Stochastic exponential, exponential martingales and complete/incomplete markets

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Stochastic exponential

- Let $d = 1$ and consider the process $Z = (Z(t), t \geq 0)$ solution of the SDE:

$$dZ(t) = Z(t-)\, dY(t), \tag{1}$$

where Y is a Lévy-type stochastic integral, of the type:

$$dY(t) = G(t)\, dt + F(t)\, dB(t) + \int_{|x|<1} H(t,x)\, \widetilde{N}(dt,dx) \tag{2}$$

$$+ \int_{|x|\geq 1} K(t,x)\, N(dt,dx). \tag{3}$$

- The solution of (1) is the "stochastic exponential" or "Doléans-Dade exponential":

$$Z(t) = \mathcal{E}_Y(t) = \exp\left\{ Y(t) - \frac{1}{2}[Y_c, Y_c](t) \right\} \prod_{0\leq s\leq t} (1 + \Delta Y(s))\, e^{-\Delta Y(s)}. \tag{4}$$

- We require that (assumption):

$$\inf \{\Delta Y(t), t \geq 0\} > -1 \text{ a.s.} \tag{5}$$

Stochastic exponential

Proposition

If Y is a Lévy-type stochastic integral and (5) holds, then each $\mathcal{E}_Y(t)$ is a.s. finite.

- For a proof of this proposition, see Applebaum.
- Note that (5) also implies that $\mathcal{E}_Y(t) > 0$ a.s.
- The stochastic exponential $\mathcal{E}_Y(t)$ is the unique solution of SDE (1) which satisfies the initial condition $Z(0) = 1$ a.s.
- If (5) does not hold then $\mathcal{E}_Y(t)$ may take negative values.

Stochastic exponential

- Alternative form of (4):

$$\mathcal{E}_Y(t) = e^{S_Y(t)}, \tag{6}$$

where

$$\begin{aligned} dS_Y(t) &= F(t)\,dB(t) + \left(G(t) - \frac{1}{2}F(t)^2\right)dt \\ &+ \int_{|x|\geq 1} \log(1 + K(t,x))\,N(dt,dx) + \int_{|x|<1} \log(1 + H(t,x))\,\widetilde{N}(dt,dx) \\ &+ \int_{|x|<1} \left(\log(1 + H(t,x)) - H(t,x)\right)\nu(dx)\,dt \end{aligned} \tag{7}$$

Stochastic exponential

Theorem

$$d\mathcal{E}_Y(t) = \mathcal{E}_Y(t)\, dY(t)$$

- Exercise: Prove the previous theorem by applying the Itô formula to (7) (see Applebaum).

Stochastic exponentials

Stochastic exponential

- Example 1: If $Y(t) = \sigma B(t)$, where $\sigma > 0$ and B is a BM, then

$$\mathcal{E}_Y(t) = \exp\left\{\sigma B(t) - \frac{1}{2}\sigma^2 t\right\}.$$

- Example 2: If $Y = (Y(t), t \geq 0)$ is a compound Poisson process: $Y(t) = X_1 + \cdots + X_{N(t)}$ then

$$\mathcal{E}_Y(t) = \prod_{i=1}^{N(t)} (1 + X_i)$$

Stochastic exponential

- Example 3: If $Y(t) = \mu t + \sigma B(t) + J(t)$ (jump-diffusion model), where $\sigma > 0$, B is a BM, and $J = (J(t), t \geq 0)$ is a compound Poisson process: $J(t) = X_1 + \cdots + X_{N(t)}$,then

$$\mathcal{E}_Y(t) = \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right\} \prod_{i=1}^{N(t)} (1 + X_i).$$

Stochastic exponential

- Let X be a Lévy process with characteristics (b, σ, ν) and Lévy-Itô decomposition $X(t) = bt + \sigma B(t) + \int_{|x|<1} x\widetilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx)$.
- When can $\mathcal{E}_X(t)$ be written as $\exp(X_1(t))$ for a certain Lévy process X_1 and vice-versa?
- By (6) and (7) we have $\mathcal{E}_X(t) = e^{S_X(t)}$ with

$$S_X(t) = \sigma B(t) + \int_{|x|\geq 1} \log(1 + x) N(t, dx) + \int_{|x|<1} \log(1 + x) \widetilde{N}(t, dx)$$
$$+ t\left[b - \frac{1}{2}\sigma^2 + \int_{|x|<1} (\log(1 + x) - x) \nu(dx)\right]. \quad (8)$$

Stochastic exponential

- Comparing the Lévy-Itô decomposition with (8), we have

Theorem

If X is a Lévy process with each $\mathcal{E}_X(t) > 0$, then $\mathcal{E}_X(t) = \exp(X_1(t))$ where X_1 is a Lévy process with characteristics (b_1, σ_1, ν_1) given by:

$$\nu_1 = \nu \circ f^{-1}, \quad f(x) = \log(1+x).$$
$$b_1 = b - \frac{1}{2}\sigma^2 + \int_{\mathbb{R}-\{0\}} \left[\log(1+x)\mathbf{1}_{]-1,1[}(\log(1+x)) - x\mathbf{1}_{]-1,1[}(x)\right]\nu(dx),$$
$$\sigma_1 = \sigma.$$

Conversely, there exists a Lévy process X_2 with characteristics (b_2, σ_2, ν_2) such that $\exp(X(t)) = \mathcal{E}_{X_2}(t)$, where

$$\nu_2 = \nu \circ g^{-1}, \quad g(x) = e^x - 1$$
$$b_2 = b + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}-\{0\}} \left[(e^x-1)\mathbf{1}_{]-1,1[}(e^x-1) - x\mathbf{1}_{]-1,1[}(x)\right]\nu(dx),$$
$$\sigma_2 = \sigma.$$

Exponential martingales

- Lévy-type stochastic integral:

$$dY(t) = G(t)\,dt + F(t)\,dB(t) + \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx)$$
$$+ \int_{|x|\geq 1} K(t,x)\,N(dt,dx).$$

- When is Y a martingale?
- Assumptions:
- (M1) $\mathbb{E}\left[\int_0^t \int_{|x|\geq 1} |K(s,x)|^2 \nu(dx)\,ds\right] < \infty$ for each $t > 0$
- (M2) $\int_0^t \mathbb{E}[|G(s)|]\,ds < \infty$ for each $t > 0$.

Exponential martingales

- Then

$$\int_0^t \int_{|x|\geq 1} K(s,x)\, N(ds,dx) = \int_0^t \int_{|x|\geq 1} K(s,x)\, \widetilde{N}(ds,dx) \qquad (9)$$

$$+ \int_0^t \int_{|x|\geq 1} K(s,x)\, \nu(dx)\, ds. \qquad (10)$$

and the compensated integral is a martingale.

Theorem

With assumptions (M1) and (M2), Y is a martingale if and only if

$$G(t) + \int_{|x|\geq 1} K(t,x)\, \nu(dx) = 0 \quad \text{(a.s.) for a.a. } t \geq 0.$$

Exponential martingales

Exponential martingales

- Let us consider the process $e^Y = \left(e^{Y(t)}, t \geq 0\right)$.
- By Itô's formula, we have that

$$e^{Y(t)} = 1 + \int_0^t e^{Y(s-)} F(s)\, dB(s) + \int_0^t \int_{|x|<1} e^{Y(s-)} \left(e^{H(s,x)} - 1\right) \widetilde{N}(ds,dx)$$

$$+ \int_0^t \int_{|x|\geq 1} e^{Y(s-)} \left(e^{K(s,x)} - 1\right) \widetilde{N}(ds,dx)$$

$$+ \int_0^t e^{Y(s-)} \left(G(s) + \frac{1}{2} F(s)^2 + \int_{|x|<1} \left(e^{H(s,x)} - 1 - H(s,x)\right) \nu(dx) \right.$$

$$\left. + \int_{|x|\geq 1} \left(e^{K(s,x)} - 1\right) \nu(dx) \right) ds \qquad (11)$$

Exponential martingales

Theorem

e^Y is a martingale if and only if

$$G(s)+\frac{1}{2}F(s)^2+\int_{|x|<1}\left(e^{H(s,x)}-1-H(s,x)\right)\nu(dx)$$
$$+\int_{|x|\geq 1}\left(e^{K(s,x)}-1\right)\nu(dx)=0 \qquad (12)$$

a.s. and for a.a. $s\geq 0$.

- Therefore, if e^Y is a martingale then

$$e^{Y(t)}=1+\int_0^t e^{Y(s-)}F(s)\,dB(s)+\int_0^t\int_{|x|<1}e^{Y(s-)}\left(e^{H(s,x)}-1\right)\widetilde{N}(ds,dx)$$
$$+\int_0^t\int_{|x|\geq 1}e^{Y(s-)}\left(e^{K(s,x)}-1\right)\widetilde{N}(ds,dx).$$

Exponential martingales

Exponential martingales

- If e^Y is a martingale then $\mathbb{E}\left[e^{Y(t)}\right]=1$ for all $t\geq 0$ and e^Y is called an exponential martingale.
- if Y is an Itô process: $Y(t)=\int_0^t G(s)\,ds+\int_0^t F(s)\,dB(s)$ then (12) is $G(t)=-\frac{1}{2}F(t)^2$ and

$$e^{Y(t)}=\exp\left(\int_0^t F(s)\,dB(s)-\frac{1}{2}\int_0^t F(s)^2\,ds\right).$$

Change of Measure - Girsanov's Theorem

- Let P and Q be two different probability measures. Q_t and P_t are the measures restricted to $(\Omega, \mathcal{F}_t)$.
- Let e^Y be an exponential martingale and define Q_t by

$$\frac{dQ_t}{dP_t} = e^{Y(t)}.$$

- Fix an interval $[0, T]$ and define $P = P_T$ and $Q = Q_T$.

Lemma

$M = (M(t), 0 \leq t \leq T)$ is a Q-martingale if and only if $Me^Y = (M(t)e^{Y(t)}, 0 \leq t \leq T)$ is a P-martingale.

Change of Measure - Girsanov's Theorem

Change of Measure - Girsanov's Theorem

- Let Y be an Itô process (or Brownian integral) and $e^{Y(t)} = \exp\left(\int_0^t F(s)\, dB(s) - \frac{1}{2}\int_0^t F(s)^2\, ds\right)$.
- Define a new process

$$B_Q(t) = B(t) - \int_0^t F(s)\, ds.$$

Theorem

(Girsanov): B_Q is a Q-Brownian motion.

- Generalization of Girsanov: Let M be a martingale of the form $M(t) = \int_0^t \int_A L(x,s)\, \widetilde{N}(ds, dx)$, with L predictable. Then

$$N(t) = M(t) - \int_0^t \int_A L(s,x)\left(e^{H(s,x)} - 1\right) \nu(dx)\, ds$$

is a Q-martingale.

Option pricing

- Stock price: $S = (S(t), t \geq 0)$.
- Contingent claims with maturity date T: Z is a non-negative $\mathcal{F}_T$ measurable r.v. representing the payoff of the option.
- European call option: $Z = \max\{S(T) - K, 0\}$
- American call option: $Z = \sup_{0 \leq \tau \leq T} [\max\{S(\tau) - K, 0\}]$
- We assume that the interest rate r is constant.
- Discounted stock price process: $\widetilde{S} = \left(\widetilde{S}(t), t \geq 0\right)$ with $\widetilde{S}(t) = e^{-rt}S(t)$.
- Portfolio: $(\alpha(t), \beta(t))$, $\alpha(t)$ is the number of shares and $\beta(t)$ the number of riskless assets (bonds).
- Portfolio value: $V(t) = \alpha(t) S(t) + \beta(t) A(t)$
- A portfolio is said to be replicating if $V(T) = Z$.

Option pricing

- Self-financing portfolio: $dV(t) = \alpha(t)\, dS(t) + r\beta(t) A(t)\, dt$.
- A market is said to be complete if every contingent claim can be replicated by a self-financing portfolio.
- An arbitrage opportunity exists if the market allows risk-free profit. The market is arbitrage free if there exists no self-financing strategy for which $V(0) = 0$, $V(T) \geq 0$ and $P(V(T) > 0) > 0$.

Theorem

(Fundamental Theorem of Asset Pricing 1) If the market is free of arbitrage opportunities, then there exists a probability measure Q, which is equivalent to P, with respect to which the discounted process $\widetilde{S}$ is a martingale.

Option pricing

Theorem

Fundamental Theorem of Asset Pricing 2) An arbitrage-free market is complete if and only if there exists a unique probability measure Q, which is equivalent to P, with respect to which the discounted process $\widetilde{S}$ is a martingale.

- Such a Q is called a martingale measure or risk-neutral measure.
- If Q exists, but is not unique, then the market is incomplete.
- In a complete market, it turns out that we have

$$V(t) = e^{-r(T-t)}\mathbb{E}_Q\left[Z|\mathcal{F}_t\right]$$

and this is the arbitrage-free price of the claim Z at time t.

Meta-Theorem and complete/incomplete markets

- Let R be the number of random sources in a model and N be the number of risky assets.
- Meta-Theorem (see Bjork): The market is arbitrage free if and only if $N \leq R$ and the market is complete if and only if $N \geq R$
- The standard Black-Scholes model with one risky asset is arbitrage free and complete ($N = R = 1$).
- In a Lévy model, in general the market is incomplete, except in some very particular cases.

Stock price as a Lévy process

- Return:

$$\frac{\delta S(t)}{S(t)} = \sigma \delta X(t) + \mu \delta t,$$

where $X = (X(t), t \geq 0)$ is a Lévy process and $\sigma > 0, \mu$ are parameters called the volatility and stock drift.

- Itô calculus SDE:

$$\begin{aligned} dS(t) &= \sigma S(t-) \, dX(t) + \mu S(t-) \, dt \\ &= S(t-) dZ(t), \end{aligned}$$

where $Z(t) = \sigma X(t) + \mu t$.

- Then $S(t) = \mathcal{E}_{Z(t)}$ is the stochastic exponential of Z.

Stock price as a Lévy process

- When X is a standard Brownian motion B, we obtain the geometric Brownian motion

$$S(t) = \exp\left(\sigma B(t) + \left(\mu - \frac{1}{2}\sigma^2\right) t\right)$$

- idea: Let X be a Lévy process. In order for stock prices to be non-negative, (5) yields $\Delta X(t) > -\sigma^{-1}$ (a.s.) for each $t > 0$. Denote $c = -\sigma^{-1}$.
- We impose $\int_{(c,-1]\cup[1,+\infty)} x^2 \nu(dx) < \infty$. This means that each $X(t)$ has first and second moments (reasonable for stock returns).
- By the Lévy-Itô decomposition,

$$X(t) = mt + kB(t) + \int_c^\infty x \widetilde{N}(t, dx),$$

where $k \geq 0$ and $m = b + \int_{(c,-1]\cup[1,+\infty)} x\nu(dx)$ (in terms of the earlier parameters).

Stock price as a Lévy process

- Representing $S(t)$ as the stochastic exponential $\mathcal{E}_{Z(t)}$, we obtain from (7) that

$$d(\log(S(t))) = k\sigma dB(t) + \left(m\sigma + \mu - \frac{1}{2}k^2\sigma^2\right) dt$$
$$+ \int_c^\infty \log(1+\sigma x)\widetilde{N}(dt, dx) + \int_c^\infty (\log(1+\sigma x) - \sigma x)\nu(dx)\,dt$$

- There are a number of explicit mathematically tractable and realistic models: variance-gamma, normal inverse Gaussian, hyperbolic, etc.

Change of measure

- we seek to find measures Q, which are equivalent to P, with respect to which the discounted stock process $\widetilde{S}$ is a martingale.
- Let Y be a Lévy-type stochastic integral of the form:

$$dY(t) = G(t)\,dt + F(t)\,dB(t) + \int_{\mathbb{R}-\{0\}} H(t,x)\widetilde{N}(dt, dx).$$

- Consider that e^Y is an exponential martingale (therefore, G is determined by F and H).
- Define Q by $\frac{dQ}{dP} = e^{Y(T)}$. By Girsanov theorem and its generalization:

$$B_Q(t) = B(t) - \int_0^t F(s)\,ds \text{ is a } Q\text{-BM}$$
$$\widetilde{N}_Q(t,A) = \widetilde{N}(t,A) - \nu_Q(t,A) \text{ is a } Q\text{-martingale}$$
$$\nu_Q(t,A) := \int_0^t \int_A \left(e^{H(s,x)} - 1\right)\nu(dx)\,ds.$$

Change of measure

- $\widetilde{S}(t) = e^{-rt}S(t)$ can be written in terms of these processes by:

$$d\left(\log\left(\widetilde{S}(t)\right)\right) = k\sigma dB_Q(t) + \left(m\sigma + \mu - r - \frac{1}{2}k^2\sigma^2 + k\sigma F(t)\right.$$
$$\left.+\sigma\int_{\mathbb{R}-\{0\}} x\left(e^{H(t,x)} - 1\right)\nu(dx)\right)dt + \int_c^\infty \log(1+\sigma x)\widetilde{N}_Q(dt,dx)$$
$$+\int_c^\infty (\log(1+\sigma x) - \sigma x)\nu_Q(dt,dx).$$

- Put $\widetilde{S}(t) = \widetilde{S}_1(t)\widetilde{S}_2(t)$, where

$$d\left(\log\left(\widetilde{S}_1(t)\right)\right) = k\sigma dB_Q(t) - \frac{1}{2}k^2\sigma^2 dt$$
$$+\int_c^\infty \log(1+\sigma x)\widetilde{N}_Q(dt,dx) + \int_c^\infty (\log(1+\sigma x) - \sigma x)\nu_Q(dt,dx).$$

Change of measure

- and

$$d\left(\log\left(\widetilde{S}_2(t)\right)\right) = \left(m\sigma + \mu - r + k\sigma F(t) + \right.$$
$$\left.+\sigma\int_{\mathbb{R}-\{0\}} x\left(e^{H(t,x)} - 1\right)\nu(dx)\right)dt.$$

- Apllying Itô's formula to $\widetilde{S}_1$ we obtain:

$$d\widetilde{S}_1(t) = k\sigma\widetilde{S}_1(t-)dB_Q(t) + \int_c^\infty \sigma\widetilde{S}_1(t-)x\widetilde{N}_Q(dt,dx).$$

and $\widetilde{S}_1$ is a Q-martingale.
- Therefore $\widetilde{S}$ is a Q-martingale if and only if

$$m\sigma + \mu - r + k\sigma F(t) + \sigma\int_{\mathbb{R}-\{0\}} x\left(e^{H(t,x)} - 1\right)\nu(dx) = 0 \text{ a.s.} \qquad (13)$$

Change of measure

- Equation (13) clearly has an infinite number of possible solution pairs (F, H).
- There are an infinite number of possible measures Q with respect to which $\widetilde{S}$ is a martingale. So the general Lévy process model gives rise to incomplete markets.
- Example - the Brownian motion case: $\nu = 0$ and $k \neq 0$. Then there is a unique solution
$$F(t) = \frac{r - \mu - m\sigma}{k\sigma} \text{ a.s.}$$
and the market is complete (Black-Scholes model).

Change of measure

- Example - the Poisson Process case:take $k = 0$ and $\nu(x) = \lambda\delta_1(x)$. Then $X(t) = mt + \int_c^\infty x\widetilde{N}(t, dx)$, where the jump part is the standard Poisson process $N(t)$. Writing $H(t, 1) = H(t)$, we have from (13) that
$$m\sigma + \mu - r + \sigma\lambda\left(e^{H(t)} - 1\right) = 0 \text{ a.s.}$$
and
$$H(t) = \log\left(\frac{r - \mu + (\lambda - m)\sigma}{\lambda\sigma}\right).$$
In this case, the market is also complete and we obtain a martingale measure if $r - \mu + (\lambda - m)\sigma > 0$.
- In most part of the other cases (with other Lévy processes), the market is incomplete.

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