Models in Finance - Class 5

Master in Actuarial Science

João Guerra

ISEG

Stochastic differential equations

- Deterministic ordinary diff. eqs.:

$$f\left(t, x(t), x'(t), x''(t), \ldots\right) = 0, \qquad 0 \leq t \leq T.$$

- 1st order ordinary diff. eq.:

$$\frac{dx(t)}{dt} = \mu(t, x(t))$$

or

$$dx(t) = \mu(t, x(t))\, dt$$

- Discrete version

$$\Delta x(t) = x(t + \Delta t) - x(t) \approx \mu(t, x(t))\, \Delta t$$

- Example:

$$\frac{dx(t)}{dt} = cx(t)$$

has solution

$$x(t) = x(0)\,e^{ct}.$$

SDE's

- SDE in differential form

$$\begin{aligned} dX_t &= \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t, \\ X_0 &= X_0 \end{aligned} \tag{1}$$

- $\mu(t, X_t)$ is the drift coefficient, $\sigma(t, X_t)$ is the diffusion coefficient.
- SDE in integral form

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s. \tag{2}$$

- "naif" interpretation of SDE: $\Delta X_t \approx \mu(t, X_t)\, \Delta t + \sigma(t, X_t)\, \Delta B_t$. e $\Delta X_t \approx N\left(\mu(t, X_t)\, \Delta t, (\sigma(t, X_t))^2\, \Delta t\right)$.

Definition

A solution of SDE (1) or (2) is a stochastic process $\{X_t\}$ which satisfies:

1. $\{X_t\}$ is an adapted process (to Bm) and has continuous sample paths.
2. $\mathbb{E}\left[\int_0^T (\sigma(s, X_s))^2 \, ds\right] < \infty$.
3. $\{X_t\}$ satisfies the SDE (1) or (2)

- The solutions of SDE's are called diffusions or "diffusion processes".

- The process $\{X_t, t \geq 0\}$ is said to be a time-homogeneous diffusion process if:
  1. it is a Markov process.
  2. it has continuous sample paths.
  3. there exist functions $\mu(x)$ and $\sigma^2(x) > 0$ such that as $\Delta t \to 0^+$,

$$E\left[X_{t+\Delta t} - X_t | X_t = x\right] = \Delta t \mu(x) + o(\Delta t),$$

$$E\left[(X_{t+\Delta t} - X_t)^2 | X_t = x\right] = \Delta t \sigma^2(x) + o(h),$$

$$E\left[(X_{t+\Delta t} - X_t)^3 | X_t = x\right] = o(\Delta t).$$

- A diffusion is "locally" like Brownian motion with drift, but with a variable drift coefficient $\mu(x)$ and diffusion coefficient $\sigma(x)$.
- Fitting a diffusion model involves estimating the drift function $\mu(x)$ and the diffusion function $\sigma(x)$. Estimating arbitrary drift and diffusion coefficients is virtually impossible unless a very large quantity of data is to hand.
- It is more usual to specify a parametric form of the mean or the variance and to estimate the parameters.

Solving an SDE by Itô formula

- **Example**: Standard model for risky asset price (SDE):

$$dS_t = \alpha S_t dt + \sigma S_t dB_t \tag{3}$$

or

$$S_t = S_0 + \alpha \int_0^t S_s ds + \sigma \int_0^t S_s dB_s \tag{4}$$

- How to solve this SDE?
- Assume that $S_t = f(t, B_t)$ with $f \in C^{1,2}$. By Itô formula:

$$S_t = f(t, B_t) = S_0 + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, B_s) \right) ds + \tag{5}$$
$$+ \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s.$$

- Comparing (4) with (5) then (uniqueness of representation as an itô process)

$$\frac{\partial f}{\partial s}(s, B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, B_s) = \alpha f(s, B_s), \tag{6}$$

$$\frac{\partial f}{\partial x}(s, B_s) = \sigma f(s, B_s). \tag{7}$$

- Differentiating (7) we get

$$\frac{\partial^2 f}{\partial x^2}(s, x) = \sigma \frac{\partial f}{\partial x}(s, x) = \sigma^2 f(s, x)$$

and replacing in (6) we have

$$\left(\alpha - \frac{1}{2}\sigma^2\right) f(s, x) = \frac{\partial f}{\partial s}(s, x)$$

- Separating the variables: $f(s,x) = g(s)\, h(x)$, we get

$$\frac{\partial f}{\partial s}(s,x) = g'(s)\, h(x)$$

and

$$g'(s) = \left(\alpha - \frac{1}{2}\sigma^2\right) g(s)$$

wich is a linear ODE, with solution:

$$g(s) = g(0) \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) s\right]$$

- Using (7), we get $h'(x) = \sigma h(x)$ and

$$f(s,x) = f(0,0) \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) s + \sigma x\right].$$

- Conclusion:

$$S_t = f\left(t, B_t\right) = S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right] \qquad (8)$$

which is the geometric Brownian motion. Therefore $\frac{S_t}{S_0}$ has lognormal distribution with parameters $\left(\alpha - \frac{1}{2}\sigma^2\right) t$ and $\sigma^2 t$.

- Remark: Note that the solution of the SDE was obtained by solving a deterministic PDE (partial differential equation).
- Moreover

$$E\left[\frac{S_t}{S_0}\right] = e^{\alpha t}, \text{ var}\left[\frac{S_t}{S_0}\right] = e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right).$$

- Let us verify that (8) satisfies SDE (3) or (4).
- Apllying the Itô formula to $S_t = f(t, B_t)$ with

$$f(t,x) = S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma x\right],$$

we obtain

$$\begin{aligned} S_t &= S_0 + \int_0^t \left[\left(\alpha - \frac{1}{2}\sigma^2\right)S_s + \frac{1}{2}\sigma^2 S_s\right]ds + \int_0^t \sigma S_s dB_s \\ &= S_0 + \alpha \int_0^t S_s ds + \sigma \int_0^t S_s dB_s \end{aligned}$$

- or:

$$dS_t = \alpha S_t dt + \sigma S_t dB_t.$$

- **Example:** Ornstein-Uhlenbeck process (or Langevin equation):

$$dX_t = \mu X_t dt + \sigma dB_t$$

or

$$X_t = X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t dB_s.$$

- Note: in discrete form, we have

$$X_{t+1} = (1 + \mu) X_t + \sigma (B_{t+1} - B_t)$$

or

$$X_{t+1} = \phi X_t + Z_t,$$

with $\phi = 1 + \mu$ and $Z_t \sim N(0, \sigma^2)$. We have an autoregressive time series of order 1.

- **Example:** Ornstein-Uhlenbeck process (or Langevin equation):
- Let
$$Y_t = e^{-\mu t} X_t$$
or $Y_t = f(t, X_t)$ with $f(t, x) = e^{-\mu t} x$. By Itô formula,
$$Y_t = Y_0 + \int_0^t \left(-\mu e^{-\mu s} X_s + \mu e^{-\mu s} X_s + \frac{1}{2} \sigma^2 \times 0 \right) ds + \int_0^t \sigma e^{-\mu s} dB_s.$$
- Therefore,
$$X_t = e^{\mu t} X_0 + e^{\mu t} \int_0^t \sigma e^{-\mu s} dB_s.$$
- If $X_0 =$cte., this process is called the Ornstein-Uhlenbeck process.

- **Example:** The Geometric Brownian motion (again)
- Let

$$dS_t = \alpha S_t dt + \sigma S_t dB_t \tag{9}$$

or

$$S_t = S_0 + \alpha \int_0^t S_s ds + \sigma \int_0^t S_s dB_s. \tag{10}$$

- Assumption

$$S_t = e^{Z_t}.$$

or

$$Z_t = \ln\left(S_t\right).$$

- By the Itô formula, with $f(x) = \ln(x)$, we have

$$dZ_t = \frac{1}{S_t} dS_t + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)(dS_t)^2$$
$$= \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t.$$

- That is $Z_t = Z_0 + \left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t$and

$$S_t = S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right].$$

- In general, the solution of the homogeneous linear SDE

$$dX_t = \mu(t) X_t dt + \sigma(t) X_t dB_t$$

is

$$X_t = X_0 \exp\left[\int_0^t \left(\mu(s) - \frac{1}{2}\sigma(s)^2\right) ds + \int_0^t \sigma(s) dB_s\right].$$

Ornstein-Uhlenbeck process with mean reversion

$$dX_t = a\left(m - X_t\right)dt + \sigma dB_t,$$
$$X_0 = x.$$

$a, \sigma > 0$ and $m \in \mathbb{R}$.

- Solution of the associated ODE $dx_t = -ax_t dt$ is $x_t = xe^{-at}$.
- Consider the variable change $X_t = Y_t e^{-at}$ or $Y_t = X_t e^{at}$.
- By the Itô foemula applied to $f\left(t, x\right) = xe^{at}$, we have

$$Y_t = x + m\left(e^{at} - 1\right) + \sigma \int_0^t e^{as} dB_s.$$

Ornstein-Uhlenbeck process with mean reversion

- Therefore

$$X_t = m + (x - m) e^{-at} + \sigma e^{-at} \int_0^t e^{as} dB_s.$$

- This is a Gaussian process, since the random part is $\int_0^t f(s)\, dB_s$, where f is deterministic, so it is a Gaussian process.
- Mean:

$$E[X_t] = m + (x - m) e^{-at}$$

Ornstein-Uhlenbeck process with mean reversion

- Covariance: By Itô isometry

$$\begin{aligned}
\mathrm{Cov}\left[X_t, X_s\right] &= \sigma^2 e^{-a(t+s)} E\left(\int_0^t e^{ar} dB_r\right)\left(\int_0^s e^{ar} dB_r\right) \\
&= \sigma^2 e^{-a(t+s)} \int_0^{t \wedge s} e^{2ar} dr \\
&= \frac{\sigma^2}{2a}\left(e^{-a|t-s|} - e^{-a(t+s)}\right).
\end{aligned}$$

- Note that

$$X_t \sim N\left[m + (x-m)\, e^{-at}, \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)\right].$$

Ornstein-Uhlenbeck process with mean reversion

- When $t \to \infty$, the distribution of X_t converges to

$$\nu := N\left[m, \frac{\sigma^2}{2a}\right].$$

  which is the invariant or stationary distribution.
- Note that if X_0 has distribution ν then the distribution of X_t will be ν for all t.

Financial applications of the Ornstein-Uhlenbeck process with mean reversion

- Vasicek model for interest rate:

$$dr_t = a\left(b - r_t\right)dt + \sigma dB_t,$$

with a, b, σ real constants.

- Solution:

$$r_t = b + \left(r_0 - b\right)e^{-at} + \sigma e^{-at}\int_0^t e^{as}dB_s.$$

Financial applications of the Ornstein-Uhlenbeck process with mean reversion

- Black-Scholes model with stochastic volatility: assume that volatility $\sigma(t) = f(Y_t)$ is a function of anOrnstein-Uhlenbeck process with mean reversion :

$$dY_t = a(m - Y_t)\,dt + \beta dW_t,$$

where $\{W_t, 0 \leq t \leq T\}$ is a sBm.

- The SDE which models the asset price evolution is

$$dS_t = \alpha S_t dt + f(Y_t)\, S_t dB_t$$

where $\{B_t, 0 \leq t \leq T\}$ is a sBmand the sBm's W_t and B_t may be correlated, i.e.,

$$E[B_t W_s] = \rho(s \wedge t).$$

Important theoretical result

- Useful theoretical result:

Let $f : [0, +\infty) \to \mathbb{R}$ be a deterministic function. Then

1. $M_t = \exp\left(\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t (f(s))^2 ds\right)$ is a martingale
2. $\int_0^t f(s)dB_s$ has a normal distribution with mean 0 and variance $\int_0^t (f(s))^2 ds$.

- Part 1 is a simple generalization of the fact that $\exp\left(\lambda B_t - \frac{1}{2}\lambda^2 t\right)$ is a martingale.
- Part 2 follows from 1, because martingales have constant mean and $E[M_0] = 1$ and $E\left[\exp\left(\lambda\int_0^t f(s)dB_s\right)\right] = \exp\left(\frac{1}{2}\lambda^2\int_0^t (f(s))^2 ds\right)$, which is the moment generating function of the $N\left(0, \int_0^t (f(s))^2 ds\right)$ distribution.

AR(1) and mean reverting OU process

- Consider the AR(1) process:

$$X_t = \phi X_{t-1} + Z_t,$$

with $Z_t \sim N\left(0, \sigma_e^2\right)$ and t is the discrete time.

- Then

$$E\left[X_t\right] = \phi^t X_0,$$

$$Var\left[X_t\right] = \sigma_e^2 \frac{\left(1-\phi^{2t}\right)}{1-\phi^2}.$$

- These coincide with the values of the mean-reverting Ornstein-Uhlenbeck with $m = 0$ if we put $\phi = e^{-a}$ and $\frac{\sigma_e^2}{1-\phi^2} = \frac{\sigma^2}{2a}$.
- The mean-reverting Ornstein-Uhlenbeck process is the continuous equivalent of a AR(1) process such as sBm is the continuous equivalent of a random walk.