
  - Liquidity premium
  - Market segmentation

- The **expectations theory** postulates that long term rates depend on the current short term rates and the expectations on their future path.
- Let us assume that an investor has 2 investment alternatives for an horizon = *T*:
  - A long term (zero-coupon) bond, with maturity = *T*;
  - A set of bonds with short term maturities (=1), with the last investment done at T-1. The investment in these several bonds can be done by rolling over the initial investment.
- The expected returns for these 2 alternatives must be equal (being r(t,T) the yield in time = t of a bond maturing at a later period T):
  - $[1+r(t,T)]^T = (1+r(t,1)) \times (1+E(t)\ (r(t+1),1)) \times (1+E(t)\ (r(t+2),1)) \times \dots \times (1+E(t)\ (r(T-1),1))$
  - $r(t,T)=[(1+r(t,1)) \times (1+E(t)\ (r(t+1),1)) \times (1+E(t)\ (r(t+2),1)) \times \dots \times (1+E(t)\ (r(T-1),1))]^{1/n}-1$

- If one assumes that there is no risk premium (i.e. investors are risk-neutral regarding investing in short or in long term interest rates), expected interest rates are equal to forward rates.
- According to this theory, the yield curve may assume different shapes and positively (negatively) sloped curves correspond to expectations of short tem interest rate increases (decreases).
- Therefore, changes in yield curves are interpreted as changes in market expectations.
- **2 versions of the expectations theory**:

(i) pure – there is no risk premium => forward rates correspond to the expected future interest rates => $f_t^j = E_t(s_{t+j})$

(ii) non-pure – there is risk premium, but it's constant along time => forward rates do not correspond to expected future interest rates, but changes in forward rates correspond to changes in expectations about future interest rates.

- In our previous example, the maturity of the long-term bond was equal to the investment period.
- However, the same rationale can be developed by using bonds with larger maturities than the investment horizon, e.g. assuming that the long-term bond has a maturity of 30 years, while the investment horizon is 10 years.

- In this case, the return from investing in the 30-year maturity bond would have to be obtained by selling this bond after 10 years.

- The **preferred habitat theory** sustains that investors have preferred maturities, but they accept to invest in different maturities if they are compensated for that.

- The risk premium paid to attract investors to maturities different from those preferred do not necessarily increase with the maturity.

⬇

- Moves in the yield curve do not correspond necessarily to changes in investors' expectations about the future path of short-term interest rates and the yield curve may have different shapes.

⬇

- **Forward rates (or their changes) cannot be used to gauge expectations about future interest rates**.

- **Liquidity premium theory** is a particular case of the preferred habitat theory
- Investors always prefer short to long maturities
- Investors always demand a premium to invest in longer maturities
- Long term interest rates > short term interest rates
- The yield curve will always be positively sloped (unless we assume that long-term rates still reflect interest rate expectations and these point to sharp decreases)
- A positively sloped curve is usually considered as a regular curve, given that investors tend to be risk-averse => premium to invest in longer maturities due to the uncertainty on the future path of interest rates.

- The **market segmentation theory** postulates that interest rates in each maturity stem only from the supply and demand in that maturity.
- As a consequence, there is no relationship between interest rates in different maturities and the yield curve may have very irregular shapes.
- **Main conclusions:**

(i) the yield curve shape is explained by a mix of all these theories, even though market participants usually consider that a normal yield curve is a positively sloped one.

(ii) the risk premium is usually considered as increasing with maturities.

(iii) even though the risk premium is not nil, changes in long-term interest rates may be considered as changes in expectations on future short-term interest rates' behavior if one assumes that risk premium is constant along time, which tends to happen, at least, in short periods of time.

3. Hedging Interest Rate Risk

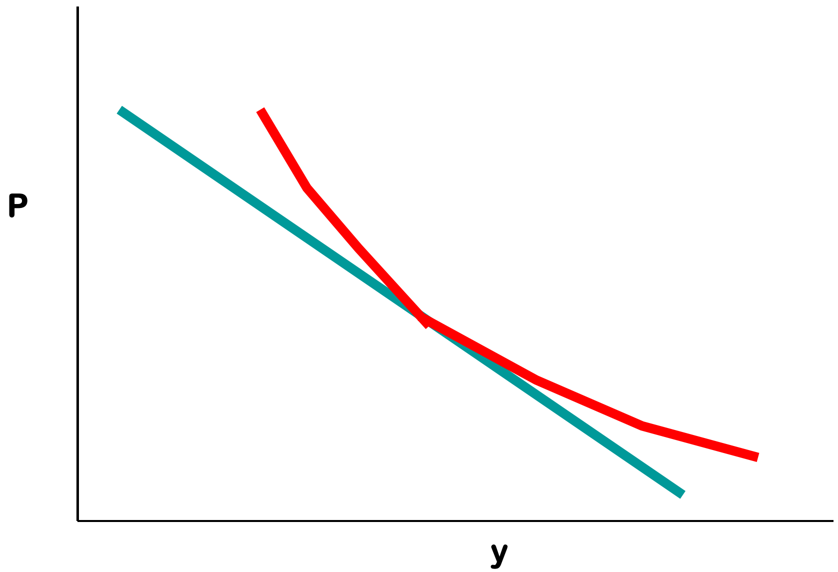
- Basic principle: attempt to reduce as much as possible the dimensionality of the problem, i.e. to hedge risk with as few factors as possible.

- First step: **duration hedging**
  - Consider only one risk factor
  - Assume only small changes in the risk factor

- Beyond duration: **convexity hedging**
  - Relax the assumption of small interest rate changes

3.1. DURATION

- We will study the sensitivity of the bond price to changes in yield - Interest rate risk:
  - Rates change from *y* to *y+dy*
  - *dy* - small variation in yields, e.g. 1 bp (e.g., from 5% to 5.01%)
  - *dP* - variation in bond price due to *dy*
- The relationship between the bond prices and the yields is not linear.
- However, for small changes in yields, a good proxy for *dP* is the first derivative of the bond price in order to *y*.



- With **continuously compounded interest rates** and assuming a flat yield curve (same yields for all maturities), we have:

$$P^c = FVe^{-yT} + \sum_{n=1}^{T} ce^{-yn}$$

$$\frac{\partial P^c}{\partial y} = \frac{\partial[FVe^{-yT} + \sum_{n=1}^{T} ce^{-yn}]}{\partial y} = -T \cdot FVe^{-yT} - \sum_{n=1}^{T} n \cdot ce^{-yn}$$

- **Macaulay Duration (Frederick Macaulay, 1938) –** aka effective maturity: **Average maturity (measured in years) of all cash-flows weighted by the relevance of their NPV on the bond price** (while residual maturity is just the maturity of the final cash-flow), assuming the yield curve is flat.

$\Downarrow$

Macaulay Duration > 0

- Calculated as (the absolute value of) the partial derivative of the bond price with respect to yield, divided by the bond price:

$$D = \frac{\sum_{n=1}^{T} n \cdot ce^{-yn} + T \cdot FVe^{-yT}}{P^c}$$
$$= 1 \cdot \frac{ce^{-y}}{P^c} + 2 \cdot \frac{ce^{-2y}}{P^c} + 3 \cdot \frac{ce^{-3y}}{P^c} + \cdots + T \cdot \frac{ce^{-yT}}{P^c} + T \cdot \frac{FVe^{-yT}}{P^c}$$

$$\frac{\partial P^c}{\partial y} = -DP^c \Rightarrow \frac{dP^c}{P^c} = -Ddy$$

Duration corresponds to the relative (%) change in price due to a small change in yield

- With **discrete compounding interest rates**:

$$P^c = \frac{FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c}{(1+y)^n}$$

$$\frac{\partial P^c}{\partial y} = \frac{\partial \left[\frac{FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c}{(1+y)^n}\right]}{\partial y}$$

$$= -\frac{T \cdot FV(1+y)^{T-1}}{(1+y)^{2T}} - \sum_{n=1}^{T} \frac{c \cdot n(1+y)^{n-1}}{(1+y)^{2n}}$$

$$= -\frac{T \cdot FV}{(1+y)^{T+1}} - \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^{n+1}}$$

$$= -\frac{1}{(1+y)} \underbrace{\left[\frac{T \cdot FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^n}\right]}$$

Weighted-average maturity of all cash-flows (weighted by the relative weight of their NPV on the bond price) ⟵

$$D = \frac{\frac{T \cdot FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^n}}{P^c}$$

$$\frac{\partial P^c}{\partial y} = -\frac{1}{(1+y)} D P^c \Rightarrow \frac{dP^c}{P^c} = -\frac{1}{(1+y)} D dy$$

$$\frac{1}{(1+y)} D = MD$$ ⟶ **modified duration** – percentage impact (%) on bond price of a given change (percentage points) in the yield

EXAMPLE (SEE CALCULATION IN SPREADSHEET)

T = 10, *c* = 5%, *y* = 5% (bond at par)

Time of Cash Flow	Cash Flow F_n	$w_n = \frac{1}{P^C} \cdot \frac{F_n}{(1+y)^n}$	$n \cdot w_n$	$n^2 \cdot ce^{-yn}$
1	50	0,047619048	0,047619	47,56147123
2	50	0,045351474	0,090703	180,9674836
3	50	0,04319188	0,129576	387,3185894
4	50	0,041135124	0,16454	654,9846025
5	50	0,039176308	0,195882	973,5009788
6	50	0,03731077	0,223865	1333,472797
7	50	0,035534067	0,248738	1726,48582
8	50	0,033841968	0,270736	2145,024147
9	50	0,032230446	0,290074	2582,394014
10	1050	0,644608916	6,446089	3032,653299
Total			8,107822	13064,3632

$$D = \frac{\frac{T \cdot FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c \cdot n}{(1+y)^n}}{P^C} \cong 8$$

- The lower the coupon rate, the higher (and closer to residual maturity) the duration will be, as the relative weight of the final cash-flow will be higher => zero-coupon bonds have duration equal to residual maturity.
- For a given coupon rate and yield, duration increases as maturity increases: $\frac{\partial D}{\partial n} \geq 0$
- For a given maturity and coupon rate, duration increases as the yield decreases, given that the net present value of the cash-flows increase more in longer than in shorter maturities:
$$\frac{\partial D}{\partial y} \leq 0$$

DURATION HEDGING

- Principle: immunize the value of a bond portfolio with respect to changes in yield:
  - P = value of the portfolio
  - H = value of the hedging instrument
- Hedging instrument may be:
  - Bonds
  - Swaps
  - Futures
  - Options
  - FRAs

DURATION HEDGING

- Duration hedging is very simple to do, but it is only valid for small changes and parallel shifts of the yield curve.

- The impact of these small changes is often provided by a measure usually employed in financial markets – basis point value (BPV) or price value of a basis point (PVBP):

$$\text{PVBP} = |\text{ initial price} - \text{price if yield is changed by 1 basis point }|$$

- Changes in value
  - Portfolio

$$dP \approx P'(y)dy$$

  - Hedging instrument

$$dH \approx H'(y)dy$$

- Strategy: hold *q* units of the hedging instrument so that

$$dP + qdH = \left(qH'(y) + P'(y)\right)dy = 0$$

- Solution: $q = -\dfrac{P'(y)}{H'(y)} = -\dfrac{P \times D_P}{H \times D_H}$

given that P'(y)=dP/dY and dP/P = -D x dy (with countinuously compounded interest rates) ⇔dP/dy = -P x D

→ we can calculate the number of hedging instruments to implement a duration hedging strategy just by knowing the current prices of the bond and the hedging instrument, as well as both durations.

Example:

- At date t, a portfolio P has a price of €328635, a 5.143% yield and a 7.108 duration.
- Hedging instrument – bond with price = €118.786, yield = 4.779% and duration = 5.748.
- Hedging strategy involves taking a short position (i.e. selling futures contracts) as follows:

$$q = -\frac{P'(y)}{H'(y)} = -\frac{P \times D_P}{H \times D_H}$$

q = -(328635x7.108)/(118.786x5.748) = - 3421

- Therefore, 3421 units of the hedging bond should be sold.

3.2. CONVEXITY

- Considering a second order Taylor approximation:

$$\frac{dP^c}{P^c} = \underbrace{\frac{dP^c}{dy}\frac{1}{P^c}}_{-D}(dy) + \frac{1}{2}\underbrace{\frac{d^2P^c}{dy^2}\frac{1}{P^c}}_{C}(dy)^2$$

- With continuous compounding:

$$\frac{\partial^2 P^c}{\partial y^2} = T^2 \cdot FVe^{-yT} + \sum_{n=1}^{T} n^2 \cdot ce^{-yn}$$

as $\frac{\partial P^c}{\partial y} = -T \cdot FVe^{-yT} - \sum_{n=1}^{T} n \cdot ce^{-yn}$

$$C = \frac{\partial^2 P^c}{\partial y^2} \cdot \frac{1}{P^c} = \frac{T^2 \cdot FVe^{-yT} + \sum_{n=1}^{T} n^2 \cdot ce^{-yn}}{P^c} \geq 0$$

- With discrete compounding, convexity may be written as a function of MD and its first derivative in order to the yield:

$$C = \frac{\partial^2 P^c}{\partial y^2} \cdot \frac{1}{P^c} \leftrightarrow C \cdot P^c = \frac{\partial^2 P^c}{\partial y^2}$$

$$\frac{\partial P^c}{\partial y} = -MD \cdot P^c \Rightarrow C \cdot P^c = \frac{\partial(-MD \cdot P^c)}{\partial y} = \frac{\partial(-MD)}{\partial y} \cdot P^c + \frac{\partial P^c}{\partial y} \cdot (-MD)$$

$$= \frac{\partial(-MD)}{\partial y} \cdot P^c + \left(-\frac{1}{(1+y)} DP^c\right) \cdot (-MD)$$

$$= -\frac{\partial(MD)}{\partial y} \cdot P^c + (-MD \cdot P^c) \cdot (-MD) \leftrightarrow$$

$$\leftrightarrow C = -\frac{\partial(MD)}{\partial y} + MD^2$$

- As MD decreases with the yield => $C \geq 0$

unless bonds have embedded options, namely the prepayment option (Call–option hold by the issuer)

CONVEXITY- PROPERTIES

$$C = \frac{\partial^2 P^c}{\partial y^2} \cdot \frac{1}{P^c} = \frac{T^2 \cdot FVe^{-yT} + \sum_{n=1}^{T} n^2 \cdot ce^{-yn}}{P^c} \geq 0$$

- For a given maturity and yield, convexity increases when the bond provides regular payments along time => convexity increases with the coupon rate and with maturity => the yield curve assumes a convex shape in longer maturities.
- But if the coupon rate increases, the yield will also increase, bringing convexity (and duration) down.
- For a given maturity and coupon rate, convexity increases when the yield decreases.
- A bond with higher convexity is always preferred, as its price benefits more from yield decreases and its less impacted by yield increases => bonds with low coupon rates.

Duration + Convexity Hedging - Principle

- Principle: immunize the value of a bond portfolio with respect to changes in yield:
  - Denote by *P* the value of the portfolio
  - 2 hedging instruments (whose value is H_1 and H_2) because there are 2 risk factors to be hedged - parallel shifts in the yield curve and the second order effect:
  - Portfolio value variations: $$dP \approx P'(y)dy + \frac{P''(y)}{2}dy^2$$
  - Hedging instruments value variations: $$\begin{cases} dH_1 \approx H_1'(y)dy + \frac{1}{2}H_1''(y)dy^2 \\ dH_2 \approx H_2'(y)dy + \frac{1}{2}H_2''(y)dy^2 \end{cases}$$

- Strategy: hold q_1 and q_2 units of the 1st and 2nd hedging instruments so that:

$$dP + q_1 \times dH_1 + q_2 \times dH_2 = 0$$

- Solution

  - Under the assumption of unique *dy* – parallel shifts (impacting simultaneously on the price of the portfolio and the hedging instruments):

$$\begin{cases} P'(y) + q_1 H_1'(y) + q_2 H_2'(y) = 0 \\ P''(y) + q_1 H_1''(y) + q_2 H_2''(y) = 0 \end{cases}$$

→ with non-flat yield curves, the impact of parallel yield changes on bond prices will not result straight from durations or convexities

  - Under the assumption of a unique *y* – flat yield curve => the impact of yield changes on bond prices will result straight from durations and convexities:

$$\begin{cases} q_1 H_1(y) D_1 + q_2 H_2(y) D_2 = -P(y) D_p \\ q_1 H_1(y) C_1 + q_2 H_2(y) C_2 = -P(y) C_p \end{cases}$$

EXAMPLE:

- Portfolio at date t
  - Price P = € 347000
  - Yield y = 5,13%
  - Duration = 6,78
  - Convexity = 50,26

- Hedging instrument 1
  - Price H_1 = € 97962
  - Yield y_1 = 5,27%
  - Duration = 8,09
  - Convexity = 73,35

- Hedging instrument 2:
  - Price H_2 = € 108039
  - Yield y_2 = 4.10%
  - Duration = 2.82
  - Convexity = 8,18

- Optimal quantities q_1 and q_2 of each hedging instrument are given by

$$\begin{cases} q_1 H_1(y) D_1 + q_2 H_2(y) D_2 = -P(y) D_P \\ q_1 H_1(y) C_1 + q_2 H_2(y) C_2 = -P(y) C_P \end{cases}$$

$$\begin{cases} q_1 \cdot 97962 \cdot 8{,}09 + q_2 \cdot 108039 \cdot 2{,}82 = -347000 \cdot 6{,}78 \\ q_1 \cdot 97962 \cdot 73{,}35 + q_2 \cdot 108039 \cdot 8{,}18 = -347000 \cdot 50{,}26 \end{cases}$$

- Solving in order to $q1$ and $q2$:

$$\begin{cases} q_1 = -2{,}17 \\ q_2 = -2{,}07 \end{cases}$$

The investor should sell 2 units of each hedging instrument if the yield curve is (close to) flat, namely be selling 2 futures contracts with the underlying assets being bonds with the corresponding features.

DURATION + CONVEXITY HEDGING - LIMITATIONS

- It is not easy to find hedging contracts with the required features => higher costs to find tailor-made hedging instruments or any hedging will be just an approximation, even assuming that the yield curve is flat.

- Furthermore, the yield curve is not flat => more complex calculations.

- Additionally, yield curve shifts are not only parallel, but its shape also changes => even more complex calculations.

The yield curve dynamics is not fully explained by one-factor models => multifactor models are needed.

4. IR DERIVATIVES

- Forward Rate Agreements (**FRAs**)
- Interest Rate futures
- Interest Rate swaps (**IRS**)
- Interest Rate **Options:**
  - Plain vanilla Bond or short-term futures **Options**
  - Interest rate **CAPS**
  - **Swaptions**

FRAs

- A FRA is an OTC contract settle to fix the interest rate that will apply to a deposit or loan of a given amount to occur at a future date (settlement date) for a given maturity, allowing to lock-in this future interest rate.

- If the investor is risk-neutral, the FRA interest rate corresponds to the expected interest rate for its settlement date and maturity.

- Being an OTC contract, FRAs are not listed in exchange markets, even though there is public data on FRAs, from quotes by market participants.

FRAs

- FRAs are quoted for fixed times to settlement and maturities, e.g. 3 x 9 (6-month interest rate forward, with a time to settlement of three months).
- Quotes are in percentage points, as usual interest rates (e.g. 2%, 4%).
- In order to cancel an exposure to a FRA, one cannot sell the contract, but can take a short exposure on a futures contract, for a time to settlement and an interest rate maturity as close as possible.
- Just like for a bond, the value of a FRA contract is an inverse function of the interest rate: an increase in spot rates reduces the value of a FRA contract => for the buyer/holder of a long position in a FRA, interest rate increases are unfavorable.

FRAs

- 3-month FRA with a time to settlement = 3y <-> FRA (3x39)
- As the actual 3-month interest rate at T is higher than the FRA rate, the FRA buyer will lose money:

Example 4.3

Suppose that a company enters into an FRA that specifies it will receive a fixed rate of 4% on a principal of \$100 million for a 3-month period starting in 3 years. If 3-month LIBOR proves to be 4.5% for the 3-month period the cash flow to the lender will be

$$100{,}000{,}000 \times (0.04 - 0.045) \times 0.25 = -\$125{,}000$$

at the 3.25-year point. This is equivalent to a cash flow of

$$-\frac{125{,}000}{1 + 0.045 \times 0.25} = -\$123{,}609$$

Assuming simple compounding

at the 3-year point. The cash flow to the party on the opposite side of the transaction will be +\$125,000 at the 3.25-year point or +\$123,609 at the 3-year point. (All interest rates in this example are expressed with quarterly compounding.)

Source: Hull, John (2009), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 7th Edition

FRAs

- Therefore, there is a positive pay-off to the long (short) position holder when interest rates decrease (increase), being the FRA value at any given time calculated as:

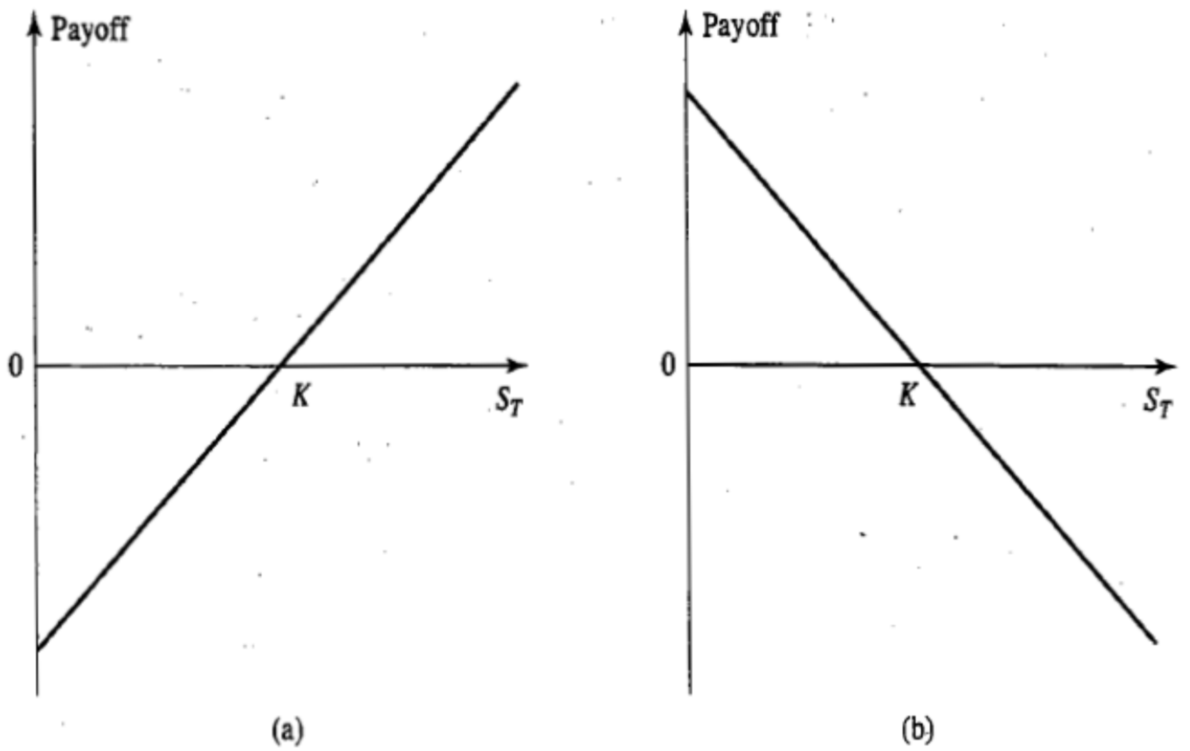
$$V_{\mathrm{FRA}} = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}$$

being:
L = principal amount
R_K = settled FRA rate
R_F = current forward rate for the corresponding time to settlement and maturity
T_1 = maturity date
T_2 = settlement date

- At the negotiation day, the FRA value is zero, as the settlement rate is agreed as the market forward rate.

FRAs

Figure 1.2 Payoffs from forward contracts: (a) long position, (b) short position. Delivery price = K; price of asset at contract maturity = S_T.



Source: Hull, John (2009), “Options, Futures and Other Derivatives”, Pearson Prenctice Hall, 7th Edition

Note: Asset price is an inverse function of the interest rate (just like a bond or a futures contract).

INTEREST RATE FUTURES

- Futures contracts are traded in exchange markets, for fixed settlement dates (and consequently different times to settlement and maturity), e.g. the 3-month Euribor futures for Dec.2016 settlement).
- Conversely, FRAs are traded for variable settlement dates and fixed times to settlement and maturity.
- In order to cancel a long position in such a contract, an investor can take a short position in (sell) the same contract.
- Usually, one can find futures contracts for short and long term interest rates.
- Short-term futures contracts are quoted as 100 minus the implied interest rates.
- Long-term futures contracts are priced as a % of the nominal value of the theoretical bond embedded in the futures contract, just like the true bonds, being the price of the theoretical bond the NPV of its cash-flows, discounted at current market rates.

INTEREST RATE FUTURES

- Therefore, an increase in the price of short-term and long-term futures contracts means that the implied interest rate is decreasing.

- Short-term interest rate futures have financial settlement, while long-term interest rate futures usually have physical settlement, through the cheapest-to-deliver bond (among the bonds considered as deliverable, i.e. proxies for the theoretical underlying bond).

- Short and long term interest rate futures are usually available for quarterly settlement dates (pre-specified days in March, June, September and December).

- However, short term futures are usually available for a longer set of settlement dates (comparing to long term interest rate futures and also FRAs), even though the most liquid contracts are those with shorter times to settlement.

IRS

• IRS are contracts that settle the exchange of fixed for variable interest rates at pre-specified dates.

• Therefore, they may be seen as a long (short) position in a fixed rate bond, a set of FRAs or interest rate futures, on one hand, and a short (long) position in a variable interest rate bond, on the other hand.

• The swap value or price corresponds to its replacement cost, i.e. the amount of money that should be paid by one counterparty to the other to cancel the contract, reflecting the dynamics of short and long term interest rates since the initial date or the last payment date.

• This also corresponds to the difference between a fixed and a floating rate bond:

(from the floating rate payer's point of view) $V_{\text{swap}} = B_{\text{fix}} - B_{\text{fl}}$

• Consequently, at the initial and all payment dates, the swap value returns to 0.

Interest Rate Options

- Interest rate volatilities have to be estimated.

- Consequently, the Dynamics of the yield curve must be assessed => Stochastic Interest Rate Models.

- **Cap** – set of put options on any interest rate to be paid in the future.

- Each of these options is a **caplet** and can be traded individually.

- **Swaption** - gives the right to enter into a swap at a future pre-specified rate.

- Contrary to the caps, the several cash-flows of the swaption cannot be traded separately.

Interest Rate Options

Problem:
We want to price, at t, a European Call, with exercise date S, and strike price K, on an underlying T-bond. $(t < S < T)$.

Naive approach: Use Black-Scholes's formula.

$$F(t,p) = pN[d_1] - e^{-r(S-t)}KN[d_2].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{S-t}}\left\{\ln\left(\frac{p}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(S-t)\right\}, \\ d_2 &= d_1 - \sigma\sqrt{S-t}. \end{aligned}$$

where

$$p = p(t,T)$$

Is this allowed?

INTEREST RATE OPTIONS

Difficulties with Black-Scholes pricing:

- The **Dynamics of the yield curve** must be assessed => Stochastic Interest Rate Models.
- **Volatility of the underlying bond varies along time and tends to 0**, while the bond is getting closer to the redemption date.
- **Short-term rates are stochastic** => implementation of dynamic bond pricing models