Exercise 13.2 **The term structure and regime switching**, donated by Rodolfo Manuelli

Consider a pure exchange economy where the stochastic process for consumption is given by

$$c_{t+1} = c_t \exp\left[\alpha_0 - \alpha_1 s_t + \varepsilon_{t+1}\right],$$

where

(i) $\alpha_0 > 0$, $\alpha_1 > 0$, and $\alpha_0 - \alpha_1 > 0$.

(ii) ε_t is a sequence of i.i.d. random variables distributed $N(\mu, \tau^2)$. Note: given this specification, it follows that $E[e^{\varepsilon}] = \exp[\mu + \tau^2/2]$.

(iii) s_t is a Markov process independent from ε_t that can take only two values, $\{0, 1\}$. The transition probability matrix is completely summarized by

$$\begin{aligned} \text{Prob}\left[s_{t+1} = 1 | s_t = 1\right] &= \pi(1), \\ \text{Prob}\left[s_{t+1} = 0 | s_t = 0\right] &= \pi(0). \end{aligned}$$

(iv) The information set at time t, Ω_t, contains $\{c_{t-j}, s_{t-j}, \varepsilon_{t-j}; j \geq 0\}$.

There is a large number of individuals with the following utility function

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u(c) = c^{(1-\sigma)}/(1-\sigma)$. Assume that $\sigma > 0$ and $0 < \beta < 1$. As usual, $\sigma = 1$ corresponds to the log utility function.

a. Compute the "short-term" (one-period) interest rate.

b. Compute the "long-term" (two-period) interest rate measured in the same time units as the rate you computed in a. (That is, take the appropriate square root.)

c. Note that the log of the rate of growth of consumption is given by

$$\log(c_{t+1}) - \log(c_t) = \alpha_0 - \alpha_1 s_t + \varepsilon_{t+1}.$$

Thus, the conditional expectation of this growth rate is just $\alpha_0 - \alpha_1 s_t + \mu$. Note that when $s_t = 0$, growth is high, and when $s_t = 1$, growth is low. Thus, loosely speaking, we can identify $s_t = 0$ with the peak of the cycle (or good times) and $s_t = 1$ with the trough of the cycle (or bad times). Assume $\mu > 0$. Go as far as you can describing the implications of this model for the cyclical behavior of the term structure of interest rates.

d. Are short term rates pro- or countercyclical?

e. Are long rates pro- or countercyclical? If you cannot give a definite answer to this question, find conditions under which they are either pro- or countercyclical, and interpret your conditions in terms of the "permanence" (you get to define this) of the cycle.

Solution

a. We use the formula derived in chapter 10. Specifically:

$$\frac{1}{R_{1t}} = E_t\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right) \tag{92}$$

$$\frac{1}{R_{1t}} = \beta\exp(-\sigma\alpha_0 + \sigma\alpha_1 s_t)E_t\left(\exp(-\sigma\varepsilon_{t+1})\right) \tag{93}$$

$$\frac{1}{R_{1t}} = \beta\exp(-\sigma\alpha_0 + \sigma\alpha_1 s_t - \sigma\mu + \sigma^2\tau^2/2), \tag{94}$$

where the last equality follows from the fact that $-\sigma\varepsilon_{t+1}$ is normal with mean $-\sigma\mu$ and variance $\sigma^2\tau^2$.

b. Using again the formula of chapter 10 we have :

$$\frac{1}{R_{2t}^2} = E_t\left(\beta^2\left(\frac{c_{t+2}}{c_{t+1}}\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right) \tag{95}$$

$$\frac{1}{R_{2t}^2} = \beta^2 E_t\left(\exp(-\sigma\alpha_0 + \sigma\alpha_1 s_{t+1} + \sigma\varepsilon_{t+2})\right)\exp(-\sigma\alpha_0 + \sigma\alpha_1 s_t + \sigma\varepsilon_{t+1}) \tag{96}$$

$$\frac{1}{R_{2t}^2} = \beta^2\exp\left(2(-\sigma\alpha_0 - \sigma\mu + \sigma^2\tau^2/2)\right)E_t\left(\exp(\sigma\alpha_1(s_t + s_{t+1}))\right). \tag{97}$$

Observe that either $s_{t+1} = s_t$, or $s_{t+1} = 1 - s_t$. Therefore, we can write:

$$E_t\left(\exp(\sigma\alpha_1 s_{t+1})\right) = \exp(\sigma\alpha_1 s_t) \times \left[\pi(s_t|s_t) + \pi(1-s_t|s_t)\exp(\sigma\alpha_1(1-2s_t))\right].$$

This yields to the following two expressions for the long rate :

$$(98) \qquad \frac{1}{R_{2t}} = \frac{1}{R_{1t}}\left[\pi(s_t|s_t) + \pi(1-s_t|s_t)\exp(\sigma\alpha_1(1-2s_t))\right]^{1/2}$$

$$(99) \qquad \frac{1}{R_{2t}} = \beta\exp(-\sigma\alpha_0 - \sigma\mu + \sigma^2\tau^2/2)$$

$$(100) \qquad \times\left[\pi(s_t|s_t)\exp(2\sigma\alpha_1 s_t) + \pi(1-s_t|s_t)\exp(\sigma\alpha_1)\right]^{1/2}.$$

c.,d. and e. Equation (98) implies that, at the peak $s_t = 0$, the long rate is smaller than the sort rate : the term structure of interest rates is downwards slopping. The intuition goes as follows. In two periods, there is a positive probability of low growth. Therefore, "long term consumption" is relatively scarcer than "short term consumption". Its price should be higher. In other words, the long term interest rate is lower than the short term interest rate.
Conversely, at the trough $s_t = 1$, the long term interest rate is higher than the short term interest rate : the term structure of interest rates is upwards slopping.

Short term interest rates are low when $s_t = 1$ (trough) and high when $s_t = 0$ (peak). Again, this is because when $s_t = 1$, the growth rate of consumption is low. Tomorrow's good is relatively scarcer than if $s_t = 1$. Therefore, tomorrow's good should have a higher price when $s_t = 1$ than when $s_t = 0$. In other words, the short term interest rate is low at a trough and high at a peak. In this precise sense, the short term interest rate is procyclical

Examination of equation (100) shows that long term interest rate is procyclical. Also, procyclicality is stronger if $\pi(s_t|s_t)$ is closer to one, i.e. if shocks are persistent.