# 2

# Stochastic Interest Rate Models

# 2.1. CONTINUOUS TIME FINANCE RECAP

**Note: Please see Hull (2018), Chap.14.**

- **Stochastic process** – any variable whose value changes over time in an uncertain way => different random trajectories for the variable.

- Discrete vs continuous time stochastic processes:
  - Discrete – the variable value can change only at certain fixed points in time
  - Continuous – changes can take place at any time

- Continuous vs discrete variables:
  - Discrete – only certain values are possible
  - Continuous – can take any value within a certain range

- Continuous-variable, continuous-time – variables can assume any value and changes can occur at any time.

# STOCHASTIC PROCESSES

- **Continuous-variable, continuous-time stochastic processes are key to understanding the pricing of options and other derivatives.**

- However, **in practice, most asset prices do not follow continuous-variable, continuous-time stochastic processes.**

- For instance, stock prices are restricted to discrete values (e.g. multiples of a cent) and changes can be observed only when the markets are open.

- Nonetheless, continuous-variable, continuous-time stochastic processes are useful for many valuation purposes.

# STOCHASTIC PROCESSES

- **Markov Stochastic Process** – stochastic process where only the current value of a variable is relevant for predicting the future => all past information is irrelevant, as it is already incorporated into today's stock price (weak form of market efficiency, while the strong form states that all relevant information is incorporated in current prices).

- The probability distribution at any particular future time is independent from the path followed by the variable in the past.
- If the weak form of market efficiency were not true, market participants could make above-average returns by interpreting the past behavior of asset prices.

# STOCHASTIC PROCESSES

- Assuming a Markov process X(t), the 1-year change ~ $\mathcal{N}(0,1)$.

- 2-year change = $\mathcal{N}(0,1)$ + $\mathcal{N}(0,1)$ = $\mathcal{N}(0,2)$, as both distributions are independent - given that this is a Markov process, the second distribution does not depend on the first.

$\Delta t$ (very small period of time) change ~ $\mathcal{N}(0, \Delta t)$

# WIENER PROCESS

A stochastic process z follows a **Wiener process (or the continuous random walk)** if it has the following properties:

Property 1. *The change $\Delta z$ during a small period of time $\Delta t$ is*

$$\Delta z = \epsilon\sqrt{\Delta t} \qquad (14.1)$$

*where $\epsilon$ has a standard normal distribution $\phi(0,1)$.*

Property 2. *The values of $\Delta z$ for any two different short intervals of time, $\Delta t$, are independent.*

It follows from the first property that $\Delta z$ itself has a normal distribution with

$$\text{mean of } \Delta z = 0$$

$$\text{standard deviation of } \Delta z = \sqrt{\Delta t}$$

$$\text{variance of } \Delta z = \Delta t$$

The second property implies that $z$ follows a Markov process.

Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

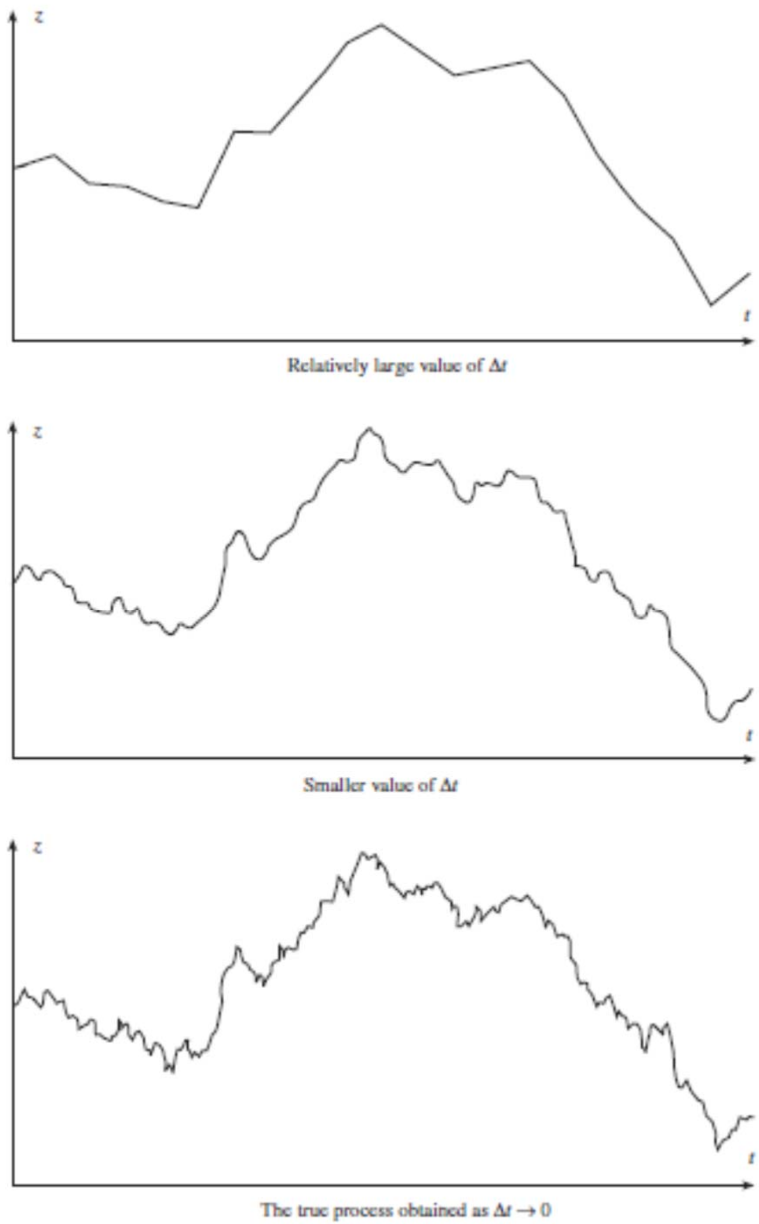
- Therefore , a Wiener process is a Markov process with its change having:
  - **mean (drift) = 0** => the expected value of any future outcome is equal to the current value (Martingale): z=25 => 1 year after, z ~ N(25,1); 5 years after, z ~ N(25,5)
  - **variance (variance rate) = 1** => uncertainty (standard-deviation) is proportional to the square root of time.

# WIENER PROCESS

**Wiener processes for different magnitudes of change in time:**

**When $\Delta$t -> 0, the path becomes much more irregular, as the size of the movement in the variable in time $\Delta$t is proportional to the $\sqrt{\Delta t}$. When $\Delta$t is small, $\sqrt{\Delta t}$ is much larger than $\Delta$t => the changes in z will be much larger than $\Delta$t, as** $\Delta z = \epsilon\sqrt{\Delta t}$

Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition



# GENERALIZED WIENER PROCESS

- Instead of a drift = 0 and a variance rate =1 as in the Wiener process (*dz*), we may have a stochastic process where the drift can assume any value ***a*** and the variance rate can be $b^2$ => **Generalized Wiener Process**.

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$$ where $a$ and $b$ are constants.

- For very small time changes Δt: $\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$

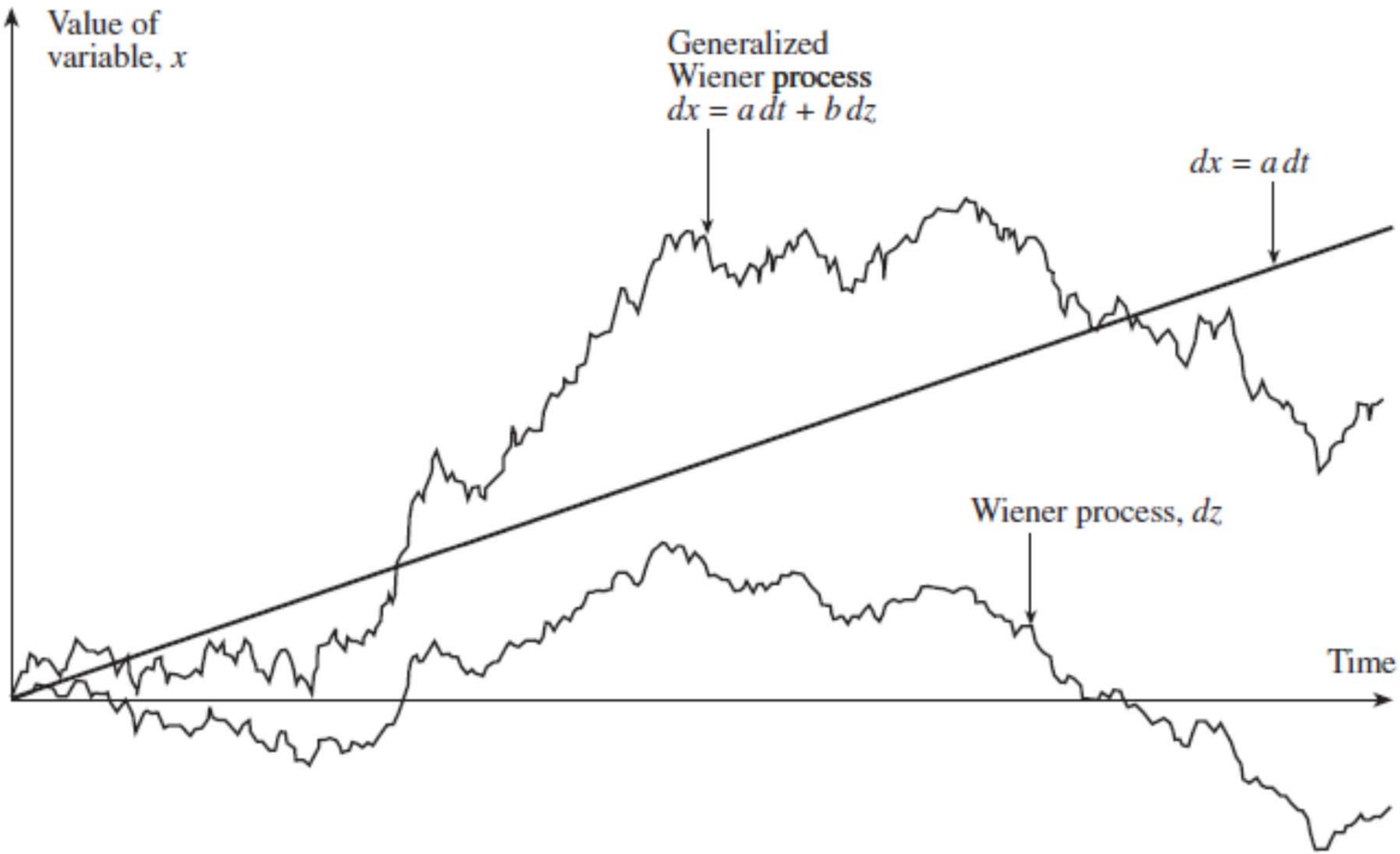
Δx ~ N, with

$$\text{mean of } \Delta x = a\,\Delta t$$
$$\text{standard deviation of } \Delta x = b\sqrt{\Delta t}$$
$$\text{variance of } \Delta x = b^2\,\Delta t$$

- The average increases in *x* are proportional to time (if there is no drift, the mean of *x* doesn't change, i.e. $\Delta x = 0$).

# GENERALIZED WIENER PROCESS

**Figure 14.2** Generalized Wiener process with $a = 0.3$ and $b = 1.5$.




Source: Hull, John (2018), “Options, Futures and Other Derivatives”, Pearson Prenctice Hall, 10th Edition

# ITÔ PROCESS

- Definition: Generalized Wiener process with average and standard-deviation as functions of the underlying variable and time (instead of constant along time):

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

- For small time intervals, we may assume that the average and the standard-deviation don't change (we're assuming that the drift and the variance rate don't change between *t* and *t+Δt*):

$$\Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$$

- This is still a Markov process, as *a* and *b* only depend on the current value of x, not on previous values.

# ITÔ PROCESS

- It may be tempting to assume that a stock price follows a generalized Wiener process (constant drift and variance).
- However, this assumption is not valid, having in mind that investors require or expect a given level of returns (as a % variation) regardless the price level, i.e. for higher prices, expected changes will also be higher.

⬇

- One can replace the assumption of constant expected drift by the assumption of constant expected returns (i.e. constant expected drift divided by the stock price ⇔ variable drift along time).

# ITÔ PROCESS

- If $S$ is the stock price at time $t$ => *expected drift rate in $S$ (i.e. a(x,t))* must be $\mu S$ (being $\mu$ constant, corresponding to the expected rate of return on the stock, expressed in decimal form).

- In a short interval of time $\Delta$t, the expected increase in $S$ is $\mu S\Delta$t, i.e the expected rate of return on the stock, times the stock price, times the time interval:

$$\Delta S = \mu S \Delta t \qquad\qquad \Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$$

- If **$\Delta$t -> 0 =>**

$$dS = \mu S dt \Leftrightarrow \frac{dS}{S} = \mu dt$$

- This corresponds to the price of an asset following a continuously compounding process (under no uncertainty, being $\mu$ = risk-free rate in a risk-neutral world): $S_T = S_0 e^{\mu T}$

## GEOMETRIC BROWNIAN MOTION

- Given that in practice there is uncertainty, a reasonable assumption is that the variability of the percentage return ($\sigma$) in a short period of time $\Delta t$ is the same regardless the stock price.

- An investor is as uncertain about his return when the stock price is high or low.

- Accordingly, the standard deviation of the change in a short period of time must be proportional to the stock price, as the standard deviation for the percentage change is constant – **Geometric Brownian Motion**:

$$dS = \mu S dt + \sigma S dz \Leftrightarrow$$

$$\frac{dS}{S} = \mu dt + \sigma dz$$

# GEOMETRIC BROWNIAN MOTION

- Example:

Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding. In this case, $\mu = 0.15$ and $\sigma = 0.30$. The process for the stock price is

$$\frac{dS}{S} = 0.15\,dt + 0.30\,dz$$

If $S$ is the stock price at a particular time and $\Delta S$ is the increase in the stock price in the next small interval of time, the discrete approximation to the process is

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\epsilon\sqrt{\Delta t}$$

where $\epsilon$ has a standard normal distribution. Consider a time interval of 1 week, or 0.0192 year, so that $\Delta t = 0.0192$. Then the approximation gives

$$\frac{\Delta S}{S} = 0.15 \times 0.0192 + 0.30 \times \sqrt{0.0192}\,\epsilon$$

or

$$\Delta S = 0.00288S + 0.0416S\epsilon$$

Source: Hull, John (2018), “Options, Futures and Other Derivatives”, Pearson Prenctice Hall, 10th Edition

# GEOMETRIC BROWNIAN MOTION

- Monte Carlo simulation:

A path for the stock price over 10 weeks can be simulated by sampling repeatedly for $\epsilon$ from $\phi(0, 1)$ and substituting into equation (14.10). The expression =RAND() in Excel produces a random sample between 0 and 1. The inverse cumulative normal distribution is NORMSINV. The instruction to produce a random sample from a standard normal distribution in Excel is therefore =NORMSINV(RAND()). Table 14.1 shows one path for a stock price that was sampled in this way. The initial stock price is assumed to be \$100. For the first period, $\epsilon$ is sampled as 0.52. From equation (14.10), the change during the first time period is

$$\frac{\Delta S}{S} = 0.15 \times 0.0192 + 0.30 \times \sqrt{0.0192}\,\epsilon$$

$$\Delta S = 0.00288S + 0.0416S\epsilon$$

**Table 14.1** Simulation of stock price when $\mu = 0.15$ and $\sigma = 0.30$ during 1-week periods.

| *Stock price at start of period* | *Random sample for $\epsilon$* | *Change in stock price during period* |
|---|---|---|
| 100.00 | 0.52 | 2.45 |
| 102.45 | 1.44 | 6.43 |
| 108.88 | −0.86 | −3.58 |
| 105.30 | 1.46 | 6.70 |
| 112.00 | −0.69 | −2.89 |
| 109.11 | −0.74 | −3.04 |
| 106.06 | 0.21 | 1.23 |
| 107.30 | −1.10 | −4.60 |
| 102.69 | 0.73 | 3.41 |
| 106.11 | 1.16 | 5.43 |
| 111.54 | 2.56 | 12.20 |

Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

# GEOMETRIC BROWNIAN MOTION

- Correlated processes

$$dx_1 = a_1\,dt + b_1\,dz_1 \quad \text{and} \quad dx_2 = a_2\,dt + b_2\,dz_2$$

$$\Delta x_1 = a_1\,\Delta t + b_1\,\epsilon_1\sqrt{\Delta t} \quad \text{and} \quad \Delta x_2 = a_2\,\Delta t + b_2\,\epsilon_2\sqrt{\Delta t}$$

If $x_1$ and $x_2$ have a nonzero correlation $\rho$, then the $\epsilon_1$ and $\epsilon_2$ that are used to obtain movements in a particular period of time should be sampled from a bivariate normal distribution. Each variable in the bivariate normal distribution has a standard normal distribution and the correlation between the variables is $\rho$. In this situation, we would refer to the Wiener processes $dz_1$ and $dz_2$ as having a correlation $\rho$.

$$\epsilon_1 = u \quad \text{and} \quad \epsilon_2 = \rho u + \sqrt{1-\rho^2}\,v$$

Source: Hull, John (2018), “Options, Futures and Other Derivatives”, Pearson Prenctice Hall, 10th Edition

# Itô's Lemma

- An option price ($G$) is a function of the underlying asset's price and time.
- Therefore, it is important to understand the behavior of functions of stochastic variables.
- An important result was discovered by K. Itô in 1951 and is known as **Itô's lemma**.

[6] See K. Itô, "On Stochastic Differential Equations," *Memoirs of the American Mathematical Society*, 4 (1951): 1–51.

- Assuming that a variable x follows an Itô process:

$$dx = a(x, t)\,dt + b(x, t)\,dz$$

where $dz$ is a Wiener process and $a$ and $b$ are functions of $x$ and $t$. The variable $x$ has a drift rate of $a$ and a variance rate of $b^2$. Itô's lemma shows that a function $G$ of $x$ and $t$ follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$

# ITÔ'S LEMMA

- Thus, G also follows an Itô process with a drift rate

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

and a variance rate of

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2$$

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

- Assuming that the stock price follows a Geometric Brownian Motion, with constant $\mu$ and $\sigma$:

$$dS = \mu S dt + \sigma S dz$$

- From Ito's Lemma it follows that

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S\,dz$$

- … in line with $dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$

# ITÔ'S LEMMA

- Therefore, both *S* and *G* are affected by the same volatility source – *dz*.

- This is in line with the Black-Scholes option pricing formula, as G (the option price) is determined by the instantaneous volatility of the returns of the underlying asset price.

# APPLICATION TO FORWARD CONTRACTS

- Forward: $F_0 = S_0 e^{rT}$

- Forward at *t*: $F = Se^{r(T-t)}$

- Ito's Lemma => $dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S\, dz$

- The stochastic process of *F* can be defined calculating the derivatives of *F* in order to *S* and *t*:

$$\frac{\partial F}{\partial S} = e^{r(T-t)}, \qquad \frac{\partial^2 F}{\partial S^2} = 0, \qquad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}$$

Substituting $F$ for $Se^{r(T-t)}$

$$dF = \left[e^{r(T-t)}\mu S - rSe^{r(T-t)}\right] dt + e^{r(T-t)}\sigma S\, dz$$ => $$dF = (\mu - r)F\, dt + \sigma F\, dz$$

- Like S, F follows a GMB, with the same volatility and a trend of $\mu$-*r* (instead of $\mu$).

# PROBABILITY DISTRIBUTION

- From the stochastic process of the rate of returns,

$$\frac{dS}{S} = \mu dt + \sigma dz$$

- Its distribution gets

$$\frac{\Delta S}{S} \sim \phi(\mu \Delta t, \sigma^2 \Delta t)$$

- Assuming $G = \ln S$ , since $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, $\frac{\partial G}{\partial t} = 0$ , it follows from the Itô's lemma that

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dz$$

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S\, dz$$

# PROBABILITY DISTRIBUTION

Since $\mu$ and $\sigma$ are constant, this equation indicates that $G = \ln S$ follows a generalized Wiener process. It has constant drift rate $\mu - \sigma^2/2$ and constant variance rate $\sigma^2$. The change in $\ln S$ between time 0 and some future time $T$ is therefore normally distributed, with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$. This means that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

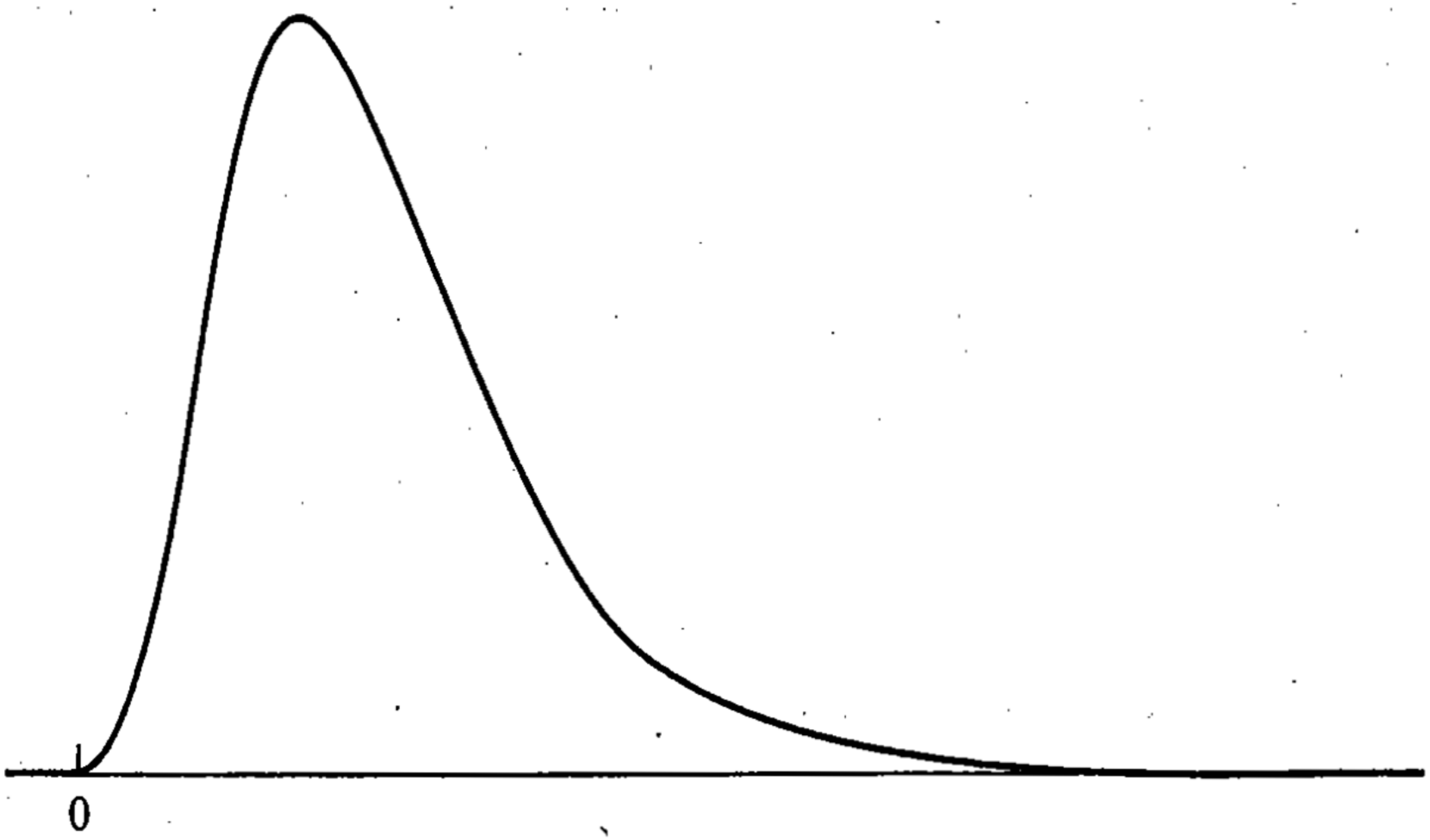
or

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

- This equation shows that ln $S_T$ is normally distributed (and $S_T$ has a log normal distribution), with a standard deviation $\sigma\sqrt{T}$ that is proportional to the square root of time **=> the growth rate of the asset price is normally distributed => the asset price is lognormally distributed.**

# PROBABILITY DISTRIBUTION

**Figure 13.1** Lognormal distribution.




Source: Hull, John (2009), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 7th Edition