Models in Finance - Slides 7 - Stochastic Interest Rate models

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Stochastic Interest Rate models - Introduction

- Financial Contracts are often of a long-term nature.
- There may be considerable uncertainty about economic and investment conditions over the duration of the contract.
- One should adopt a conservative basis when calculations are performed with deterministic fixed rates of interest (example: determine premium rates on the basis of one fixed rate of return - one should use a conservative basis for the rate to be used in any calibration).

Introduction to stochastic interest rates

- Alternative approach to deterministic interest rates: use stochastic interest rate models.
- Example: Investor wishes to invest a sum of P into a fund with compound investment rate growth at a constant rate for n years. This investment return is not known. It could be assumed that the mean rate will apply. However, the accumulated value using the mean rate of return will not equal the mean accumulated value:

$$P\left(1+\sum_{j=1}^{k} i_j p_j\right)^n \neq P\sum_{j=1}^{k}\left(p_j\left(1+i_j\right)^n\right),$$

  where i_j is the jth of k possible investment rates of return and p_j is the associated probability. ,
- A more flexible model is provided by assuming that each single year, the annual yield is independent from yields in previous years and being determined by a given probability distribution: independent annual rates of return.

Independent annual rates of return

- Consider time interval $[0, n]$ subdivided into periods $[0, 1], [1, 2], ..., [n-1, n]$. Let i_t be the yield obtainable over the period $[t-1, t]$. Assume the money is invested at the beginning of each year.
- Let F_t denote the accumulated amount at time t of all money invested before t and let P_t be the amount invested at time t.
- Then

$$F_t = (1 + i_t)(F_{t-1} + P_{t-1}). \tag{1}$$

- A single investment of 1 at time 0 will accumulate at time n to

$$S_n = (1 + i_1)(1 + i_2) \cdots (1 + i_n). \tag{2}$$

Independent annual rates of return

- Similarly, a series of annual investments of 1 at times $0, 1, 2, ..., n-1$, will accumulate at time n to:

$$\begin{aligned} A_n = & \left(1+i_1\right)\left(1+i_2\right)\cdots\left(1+i_n\right) \\ & +\left(1+i_2\right)\left(1+i_3\right)\cdots\left(1+i_n\right) \\ & +\left(1+i_3\right)\cdots\left(1+i_n\right) \\ & \vdots \\ & +\left(1+i_n\right). \end{aligned} \tag{3}$$

- A_n and S_n are random variables, each with its own probability distribution. Example: if the yields i_t can take the values $0.02, 0.04$ and 0.06, each with probability $1/3$, then the value of S_n will be between $(1.02)^n$ and $(1.06)^n$, each of these extreme values will occur with probability $(1/3)^n$.

Moments of Sn

- In general, a theoretical analysis of the distribution functions of A_n and S_n may be difficult. It is often useful to use simulation techniques for practical problems. However, the moments of A_n and S_n can be found relatively simply in terms of the moments of the distribution for the yield each year.
- From Eq. (2), we obtain

$$\mathbb{E}\left[(S_n)^k\right] = \mathbb{E}\left[\prod_{t=1}^{n}(1+i_t)^k\right] = \prod_{t=1}^{n}\mathbb{E}\left[(1+i_t)^k\right]$$

- For example, assume that the yield each year has mean j and variance s^2. Then

$$\mathbb{E}\left[S_n\right] = \prod_{t=1}^{n}\mathbb{E}\left[(1+i_t)\right] = (1+j)^n . \tag{4}$$

Moments of Sn

- Assume that the yield each year has mean j and variance s^2. Then

$$\mathbb{E}\left[(S_n)^2\right] = \prod_{t=1}^{n}\mathbb{E}\left[(1+i_t)^2\right] = \left(1+2\mathbb{E}\left[i_t\right]+\mathbb{E}\left[i_t^2\right]\right)^n$$
$$= \left(1+2j+j^2+s^2\right)^n.$$

- Moreover, from the previous computations is easy to see that

$$Var\left[S_n\right] = \left(1+2j+j^2+s^2\right)^n - (1+j)^{2n}. \tag{5}$$

- These arguments can be extended to derive the higher moments of S_n in terms of the higher moments of i_t.

Moments of An

- From Eq. (3), one obtains that
$$A_n = (1 + i_n)(1 + A_{n-1}). \tag{6}$$
- It is easy to see that i_n and A_{n-1} are independent.
- The previous equation allows to obtain a recurrence relation from which the moments of A_n may be obtained.
- Example: we will obtain the mean and variance of A_n.
- Let $\mu_n = \mathbb{E}[A_n]$ and $m_n = \mathbb{E}[A_n^2]$. Consider again that the mean and variance of i_t are j and s^2, respectively.
- Since
$$A_1 = 1 + i_1,$$
it follows that
$$\mu_1 = 1 + j,$$
$$m_1 = 1 + 2j + j^2 + s^2.$$

Moments of An

- Taking expectations of Eq. (6), we obtain

$$\mu_n = (1+j)\,(1+\mu_{n-1})\,.$$

- This equation combined with initial value $\mu_1 = 1+j$ has the solution

$$\mu_n = (1+j) + (1+j)^2 + \cdots + (1+j)^n = (1+j)\,\frac{(1+j)^n - 1}{j}. \quad (7)$$

- In actuarial notation:

$$\mu_n = \ddot{s}_{\overline{n|}j} \quad \text{(at rate } j\text{)}.$$

Moments of An

- Since
$$A_n^2 = \left(1 + 2i_n + i_n^2\right)\left(1 + 2A_{n-1} + A_{n-1}^2\right),$$
by taking expectations we obtain
$$m_n = \left(1 + 2j + j^2 + s^2\right)\left(1 + 2\mu_{n-1} + m_{n-1}\right). \tag{8}$$

- As the value of μ_n is known, the previous equation provides a recurrence relation for the calculations of $m_2, m_3, m_{4,\ldots}$
-
$$Var\left[A_n\right] = m_n - \mu_n^2. \tag{9}$$

- These arguments can be extended to provide recurrence relations for the higher moments of A_n.

Example

- Company considers that on average will earn interest at the rate of 4% p.a. However, in any one year the yield is equally likely to take any value between 2% and 6%. For both single and annual premium accumulations with term of 5 years and single (or annual investment) of 1 Euro, find the mean accumulation and the standard deviation of the accumulation at maturity date.

- Solution: Annual rate of return uniformly distributed on the interval $[0.02, 0.06]$. The mean annual rate of interest is $j = 0.04$. The variance of the annual rate of return is

$$s^2 = \int_{0.02}^{0.06} \frac{(x - 0.04)^2}{0.06 - 0.02} dx = 25 \frac{(0.06 - 0.04)^3 - (0.02 - 0.04)^3}{3}$$
$$= 1.333 \times 10^{-4}.$$

Example

- We have to find $\mathbb{E}[A_n], (Var[A_n])^{1/2}, \mathbb{E}[S_n], (Var[S_n])^{1/2}$ for $n = 5$.
- By Eq. (4) and Eq. (5),

$$\mathbb{E}[S_5] = (1+0.04)^5 = 1.2167,$$

$$(Var[S_5])^{1/2} =$$

$$= \left[\left(1 + 0.08 + (0.04)^2 + 1.333 \times 10^{-4}\right)^5 - (1+0.04)^{10}\right]^{1/2}$$

$$= 0.03021.$$

- By Eq. (7) and by Eq. (8)-(9)

$$\mathbb{E}[A_5] = \mu_5 = (1+0.04)\frac{(1+0.04)^5 - 1}{0.04} = 5.633,$$

$$(Var[A_5])^{1/2} = 0.09443.$$