PART II

INTEREST RATE MODELS

1

STATIC INTEREST RATE MODELS

1.1. INTRODUCTION

FUNDAMENTAL ASSET PRICING FORMULA

$$P = \sum_{t=1}^{N} \frac{C_t}{(1+s_t)^t} + \frac{M}{(1+s_N)^N}$$

Main question: Where do we get s_t from?

- Any relevant information concerning how to price a financial asset must be primarily obtained from market sources

- Spot rate - annualized rate of a pure discount (or zero-coupon) bond

- As there aren't enough zero-coupon bonds for most countries and currencies, this information will have to be extracted from coupon-paying bonds.

1.2. Fitting the TS of Interest Rates

1.2.1. Direct Methods

- Bootstrapping method
- Carleton and Cooper (1976)
- Interpolation methods:
  - Linear
  - polynomial

BOOTSTRAPPING

- Consider 2 securities (nominal value = 100€):
  - 1-year pure discount bond selling at 95€.
  - 2-year 8% bond selling at 99€, with annual coupon payments.

- 1-year spot rate:

$$95 = \frac{100}{(1+R_{0,1})}; \; R_{0,1} = 5.26\%$$

- 2-year spot rate:

$$99 = \frac{8}{(1+R_{0,1})} + \frac{108}{(1+R_{0,2})^2} = \frac{8}{1.0526} + \frac{108}{(1+R_{0,2})^2};$$

$$R_{0,2} = 8.7\%$$

- The same type of reasoning can be developed for any number of bonds, e.g. 4 bonds (p(0,*k*) is the current discount factor for cash-flows to be paid *k* years from now):

	Coupon	Maturity (year)	Price
Bond 1	5	1	101
Bond 2	5.5	2	101.5
Bond 3	5	3	99
Bond 4	6	4	100

- Solve the following system:

$$101 = 105\, p(0,1)$$
$$101.5 = 5.5\, p(0,1) + 105.5\, p(0,2)$$
$$99 = 5\, p(0,1) + 5\, p(0,2) + 105\, p(0,3)$$
$$100 = 6\, p(0,1) + 6\, p(0,2) + 6\, p(0,3) + 106\, p(0,4)$$

- and obtain

p(0,1)=0.9619, p(0,2)=0.9114, p(0,3)=0.85363, p(0,4)= 0.7890

R(0,1)=3.96%, R(0,2)=4.717%, R(0,3)=5.417%, R(0,4)=6.103%

- A usual practical way to estimate the yield curve involves the employment of interbank money market rates for several maturities:

Maturity	ZC
Overnight	4.40%
1 month	4.50%
2 months	4.60%
3 months	4.70%
6 months	4.90%
9 months	5.00%
1 year	5.10%

	Coupon	Maturity (years)	Price
Bond 1	5%	1 y and 2 m	103.7
Bond 2	6%	1 y and 9 m	102
Bond 3	5.50%	2 y	99.5

- 1 year and 2 months rate

  x=5.41%

$$103.7 = \frac{5}{(1+4.6\%)^{1/6}} + \frac{105}{(1+x)^{1+1/6}}$$

CONCLUSIONS

- If one can find different bonds with coincident cash-flow dates and one of them only has one remaining cash-flow date, then one can get the spot rates directly.

- These rates are not yields (except for the shortest bond) and consequently they do not face their consistency problems.

- Therefore, **we have a single spot rate for each maturity** and the yield curve may have any shape.

- One can also calculate spot rates by using money market rates.

Carlton and Cooper (1976)

- Estimation of the discount factors by OLS method if the number of bonds is larger than the number of discount factors to be estimated.

$$\underset{(ix1)}{P} = \underset{(ixt)}{CF} \cdot \underset{(tx1)}{d}$$

Where

i = 1, …, *k* - riskless government bonds considered

t = 1,…, *n* - the cash-flows for which the discount factors are to be calculated.

P = vector of the prices of the *i* bonds (a column vector with *i* rows);

CF = matrix of the cash-flows of the *i* bonds for the *t* cash-flows (*i* rows and *t* columns);

d = vector of the discount factors for the *t* cash-flows (a column vector with *t* cash-flows).

CARLTON AND COOPER (1976)

- This method has several drawbacks:

(i) it only allows the estimation of some points of the discount function (for the maturities of the cash-flows considered);

(ii) it does not impose any smoothness on the discount function, allowing meaningless shapes; and

(iii) It faces multicolinearity problems resulting from the linear dependence between the cash-flows of, at least, some of the securities considered.

INTERPOLATION - LINEAR

- Interpolations may be useful if we don't have all market information required to get spot rates for the same maturities.
- Simplest approach - linear interpolations:
  - Assuming that we know discount rates for maturities t_1 and t_2, the rate for maturity t, being $t_1 < t < t_2$, corresponds to the weighted average of the adjacent rates, being the weights higher for the maturity closer to t (e.g. if $t = t_2$, t_1 will not have any relevance to calculate t):

$$R(0,t) = \frac{(t_2 - t)R(0,t_1) + (t - t_1)R(0,t_2)}{(t_2 - t_1)}$$

  - Linear interpolations provide good proxies for near maturities.
  - However, for distant maturities, the shape of the resulting yield curve tends to be kinked.
  - By definition, linear interpolation doesn't allow to get estimates for maturities longer than those observed.

INTERPOLATION – POLYNOMIAL

- **Polynomial interpolations of the interest rates** allow to obtain smoother yield curves, with interest rates as polynomial functions of maturities.
- A very common polynomial interpolation is the cubic => one can estimate the full term structure just by knowing the spot rates for 4 maturities.
- Therefore, if $R(0, t_1)$, $R(0, t_2)$, $R(0, t_3)$ and $R(0, t_4)$ are known, one can solve the following system in order to the 4 coefficients of the 3rd order polynomial.

$$\begin{cases} R(0,t_1) = at_1^3 + bt_1^2 + ct_1 + d \\ R(0,t_2) = at_2^3 + bt_2^2 + ct_2 + d \\ R(0,t_3) = at_3^3 + bt_3^2 + ct_3 + d \\ R(0,t_4) = at_4^3 + bt_4^2 + ct_4 + d \end{cases} \Rightarrow R = T \cdot A \text{ , being}$$

$$R = \begin{bmatrix} R(0,1) \\ R(0,2) \\ R(0,3) \\ R(0,4) \end{bmatrix}, T = \begin{bmatrix} t_1^3 & t_1^2 & t_1 & 1 \\ t_2^3 & t_2^2 & t_2 & 1 \\ t_3^3 & t_3^2 & t_3 & 1 \\ t_4^3 & t_4^2 & t_4 & 1 \end{bmatrix}, A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

- If one uses more than 4 spot rates, these coefficients are estimated by econometric techniques (as we will have degrees of freedom), e.g. ordinary least squares (as the functions are linear in the coefficients).
- Otherwise, $R = T \cdot A \Leftrightarrow A = T^{-1} \cdot R$

EXAMPLE

- The calculation of *a*, *b*, *c* and *d* allows to obtain the spot rate for any maturity *t*: $R(0,t) = at^3 + bt^2 + ct + d$
- Assuming the following rates are known:
  - R(0,1) = 3%
  - R(0,2) = 5%
  - R(0,3) = 5.5%
  - R(0,4) = 6%

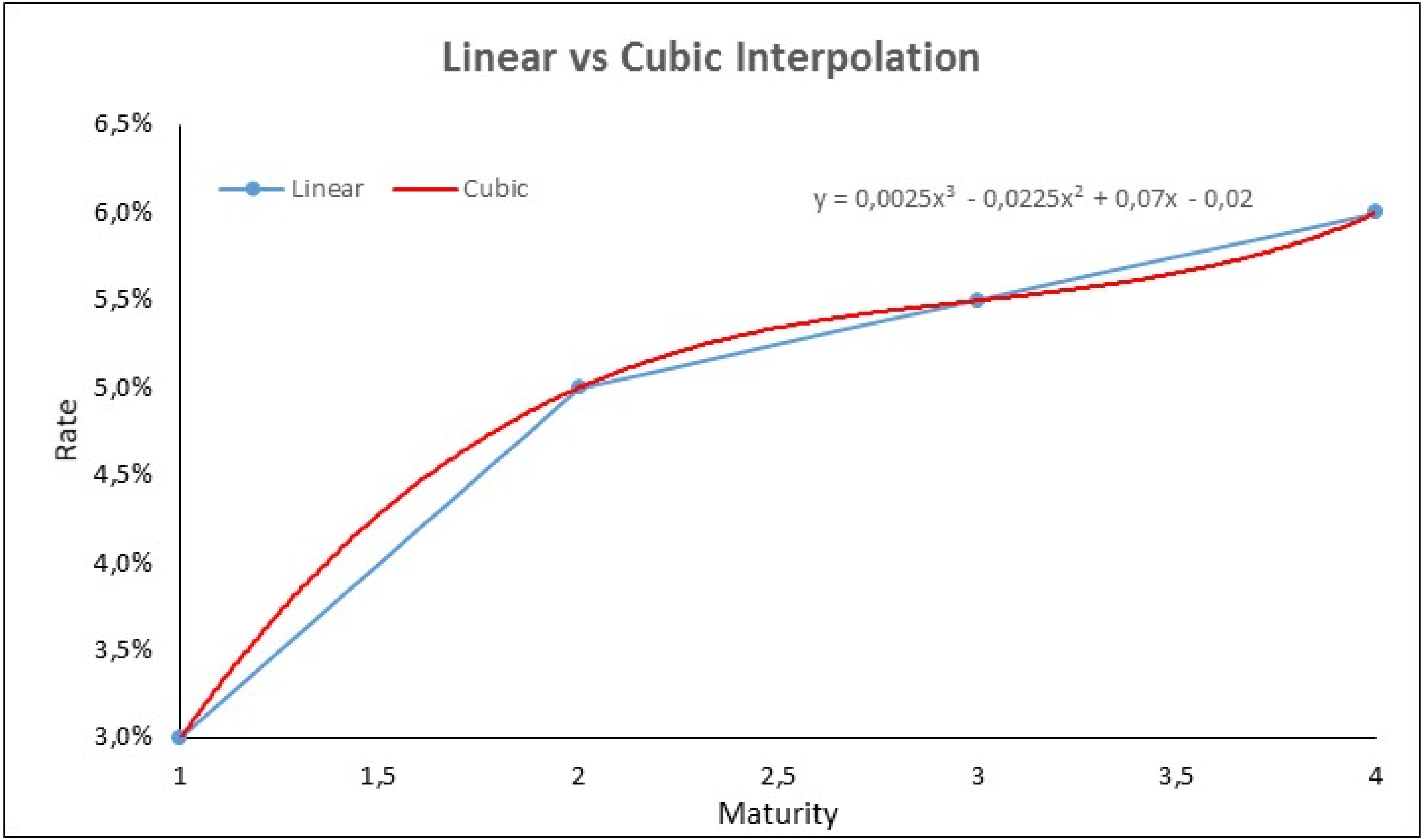
$$\begin{cases} R(0,1) = a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d \\ R(0,2) = a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d \\ R(0,3) = a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d \\ R(0,4) = a \cdot 4^3 + b \cdot 4^2 + c \cdot 4 + d \end{cases}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3\% \\ 5\% \\ 5.5\% \\ 6\% \end{pmatrix} = \begin{pmatrix} 0.0025 \\ -0.0225 \\ 0.07 \\ -0.02 \end{pmatrix}$$

- Goal - Compute the 2.5 year rate:

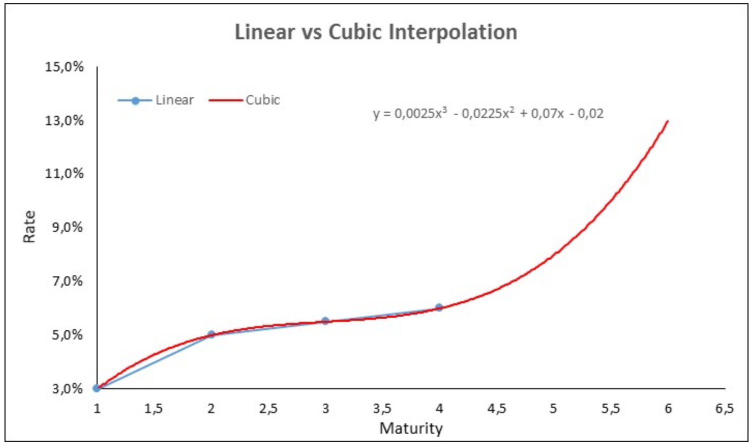
$R(0,2.5) = a \times 2.5^3 + b \times 2.5^2 + c \times 2.5^1 + d = 5.34375\%$

ILLUSTRATION: LINEAR VERSUS CUBIC



CONCLUSIONS

- The resulting spot curve using 3rd order polynomial methods tends to be too irregular, namely when:
  - one uses it to estimate a rate for a maturity much higher than the maximum maturity used to calculate the polynomial coefficients (e.g. in the previous example the 10-year would be 93%!)
  - the difference between two consecutive maturities is too large.



- **Polynomial splines** improve the adjustment, by allowing different specifications for the polynomials in different maturity buckets.
- Nonetheless, the explosive behavior of the resulting curves is kept.

1.2.2. Spline Methods

Polynomial Functions

- Discount factors (*p*) as polynomial functions of the maturity (*s*), with all coefficients differing in the different maturity buckets:

$$p(s) = \begin{cases} p_0(s) = d_0 + c_0 s + b_0 s^2 + a_0 s^3, s \in [0,5] \\ p_5(s) = d_1 + c_1 s + b_1 s^2 + a_1 s^3, s \in [5,10] \\ p_{10}(s) = d_2 + c_2 s + b_2 s^2 + a_2 s^3, s \in [10,20] \end{cases}$$

- Imposing continuity constraints and given the fact that the discount factor for zero maturity is 1, the number of parameters is reduced:

$$p_0(5) = p_5(5)$$
$$p_5(10) = p_{10}(10)$$
$$p_0(0) = 1$$

- The number of parameters may be even further reduced if it is assumed that only one of the parameters is different in the several maturity buckets => McCulloch (1971, 1975) splines.

POLYNOMIAL SPLINES

- Dividing the maturity spectrum in *k-2* intervals, with *k-3* vertices, the discount function can be defined as a cubic function, adding a factor (spline) to the 3rd order component, being k = No. of parameters:

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + \sum_{h=1}^{k-3} a_{4,h+1}(t - t_h)^3 \cdot D_h(t)$$

where $D_h(t)$ for h=1,2,…, k-3 are functions defined on the basis of the vertices of the intervals, as follows:

$D_h(t)$ =0, if $t<t_h$, $D_h(t)$ =1, if $t \geq t_h$, for h=1,….,k-3.

- The discount function is continuous ⇔ for all vertices, the values for the discount function are given as: $d(t) = a_0 + \sum_{h=1}^{k} a_h g_h(t)$

POLYNOMIAL SPLINES

- **How to choose the number of parameters/intervals and the vertices:**

  – McCulloch proposes *k* = square root of the number of observations (bonds), rounded to the nearest integer, with the vertices chosen to ensure all intervals have the same No. observations (or the difference between the No. observations in each interval is not higher than 1).

  – Alternative methodology (used more often) - fixing the vertices of the intervals in maturity dates corresponding to the maturities in which the market is traditionally “divided”: 1, 3, 5 and 10 years.

POLYNOMIAL SPLINES

- If the vertices of the intervals correspond to the maturities in which the market is traditionally "divided" - 1, 3, 5 and 10 years – we have:
  - No. Intervals: k-2 = 5 (0-1, 1-3, 3-5, 5-10 and > 10y)
  - No. Vertices: k-3 = 4 (1, 3, 5 and 10)

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + \sum_{h=1}^{k-3} a_{4,h+1}(t - t_h)^3 \cdot D_h(t)$$

$$d(t) = 1 + a_{2,1}t + a_{3,1}t^2 + a_{4,1}t^3 + a_{4,2}(t-1)^3 \cdot D_1(t) + a_{4,3}(t-3)^3 \cdot D_2(t)$$
$$+ a_{4,4}(t-5)^3 \cdot D_3(t) + a_{4,5}(t-10)^3 \cdot D_4(t)$$

$D_1(t) = 0$, if *t<1*, $D_1(t) = 1$, if $t \geq 1$ $D_2(t) = 0$, if *t<3*, $D_2(t) = 1$,if $t \geq 3$
$D_3(t) = 0$, if *t<5*, $D_3(t) = 1$, if $t \geq 5$ $D_4(t) = 0$, if *t<10*, $D_4(t) = 1$,
if $t \geq 10$

POLYNOMIAL SPLINES

- The method of polynomial splines provides us better estimates in sample, i.e. up to the longest observed maturity, comparing to polynomial functions.
- However, the estimation problems outside the sample remain, as the discount function tends to assume irregular shapes from the longest maturity onwards, and it may even become negative.
- Whenever the yield curve assumes complex shapes, the use of a high number of parameters leads the estimated curve to adjust excessively to outliers => yield curve becomes even more irregular.
- This is particularly inconvenient if the objective is, as it usually happens, the estimation of the term structure of interest rates for a fixed or standardised range of maturities, or to calculate forward rates.
- Therefore, more complex specifications will be required.

1.2.3. Deterministic Methods

- 3 steps:

  - **Step 1**: select a set of K bonds with prices P^j paying cash-flows $F^j(t_i)$ at dates $t_i > t$
  - **Step 2**: select a deterministic interest rate model for the functional form of the discount factors $p(t,t_i;\beta)$, or the discount rates $R(t,t_i;\beta)$ (or alternatively spot or forward rates), where β is a vector of unknown parameters, and generate prices.

$$\hat{P}^j(t) = \sum_{i=1}^{N} CF^j(t_i) p(t,t_i;\beta) = \sum_{i=1}^{N} CF^j(t_i) e^{-(t_j - t)R(t,t_i;\beta)}$$

  - **Step 3**: estimate the parameters β as the ones making the theoretical prices as close as possible to market prices:

$$\beta = \arg\min \sum_{j=1}^{K} \left(\hat{P}^j(t) - P^j(t) \right)^2$$

- **Key advantages:**

  – Parsimonious models, i.e. do not involve many parameters

  – Ensure stable functions

  – Adjust to many possible shapes of the TS

  – Some parameters have economic interpretation

NELSON AND SIEGEL (1987)

- Nelson and Siegel (1987) proposed to fit the term structure using a flexible, smooth parametric function.
- They demonstrated that the proposed model is capable of capturing many of the typically observed shapes that the yield curve assumes over time.
- The resulting Nelson-Siegel forward curve can be assumed to correspond to a 3 unobserved factor model (as pointed out in Diebold and Li (2005)):

$$ {}_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau)} + \beta_2 \cdot \left[(m/\tau) \cdot e^{(-m/\tau)} \right] $$

$$ s_m = \beta_0 + (\beta_1 + \beta_2) \cdot \left[1 - e^{(-m/\tau)} \right] / (m/\tau) - \beta_2 \cdot \left[e^{(-m/\tau)} \right] $$

$$ d_m = e^{\left[-\beta_o m - (\beta_1 + \beta_2)\tau \left(1 - e^{-\frac{m}{\tau}} \right) + \beta_2 m \cdot e^{\left(-\frac{m}{\tau} \right)} \right]} $$

β_0 : level parameter - the long-term spot or instantaneous forward rate (lim *s* or *f* , *m*-> ∞)

$\beta_0 + \beta_1$: short-term rate (lim *s* or *f* , *m*-> *0*)

β_1 : (-) slope parameter

β_2: curvature parameter

τ : influences the speed of convergence of the curve towards the asymptotic value.

$\left(1-\frac{\beta_1}{\beta_2}\right)\tau$: point of inflection of the slope of the forward curve

$\left(2-\frac{\beta_1}{\beta_2}\right)\tau$: point of inflection of the concavity of the forward curve

SVENSSON (1994)

- Nelson-Siegel model faces estimation difficulties whenever the yield curve has more than one point of inflection of the slope or concavity.
- This is usually observed after disturbances in money markets.

↓

- Several more flexible NS specifications have been proposed in the literature to improve the fit to more complex shapes, namely with multiple inflection points, introducing additional factors and parameters.
- A popular term-structure estimation method among central banks (see BIS, 2005) to address is the 4-factor Svensson (1994) model, that allows to lead with two changes in the slope or in the concavity.

↓

- Svensson (1994) proposes to increase the flexibility and fit of the NS model by adding a second hump-shape factor with a separate decay parameter.

SVENSSON (1994)

- The resulting 4-factor forward curve is given by:

$$ {}_{m}f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau_1) \cdot e^{(-m/\tau_1)} \right] + \beta_3 (m/\tau_2) e^{(-m/\tau_2)} $$

- Thus, the spot rate will be given by the following expression:

$$ \begin{aligned} s_m &= b_0 + b_1 \cdot \left[1 - e^{(-m/\tau_1)} \right] / (m/\tau_1) + \\ &+ b_2 \cdot \left\{ \left[1 - e^{(-m/\tau_1)} \right] / (m/\tau_1) - e^{(-m/\tau_1)} \right\} + \\ &+ b_3 \cdot \left\{ \left[1 - e^{(-m/\tau_2)} \right] / (m/\tau_2) - e^{(-m/\tau_2)} \right\} \end{aligned} $$

β_0 : level parameter - the long-term rate

$\beta_0 + \beta_1$: short-term rate

β_1 : (-) slope parameter

β_2, β_3: curvature parameters

τ_1, τ_2 : influences the speed of convergence of the curve towards the asymptotic value.

PROPERTIES

- Even though the Svensson method is more adequate to estimate the term structure of interest rates for monetary policy purposes, given its higher adjustment capacity in the segment of the shorter maturities, **when the yield curve assumes simple shapes in the short segment, the estimation by the NS method seems preferable since it is more parsimonious.**

- In fact, the NS model is a restricted version of the Svensson model with the restriction $\beta_3 = 0$ and/or $\tau_2 \rightarrow 0$. Thus, we can test the null hypothesis corresponding to those restrictions:

$$H_0: \quad \beta_0 = \beta_1 = \ldots = \beta_q = 0$$

where: v = likelihood function of the adjustment with restrictions; v^* = likelihood function of the adjustment without restrictions; q = number of restrictions.

- The test is based on the following log-likelihood ratio test:

$$\lambda = -2 \cdot (\ln v - \ln v^*) \approx \chi^2(q)$$

PROPERTIES

- In this case, *v* corresponds to the likelihood function of the NS model (the restricted model), while *v** is the likelihood function of the Svensson model.

- Thus, if the logarithm of the likelihood function of the Svensson model is large enough (i.e., is statistically above that of the NS model), the Svensson model will be selected.

- **Ho is rejected if $\lambda > \chi^2 \Leftrightarrow$ Svensson model must be chosen.**

- A potential problem with the Svensson model is that it is highly non-linear, which can make the estimation of the model difficult (see Bolder and Stréliski (1999) for a discussion).

- Nonetheless, one can implement it even in a spreadsheet!

BJÖRK AND CHRISTENSEN

- One alternative model to Nelson-Siegel and Svensson was developed by Bjork, T. and Christensen B.J. (1999): "Interest rate dynamics and consistent forward rate curves", Mathematical Finance.

- Björk and Christensen (1999) proposed a model very similar to Svensson, by also adding a 4th factor to the instantaneous forward curve, but with a different specification for this 4th factor, that depends on a parameter (τ) that is the same in the 3rd factor:

$$f_t(\tau) = \beta_{1,t} + \beta_{2,t}\exp\left(-\frac{\tau}{\lambda_t}\right) + \beta_{3,t}\left(\frac{\tau}{\lambda_t}\right)\exp\left(-\frac{\tau}{\lambda_t}\right) + \beta_{4,t}\exp\left(-\frac{2\tau}{\lambda_t}\right)$$

$$_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau_1)\cdot e^{(-m/\tau_1)}\right] + \beta_3 (m/\tau_2) e^{(-m/\tau_2)}$$

Svensson (1994)

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left[\frac{1-\exp\left(-\frac{\tau}{\lambda_t}\right)}{\left(\frac{\tau}{\lambda_t}\right)}\right] + \beta_{3,t}\left[\frac{1-\exp\left(-\frac{\tau}{\lambda_t}\right)}{\left(\frac{\tau}{\lambda_t}\right)} - \exp\left(-\frac{\tau}{\lambda_t}\right)\right]$$

$$+ \beta_{4,t}\left[\frac{1-\exp\left(-\frac{2\tau}{\lambda_t}\right)}{\left(\frac{2\tau}{\lambda_t}\right)}\right]$$

The 4th component, resembles the 2nd component, as it also mainly affects short-term maturities.
The difference is that it decays to zero at a faster rate.

BC (1999) FOUR-FACTOR - PROPERTIES

- The factor in $\beta_{4,t}$ can be interpreted as a second slope factor.
- As a result, the Björk and Christensen model captures the slope of the term structure by the (weighted) sum of $\beta_{2,t}$ and $\beta_{4,t}$.
- The instantaneous short rate in this case is given by :

$$y_t(0) = \beta_{1,t} + \beta_{2,t} + \beta_{4,t}$$

BLISS (1997)

- A second option to make the Nelson-Siegel more flexible is through relaxing the restriction that the slope and curvature component should be governed by the same decay parameter τ.

- Bliss (1997) estimates the term structure of interest rates with the 3-factor model that allows for 2 different decay parameters τ_1 and τ_2.

- The forward and spot curves are then given by:

$$ {}_m f_0 = \beta_0 + \beta_1 \cdot e^{(-m/\tau_1)} + \beta_2 \cdot \left[(m/\tau) \cdot e^{(-m/\tau_2)} \right] $$

$$ s_m = \beta_0 + \beta_1 \cdot \left[1 - e^{(-m/\tau_1)} \right] / (m/\tau_1) + \beta_2 \cdot \left[\left[1 - e^{(-m/\tau_2)} \right] / (m/\tau_2) - \left[e^{(-m/\tau_2)} \right] \right] $$

DIEBOLD, PIAZZESI AND RUDEBUSCH (2005)

- **Conversely, some authors argue that even the NS model has to many parameters to be estimated.**
- Litterman and Scheinkman (1991)* show that the variation in interest rates can be explained by a small number of underlying common factors, typically up to three, interpreted as level, slope and curvature.
- The 1st factor explains 89,5% of the total variance of returns, the 2nd factor for 8,5% and the 3rd for the remaining 2%.
- For this reason, Diebold, Piazzesi, and Rudebusch (2005)*2 examine a 2-factor Nelson-Siegel model, even though they recognize that more than 2 factors will "be needed in order to obtain a close fit to the entire yield curve at any point in time", e.g. for pricing derivatives.

* Litterman, Robert and José Scheinkman (1991), "Common Factors Affecting Bond Returns", Journal of Fixed Income.

*2 Diebold, Francis X., Monika Piazzesi and Glenn D. Rudebusch (2005), "Modeling Bond Yields in Finance and Macroeconomics", American Economic Review, 95, pp. 415-420.

DIEBOLD, PIAZZESI AND RUDEBUSCH (2005)

- Compared to the 3-factor Nelson-Siegel model, the 2-factor model only contains the level and slope factor => only 3 parameters have to be estimated:

$$s_m = \beta_0 + \beta_1 \cdot \left[1 - e^{(-m/\tau)}\right]/(m/\tau)$$

CONCLUSIONS

- Despite the drawback that **they lack theoretical underpinnings**, the BIS reported that 9 out of 13 central banks which report their curve estimation methods to the BIS use deterministic Interest Rate Models (BIS (2005), "Zero-coupon yield curves: technical documentation", BIS Papers, No 25, Monetary and Economic Department, October 2005).
- According to this study, **most central banks reporting data have adopted either the Nelson and Siegel (1987) model or the extended version suggested by Svensson (1994).** Exceptions are Canada, Japan, Sweden, UK and the US, which all apply variants of the "smoothing splines" method.
- Deterministic interest rate models are also widely used among market practitioners.
- Given that these models are usually non-linear in the parameters, attention has to be paid to their starting values.