Models in Finance - Slides 8 - Stochastic Interest Rate models 2

Master in Actuarial Science

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ISEG

Stochastic Interest Rate models - Dependent annual rates of return

- What if the yields $i_1, i_2, ..., i_n$ are not independent?
- Example: Dependent yields satisfying the equation

$$\mathbb{E}\left[i_t\right] = (1-k)\,\mathbb{E}\left[i_1\right] + k i_{t-1}.$$

- In this example, the shape of the distribution remains invariant. Only, the expected value is "corrected", using the last information available.
- Many other examples of dependent yields can be considered.

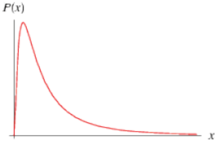
Log-normal distribution of the yields

- In general, a theoretical analysis of the distribution functions for A_n and S_n is somewhat difficult, even in the relatively simple situation when the yields each year are independent and identically distributed. There is, however, one special case one special case for which an exact analysis of the distribution function for S_n is particularly simple: the lognormal distribution of the yields.
- Suppose that the random variable $\log(1 + i_t)$ is normally distributed with mean μ and variance σ^2. In this case, the variable $1 + i_t$ is said to have a log-normal distribution with parameters μ and σ^2.

Log-normal distribution

- The probability density function is (obtained by change of variable):

$$f(x) = \frac{1}{\sigma\sqrt{2\pi x}} e^{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2}, \qquad x > 0, \sigma > 0.$$



- If X is lognormal with parameters μ and σ^2 (meaning that $\log(X) \sim N(\mu, \sigma^2)$, then

$$\mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2},$$

$$Var[X] = (\mathbb{E}[X])^2 \left(e^{\sigma^2} - 1\right) = e^{2\mu}\left(e^{2\sigma^2} - e^{\sigma^2}\right),$$

$$\mathbb{E}[\log X] = \mu, \quad Var[\log X] = \sigma^2.$$

Independent log-normal rates of return

- Suppose that the random variable $\log(1+i_t)$ is normally distributed with mean μ_t and variance σ_t^2 and that the yields are independent.
- Then, the log- accumulated amount of a single investment is

$$\log\left(S_n\right) = \sum_{t=1}^{n} \log(1+i_t). \tag{1}$$

- Moreover, since the i_t are independent, then

$$\log\left(S_n\right) \sim N\left(\sum_{t=1}^{n}\mu_t, \sum_{t=1}^{n}\sigma_t^2\right),$$

$$S_n \sim \log\text{-normal}\left(\sum_{t=1}^{n}\mu_t, \sum_{t=1}^{n}\sigma_t^2\right).$$

I.I.D. log-normal rates of return

- Consider the particular case where the n random variables $(1 + i_t)$ are independent and log-normally distributed with parameters μ and σ^2 (that is, they are i.i.d.).
- Then

$$
\begin{aligned}
\log(S_n) &\sim N\left(n\mu, n\sigma^2\right), \\
S_n &\sim \log\text{-normal}\left(n\mu, n\sigma^2\right).
\end{aligned}
$$

I.I.D. log-normal rates of return

- For the present value of the sum of 1 due at the end of n years, we have:

$$V_n = (1+i_1)^{-1}(1+i_2)^{-1}\cdots(1+i_n)^{-1},$$
$$\log V_n = -\log(1+i_1) - \log(1+i_2) - \cdots - \log(1+i_n).$$

- Therefore,

$$\log V_n \sim N\left(-n\mu, n\sigma^2\right),$$
$$V_n \sim \log\text{-normal}\left(-n\mu, n\sigma^2\right).$$

Exercises

- Exam Question - Apr 01 - Q9: The annual yields from a particular fund are independent and identically distributed. Each year, the distribution of $1+i$ is log-normal with parameters $\mu = 0.07$ and $\sigma^2 = 0.006$, where i denotes the annual yield on the fund.
  (a) Find the mean accumulation in ten years‘ time of an investment in the fund of $\$20,000$ at the end of each of the next ten years, together with $\$150,000$ invested immediately.
  (b) Find the single amount which should be invested in the fund immediately to give an accumulation of at least $\$600,000$ in ten years‘ time with probability 0.99.
- Exercise: Find the lower and upper quartiles for the accumulated value at the end of 5 years of an initial investment os 1000 Euros, assuming that the annual growth rates are i.i.d. and have a log-normal distribution with parameters $\mu = 0.075$ and $\sigma = 0.025$.
- Answer: $1401, 1511$.

Exercise

- Exercise: The random variable $(1+i)$ follows a log-normal distribution with parameters μ and σ^2. The mean of the rate of interest is 0.08 and the variance is 0.0049.
  (a) Find μ and σ^2.
  (b) Find the mean and variance of the accumulation of a unit sum of money for 12 years if the rate of interest, in each year, is log-normally distributed as above and is independent of the rate of interest in any other year.
  (c) Find the probability that a unit sum of money will accumulate to less than 2.40.
  (d) Find the probability that an investor receives a rate of return between 5% and 7% in all the years.
- Answer: (a) 0.074865, 0.0041921, (b) 2.51818, 0.32719, (c) 0.46, (d) $(0.112)^{12}$.

Exercise

- Exercise: Assume now that your goal is to obtain an accumulated value of an investment equal to av_n. The amount of the investment is not known. Further assume that the r.v. i_t will be the return on the investment in year t, with $t = 1, 2, ..., n$, and that $(1 + i_t)$ is lognormally distributed with parameters μ_t and σ_t. The n r.v. are mutually independent. In these circumstances, the present value of the investment may be considered a r.v., denote it with X. Show that the present value of av_n (the investment X) is such that

$$X \sim \text{log-normal}\left(\log\left(av_n\right) - \sum_{t=1}^{n} \mu_t, \sum_{t=1}^{n} \sigma_t^2\right).$$