Lévy-Itô decomposition and stochastic integration

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Processes of Finite Variation

- Let $\mathcal{P} = \{a = t_1 < t_2 < \cdots < t_n < t_{n+1} = b\}$ be a partition of $[a, b] \subset \mathbb{R}$.
- Variation $Var_{\mathcal{P}}[g]$ of a function g over partition $\mathcal{P}$:

$$Var_{\mathcal{P}}[g] := \sum_{i=1}^{n} |g(t_{i+1}) - g(t_i)|.$$

- If $V[g] := \sup_{\mathcal{P}} Var_{\mathcal{P}}[g] < \infty$, we say g has finite variation on $[a, b]$.
- Every non-decreasing function g has finite variation.
- A stochastic process $(X(t), t \geq 0)$ is of finite variation if the paths $(X(t)(\omega), t \geq 0)$ are of finite variation for almost all $\omega \in \Omega$.

Example - Poisson integrals

- N: Poisson random measure with intensity measure μ, let f be a measurable function and A bounded below. Let

$$Y(t) = \int_A f(x) N(t, dx).$$

- The process Y has finite variation on $[0, t]$ for each $t \geq 0$.
- Indeed:

$$Var_{\mathcal{P}}[Y] \leq \sum_{0 \leq s \leq t} |f(\Delta X(s))| \mathbf{1}_A(\Delta X(s)) < \infty \quad \text{a.s. },$$

  where $X(t)$ is the Lévy process associated to the Poisson random measure $N(t, \cdot)$.
- Necessary and sufficient condition for a Lévy process to be of finite variation: there is no Brownian part ($A = 0$ or $\sigma = 0$ in the Lévy-Khinchine formula), and

$$\int_{|x|<1} |x| \nu(dx) < \infty.$$

Lévy-Itô decomposition

- For A bounded below,

$$\int_A xN(t, dx) = \sum_{0 \leq s \leq t} \Delta X(s) \mathbf{1}_A(\Delta X(s)).$$

  is the sum of all the jumps taking values in A, up to time t.
- The sum is a finite random sum.In particular, $\int_{|x|\geq 1} xN(t, dx)$ is finite ("big jumps"). It is a compound Poisson process, has finite variation but may have no finite moments.
- If X is a Lévy process with bounded jumps then we have $E(|X(t)|^m) < \infty$ for all $m \in \mathbb{N}$. (proof: pages 118-119 of Applebaum).

Lévy-Itô decomposition

- For small jumps, let us consider compensated Poisson integrals (which are martingales): (A bounded below)

$$M(t, A) := \int_A x\widetilde{N}(t, dx).$$

- Consider the "ring-sets":

$$B_m := \left\{ x \in \mathbb{R}^d : \frac{1}{m+1} < |x| \leq \frac{1}{m} \right\},$$

$$A_n := \bigcup_{m=1}^{n} B_m.$$

- We can define

$$\int_{|x|<1} x\widetilde{N}(t, dx) := (L^2 \text{ limit}) \lim_{n \to \infty} M(t, A_n).$$

Therefore $\int_{|x|<1} x\widetilde{N}(t, dx)$ is a martingale (the L^2 limit of a sequence of martingales) .

Lévy-Itô decomposition

Theorem

(Lévy-Itô decomposition): If X is a Lévy process, then exists $b \in \mathbb{R}^d$, a Brownian motion B_A with covariance matrix A and an independent Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ such that

$$X(t) = bt + B_A(t) + \int_{|x|<1} x\widetilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx). \tag{1}$$

- Lévy-Itô decomposition in dimension 1:

$$X(t) = bt + \sigma B(t) + \int_{|x|<1} x\widetilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx). \tag{2}$$

- The 3 processes in (1) or (2) are independent. For a rigorous proof of the Lévy-Itô decomposition, see for example Applebaum (pages 121-126).

Lévy-Itô decomposition

- The Lévy-Khintchine formula is a corollary of the Lévy-Itô decomposition.

Corollary

(Lévy-Khintchine formula): If X is a Lévy process then

$$E\left[e^{i(u,X(t))}\right] = \exp\left\{t\left[i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d-\{0\}} \left[e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{|x|<1}(x)\right]\nu(dx)\right]\right\}$$

- The intensity measure μ is equal to the Lévy measure ν for X.
- $\int_{|x|<1} x\widetilde{N}(t,dx)$ is the compensated sum of small jumps (it is an L^2-martingale).
- $\int_{|x|\geq 1} xN(t,dx)$ is the sum of large jumps (may have no finite moments).

Lévy-Itô decomposition

- A Lévy process has finite variation if its Lévy-Itô decomposition is

$$\begin{aligned} X(t) &= \gamma t + \int_{x\neq 0} xN(t,dx) \\ &= \gamma t + \sum_{0\leq s\leq t} \Delta X(s), \end{aligned}$$

where $\gamma = b - \int_{|x|<1} x\nu(dx)$.

Lévy-Itô decomposition

Financial interpretation for the jump terms in the Lévy-Itô decomposition:

- if the intensity measure (μ or ν) is infinite: the stock price has "infinite activity" $\approx$ flutuations and jumpy movements arising from the interaction of pure supply shocks and pure demand shocks.
- if the intensity measure (μ or ν) is finite, we have "finite activity" $\approx$ sudden shocks that can cause unexpected movements in the market, such as a major earthquake.
- If a pure jump Lévy process (no Brownian part) has finite activity $\Longrightarrow$ then it has finite variation. The converse is false.
- The first 3 terms on the rhs of (1) have finite moments to all orders $\Longrightarrow$ if a Lévy process fails to have a moment, this is due to the "large jumps"/"finite activity" part $\int_{|x|\geq 1} xN(t, dx)$.
- $E\left[|X(t)|^n\right] < \infty$ if and only if $\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$.

Stochastic integration

- By the Lévy-Itô decomposition, a Lévy process X can be decomposed into $X(t) = M(t) + C(t)$, where

$$M(t) = B_A(t) + \int_{|x|<1} x\widetilde{N}(t, dx),$$

$$C(t) = bt + \int_{|x|\geq 1} xN(t, dx),$$

- $M(t)$ is a martingale and $C(t)$ is an adapted process of finite variation.

Stochastic integration

- Stochastic integral w.r.t. X:

$$\int_0^T F(t)\, dX_t = \int_0^T F(t)\, dM_t + \int_0^T F(t)\, dC_t. \tag{3}$$

- $\int_0^T F(t)\, dC_t$ defined by the usual Lebesgue-Stieltjes integral.
- In general, $\int_0^T F(t)\, dM_t$ requires a stochastic definition similar to Itô integral (in general, M has infinite variation).
- We define, for $E \subset \mathbb{R}$,

$$\int_0^T \int_E F(t,x)\, M(dt,dx) = \int_0^T F(t,0) dB_t + \int_0^T \int_{E-\{0\}} F(t,x)\, \widetilde{N}(dt,dx). \tag{4}$$

- Let $\mathcal{P}$ be the smallest σ-algebra with respect to which all the mappings $F : [0,T] \times E \times \Omega \to \mathbb{R}$ satisfying (1) and (2) below are measurable:
  1. For each t, $(x,\omega) \to F(t,x,\omega)$ is $\mathcal{B}(E) \times \mathcal{F}_t$ measurable.
  2. For each x and ω, $t \to F(t,x,\omega)$ is left continuous.
- $\mathcal{P}$ is called the predictable σ-algebra. A $\mathcal{P}$-measurable mapping (or process) is said predictable (predictable process)
- Let $\mathcal{H}_2$ be the linear space of mappings (or processes) $F : [0,T] \times E \times \Omega \to \mathbb{R}$ which are predictable and

$$\int_0^T \int_{E-\{0\}} \mathbb{E}\left[|F(t,x)|^2\right] \nu(dx)\, dt < \infty, \tag{5}$$

$$\int_0^T \mathbb{E}\left[|F(t,0)|^2\right] dt < \infty. \tag{6}$$

- Let F be a simple process:

$$F = \sum_{j=1}^{m} \sum_{k=1}^{n} F_k (t_j) \mathbf{1}_{(t_j, t_{j+1}]} \mathbf{1}_{A_k} \tag{7}$$

- F is predictable and its stochastic integral is defined by

$$I(F) = \sum_{j=1}^{m} \sum_{k=1}^{n} F_k (t_j) M((t_j, t_{j+1}], A_k), \tag{8}$$

where $M((t_j, t_{j+1}], A_k) = M(t_{j+1}, A_k) - M(t_j, A_k) =$ $[B(t_{j+1}) - B(t_j)] \delta_0 (A_k) + \left[\widetilde{N}(t_{j+1}, A_k - \{0\}) - \widetilde{N}(t_j, A_k - \{0\})\right]$.

Lemma

If F is simple then

$$\mathbb{E}[I(F)] = 0,$$

$$\mathbb{E}\left[(I(F))^2\right] = \int_0^T \int_{E-\{0\}} \mathbb{E}\left[|F(t,x)|^2\right] \nu(dx)\, dt + \delta_0(E) \int_0^T \mathbb{E}\left[|F(t,0)|^2\right] dt \tag{9}$$

- Exercise: Show that $\mathbb{E}[I(F)] = 0$.
- S is dense in $\mathcal{H}_2$ and the stochastic integral I can be extended to $\mathcal{H}_2$.
- For $F \in \mathcal{H}_2$ we define

$$I_t(F) = \int_0^t \int_E F(s,x) M(ds, dx)$$

and

$$\int_0^t \int_E F(s,x) M(ds, dx) = \lim_{n\to\infty} (L^2) \int_0^t \int_E F_n(s,x) M(ds, dx), \tag{10}$$

where $\{F_n, n \in \mathbb{N}\}$ is a sequence of simple processes.

- The stochastic integral $I_t(F)$ with $F \in \mathcal{H}_2$ satisfies:
  1. I_t is a linear operator
  2. $\mathbb{E}[I(F)] = 0$.
  3. $\mathbb{E}\left[(I(F))^2\right] = \int_0^T \int_{E-\{0\}} \mathbb{E}\left[|F(t,x)|^2\right] \nu(dx)\,dt + \delta_0(E) \int_0^T \mathbb{E}\left[|F(t,0)|^2\right] dt.$
  4. $\{I_t(F),\ t \in [0,T]\}$ is $\{\mathcal{F}_t\}$ adapted.
  5. $\{I_t(F),\ t \in [0,T]\}$ is a square-integrable martingale.

Poisson stochastic integrals

- The integral of a predictable process $K(t,x)$ with respect to the compound Poisson process $P_t = \int_A xN(t,dx)$ is defined by (A bounded below)

$$\int_0^T \int_A K(t,x)\,N(dt,dx) = \sum_{0 \le s \le T} K(s, \Delta P_s)\,\mathbf{1}_A(\Delta P_s). \tag{11}$$

- We can also define

$$\int_0^T \int_A H(t,x)\,\widetilde{N}(dt,dx) = \int_0^T \int_A H(t,x)\,N(dt,dx) - \int_0^T \int_A H(t,x)\,\nu(dx)\,dt \tag{12}$$

if H is predictable and satisfies (5).

Lévy type stochastic integrals

- We say Y is a Lévy type stochastic integral if

$$Y_t = Y_0 + \int_0^t G(s)\,ds + \int_0^t F(s)\,dB_s + \int_0^t \int_{|x|<1} H(s,x)\,\widetilde{N}(ds,dx) + \int_0^t \int_{|x|\geq 1} K(s,x)\,N(ds,dx), \tag{13}$$

  where we assume that the processes G, F, H and K are predictable and satisfy the appropriate integrability conditions.
- Eq. (13) can be written as

$$dY_t = G(t)\,dt + F(t)\,dB_t + \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx)$$

- Let L be a Lévy process with Lévy triplet (b, c, ν) and let X be a predictable left-continuous process satisfying (5). Then we can construct a Lévy stochastic integral Y_t by

$$dY_t = X_t dL_t.$$

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