# Stochastic integration and Itô formula

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## Lévy type stochastic integrals

- We say $Y$ is a Lévy type stochastic integral if

$$Y_t = Y_0 + \int_0^t G(s)\,ds + \int_0^t F(s)\,dB_s + \int_0^t \int_{|x|<1} H(s,x)\,\widetilde{N}(ds,dx) + \int_0^t \int_{|x|\geq 1} K(s,x)\,N(ds,dx), \quad (1)$$

  where $G$, $F$, $H$ and $K$ are processes such that the integrals are well defined.
- With stochastic differentials notation, we can write:

$$dY(t) = G(t)\,dt + F(t)\,dB(t) + \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx).$$

# Lévy type stochastic integrals

- Let $M$ be an adapted and left-continuous process. Then we can define a new process $\{Z_t, t \geq 0\}$ by

$$dZ(t) = M(t)\, dY(t)$$

or

$$dZ(t) = M(t)\, G(t)\, dt + M(t)\, F(t)\, dB(t) + M(t)\, H(t,x)\, \widetilde{N}(dt, dx) + M(t)\, K(t,x)\, N(dt, dx).$$





# Example - Lévy stochastic integrals

- $X$: Lévy process with characteristics $(b, \sigma, \nu)$ and Lévy-Itô decomposition

$$X(t) = bt + \sigma B(t) + \int_{|x|<1} x \widetilde{N}(t, dx) + \int_{|x|\geq 1} x N(t, dx).$$

Let $U \in \mathcal{H}_2(t)$ for all $t \geq 0$. and choose in (1) $F = \sigma U$, $H = K = xU$.

- The process $Y$ such that

$$dY(t) = U(t)\, dX(t)$$

is called a Lévy stochastic integral.

# Example - Ornstein Uhlenbeck (OU) process

- OU process:
$$Y(t) = e^{-\lambda t} y_0 + \int_0^t e^{-\lambda(t-s)} dX(s),$$
where $y_0$ is fixed.
- This process can be used for volatility modelling in finance.
- Exercise: Prove that if $X$ is a one-dimensional Brownian motion then $Y(t)$ is a Gaussian process with mean $e^{-\lambda t} y_0$ and variance $\frac{1}{2\lambda}\left(1 - e^{-2\lambda t}\right)$





# Example - Ornstein Uhlenbeck (OU) process

- In differential form, the OU process is the solution of the SDE:
$$dY(t) = -\lambda Y(t)\, dt + dX(t),$$
which is known as the Langevin equation (is a stochastic differential equation).
- The Langevin equation is also a model for the physical phenomenon of Brownian motion: includes the viscous drag of the medium on the particle as well as random fluctuations.

# Itô formula for Poisson stochastic integrals

- Consider the Poisson stoch. integral $W(t) = W(0) + \int_0^t \int_A K(s,x)\, N(ds,dx)$, with $A$ bounded below and $K$ predictable.

Lemma

*(Itô formula 1): If $f \in C(\mathbb{R})$ then*

$$f(W(t)) - f(W(0)) = \int_0^t \int_A [f(W(s-) + K(s,x)) - f(W(s-))]\, N(ds,dx) \quad a.s.$$



Itô formula



# Itô formula for Poisson stochastic integrals



**Sketch of the Proof:**

$$f(W(t)) - f(W(0)) = \sum_{0 \le s \le t} [f(W(s)) - f(W(s-))]$$

$$= \sum_{0 \le s \le t} [f(W(s-) + K(s,x)) - f(W(s-))]$$

$$= \int_0^t \int_A [f(W(s-) + K(s,x)) - f(W(s-))]\, N(ds,dx).$$

# Itô formula for Brownian motion

- Let $M$ be a Brownian integral with drift:

$$M(t) = \int_0^t F(s)\, dB(s) + \int_0^t G(s)\, ds.$$

- Let us define the quadratic variation process:

$$[M, M](t) = \int_0^t (F(s))^2\, ds.$$

- $$d[M, M](t) = (F(t))^2\, dt.$$





# Itô formula for Brownian motion

Theorem

*(Itô formula 2) If* $f \in C^2(\mathbb{R})$ *then*

$$f(M(t)) - f(M(0)) = \int_0^t \partial f(M(s))\, dM(s) + \frac{1}{2}\int_0^t \partial^2 f(M(s))\, d[M, M](s). \quad a.s.$$

Proof: See Applebaum

# Itô formula for Lévy type stochastic integrals

- Let

$$dY(t) = G(t)\,dt + F(t)\,dB(t) + \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx)$$

- $dY_c(t) := G(t)\,dt + F(t)\,dB(t)$
- $dY_d(t) := \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx)$





# Itô formula for Lévy type stochastic integrals

Theorem

*(Itô formula 3): If $f \in C^2(\mathbb{R})$ then*

$$f(Y(t)) - f(Y(0)) = \int_0^t \partial f(Y(s-))\,dY_c(s) + \frac{1}{2}\int_0^t \partial^2 f(Y(s-))\,d[Y_c, Y_c](s)$$
$$+ \int_0^t \int_{|x|\geq 1} [f(Y(s-) + K(s,x)) - f(Y(s-))]\,N(ds,dx)$$
$$+ \int_0^t \int_{|x|<1} [f(Y(s-) + H(s,x)) - f(Y(s-))]\,\widetilde{N}(ds,dx)$$
$$+ \int_0^t \int_{|x|<1} [f(Y(s-) + H(s,x)) - f(Y(s-))$$
$$- H(s,x)\,\partial f(Y(s-))]\,\nu(dx)\,ds$$

- Proof: see Applebaum

# Itô formula for Lévy type stochastic integrals

Theorem

*(Itô formula 4): If $f \in C^2(\mathbb{R})$ then*

$$f(Y(t)) - f(Y(0)) = \int_0^t \partial f(Y(s-))\, dY(s) + \frac{1}{2}\int_0^t \partial^2 f(Y(s-))\, d[Y_c, Y_c](s)$$
$$+ \sum_{0 \le s \le t} \left[ f(Y(s)) - f(Y(s-)) - \Delta Y(s)\, \partial f(Y(s-)) \right].$$

- Proof: see Applebaum





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