2.2. SHORT RATE MODELS

2.2.1. Interest Rate Trees

2.2.2. Continuous-time Single-factor models

2.2.3. Continuous-time Multi-Factor models

2.2.4. Modeling the Term Structure: Affine Models

2.2.1. Interest Rate Trees

- Focus: How to model the term structure by specifying the behavior of the short-term interest rate?

- Bond and interest rate derivative prices depend on the behavior of the risk-free short-term interest rate (or instantaneous short rate).

⬇

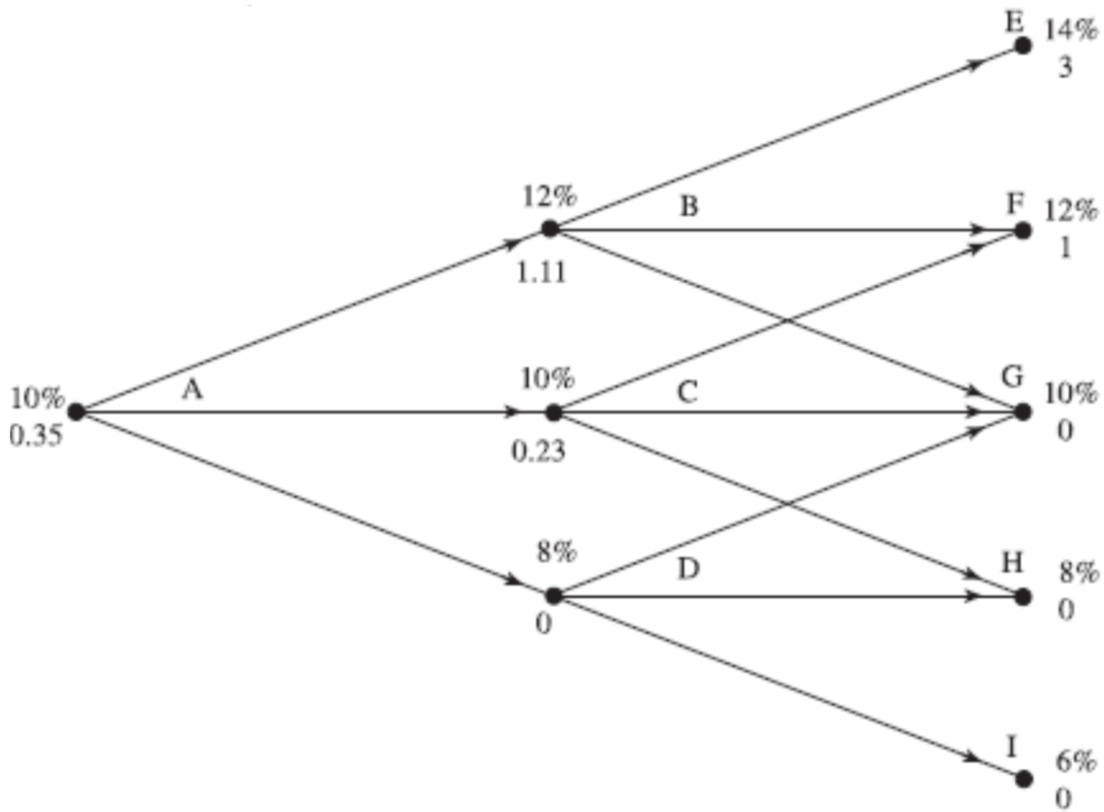
- The variable to be modeled by trees will be the instantaneous short rate.

2.2.1. Interest Rate Trees

- Why do we use trees? - A tree is a discrete-time representation of the stochastic process.
- Most trees are binomial, event though trinomial trees are recommended to value interest rate derivatives.
- At the final nodes, the value of the derivative equals its pay-off.
- At previous nodes, the value of the derivative is calculated through a rollback procedure, calculating the expected value of the derivative according to the probabilities attached to the different scenarios and discounting this expected value using the interest rate at that node.

2.2.1. Interest Rate Trees

Figure 32.4 Example of the use of trinomial interest rate trees. Upper number at each node is rate; lower number is value of instrument.



The tree is used to value a derivative that provides a payoff at the end of the second time step of

$$\max[100(R - 0.11), 0]$$

Assumption:

Probabilities of up, middle and down are 0.25, 0.5 and 0.25, respectively.

Derivative value at Node B:

$$[0.25 \times 3 + 0.5 \times 1 + 0.25 \times 0]e^{-0.12\times 1} = 1.11$$

Derivative value at Node C:

$$(0.25 \times 1 + 0.5 \times 0 + 0.25 \times 0)e^{-0.1\times 1} = 0.23$$

Derivative value at Node A:

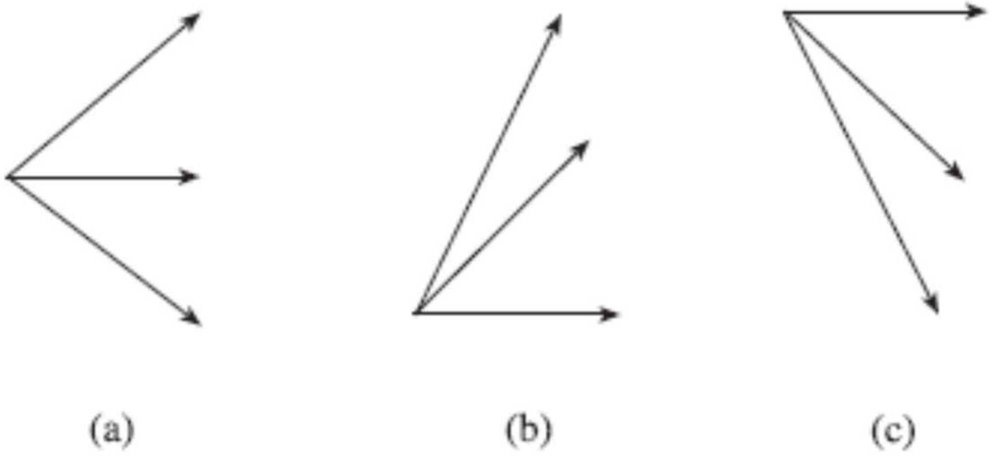
$$(0.25 \times 1.11 + 0.5 \times 0.23 + 0.25 \times 0)e^{-0.1\times 1} = 0.35$$

Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

2.2.1. INTEREST RATE TREES

- Non-standard branching

Figure 32.5 Alternative branching methods in a trinomial tree.



Source: Hull, John (2018), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 10th Edition

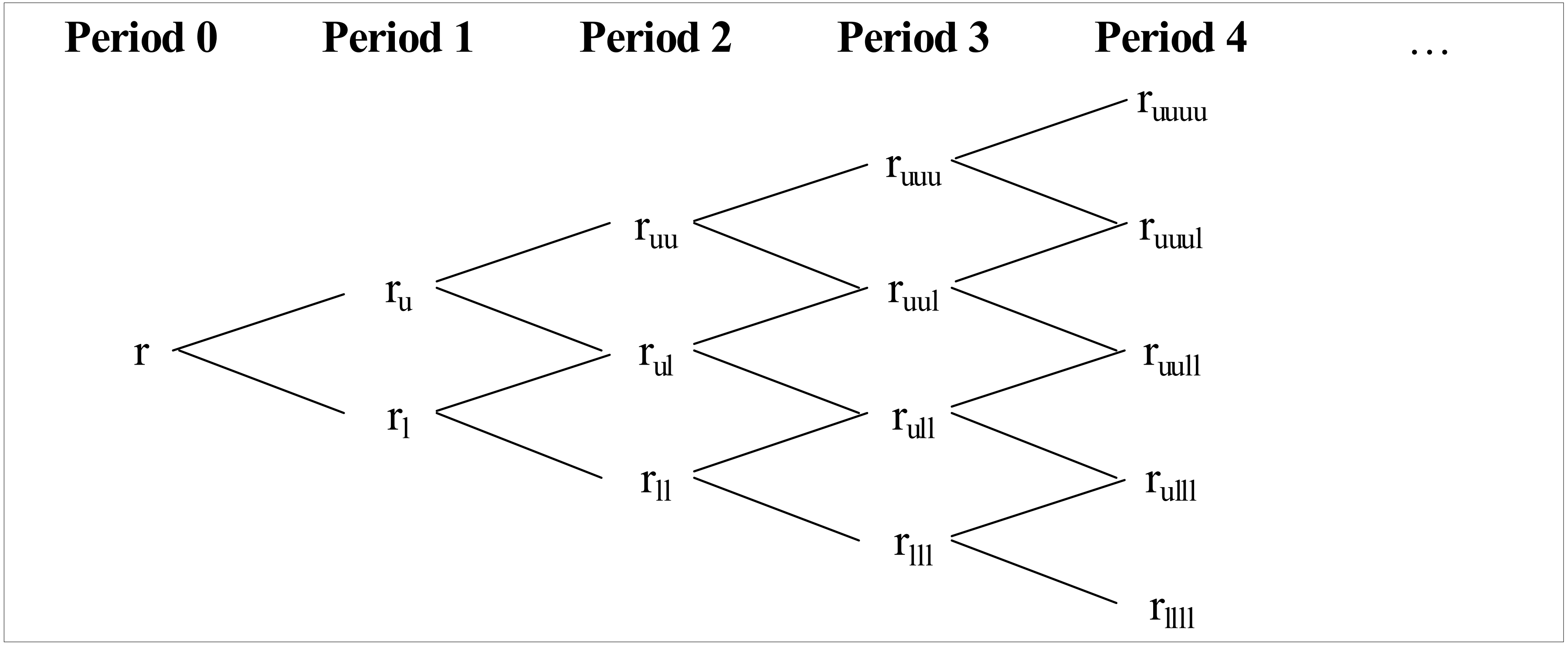
- (b) and (c) are useful to represent mean-reverting interest rates when interest rates are either very low (and are not supposed to move even lower) or very high (and are not supposed to move even higher), respectively.

2.2.1. Interest Rate Trees

- General binomial model
  - Given the current level of short-term rate r, the next-period short rate can take only two possible values: an upper value r_u and a lower value r_l, with equal probability 0.5
  - In period 2, the short-term interest rate can take on four possible values: r_{uu}, r_{ul}, r_{lu}, r_{ll}
  - More generally, in period *n*, the short-term interest rate can take on 2^n values => very time-consuming and computationally inefficient

- Recombining trees
  - Means that an upward-downward sequence leads to the same result as a downward-upward sequence (regardless being binomial or trinomial trees)
  - For example, $r_{ul} = r_{lu}$
  - Only (n+1) different values at period n

INTEREST RATE TREE - Recombining



INTEREST RATE TREE – analytical

- We may write down the binomial process as:

$$\Delta r_t \equiv r_{t+1} - r_t = \sigma \varepsilon_t$$

$$\Delta r_t \equiv r_{t+\Delta t} - r_t = \mu\left(t, \Delta t, r_t\right) + \sigma\left(t, \Delta t, r_t\right)\varepsilon_t$$

- Specific case – assuming that the drift and the variance are proportional to the time increment:

$$\Delta r_t \equiv r_{t+\Delta t} - r_t = \mu \Delta t + \sigma \sqrt{\Delta t}\,\varepsilon_t$$

- Continuous-time limit (Merton (1973)):

$$dr_t \equiv r_{t+dt} - r_t = \mu dt + \sigma dW_t$$

INTEREST RATE TREE – calibration

- **Goal:** to make the model consistent with the current term structure.
- At date 0 (working in logs):

$$\Delta \ln r_0 \equiv \ln r_{\Delta t} - \ln r_0 = \mu \Delta t + \sigma \varepsilon_0 \sqrt{\Delta t}$$

(From $\Delta r_t \equiv r_{t+\Delta t} - r_t = \mu \Delta t + \sigma \sqrt{\Delta t}\, \varepsilon_t$)

- As the uncertainty source is the path of the interest rate (up or down), the difference between interest rates in $t+\Delta t$ will be originated by the random factor (the deterministic factor will be the same if the interest rate increases or decreases):

$$\Rightarrow \ln r_u - \ln r_l = 2\sigma \sqrt{\Delta t} \quad \text{or} \quad r_u = r_l \exp\left(2\sigma \sqrt{\Delta t}\right)$$

  - If we take as given an estimate for σ and the current yield curve y_t, we iteratively find the values r_u, r_l, r_{uu}, r_{ul}, r_{lu}, r_{ll}, etc., consistent with the input data.

EXAMPLE

- Consider a 2-period tree (t=0 and t=1 => $\Delta t = 1$)
- The price 1 year from now of a 2-year Treasury bond (at the par value, i.e. coupon rate = yield) can take 2 values:

- P_u - associated with r_u (price with the interest rate increasing)
- P_l - associated with r_l (price with the interest rate decreasing):

Coupon rate, which is equal to the yield for a bond at par (at t=0).

$$P_u = \frac{100 + y_2}{1 + r_u} \quad \text{and} \quad P_d = \frac{100 + y_2}{1 + r_l}$$

r_u and r_l must be seen as the 2 possible future values in t=0 of the 1-period interest rate in t=1

NPV at *t*=1 of the future cash-flows of the bond - redemption and the last coupon (y_2, the 2-period yield at *t*=0, that corresponds to the coupon rate, as it is assumed that the bond is at par value), as in *t*=1 there is only one remaining period for the bond => the future cash-flows in *t*=1 (to be paid in t=2) are the redemption and the last coupon.

The uncertainty in *t*=0 about the bond price in *t*=1 stems from the uncertainty about the interest rate, which may have increased or decreased.

EXAMPLE

- Given that

$$\Rightarrow \ln r_u - \ln r_l = 2\sigma\sqrt{\Delta t} \quad \text{or} \quad r_u = r_l \exp\left(2\sigma\sqrt{\Delta t}\right)$$

- Taking expectations at time 0, we find an equation that can be solved for r_u and r_l , replacing r_u by the previous expression (and being Δt = 1) and taking into consideration that

- $$P_u = \frac{100 + y_2}{1 + r_u} \quad \text{and} \quad P_d = \frac{100 + y_2}{1 + r_l}$$

Bond price in t=0 is the expected value of the future cash-flows – the coupon in t=1 and the bond price also in t=1, which is the NPV at t=1 of the cash-flows to be paid in t=2.

The bond price in t=0 is also equal to 100, as it is assumed that the bond is at par.

The probability at t=0 for each bond price in t=1, with r_u or r_l, is ½.

$$100 = \frac{1}{2}\left(\frac{\dfrac{100 + y_2}{1 + r_l \exp\left(2\sigma\right)} + y_2}{1 + y_1} + \frac{\dfrac{100 + y_2}{1 + r_l} + y_2}{1 + y_1}\right)$$

$\Delta t = 1$

1st year coupon, as in t=0 we have 2 coupons ahead

Discounted by y_1 as these are cash-flows that will occur in t=1

2.2.2. CT Single Factor Models

- General expression for a single-factor continuous-time model (from the continuous time limit - Merton (1973))

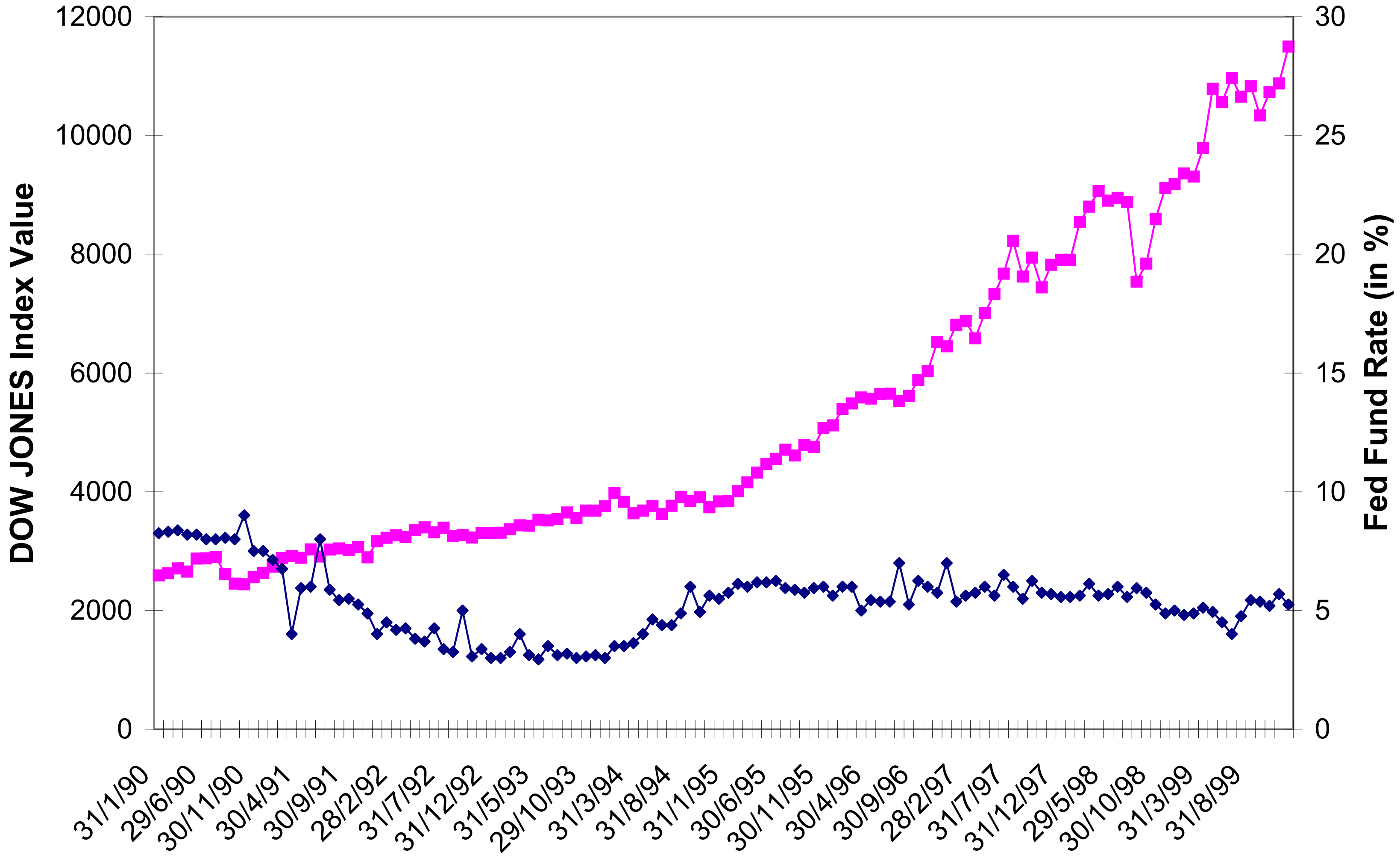
$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$$

- The term *W* denotes a Brownian motion - process with independent normally distributed increments: $dW_t = \varepsilon_t \sqrt{dt}$
  - *dW* represents the instantaneous change.
  - It is stochastic (uncertain)
  - It is a stochastic variable with a normal distribution with zero mean and variance *dt*

WHAT IS A GOOD MODEL?

- A good model is a model that is consistent with reality
- Stylized facts about the dynamics of the term structure:
  - Fact 1: (nominal) interest rates are (usually) positive
  - Fact 2: interest rates are mean-reverting
  - Fact 3: interest rates with different maturities are imperfectly correlated
  - Fact 4: the volatility of interest rates evolves (randomly) in time
- A good model should also be:
  - Tractable
  - Parsimonious

Empirical Facts 1, 2 and 4



Empirical Fact 3

	1M	3M	6M	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1M	1												
3M	0.992	1											
6M	0.775	0.775	1										
1Y	0.354	0.3	0.637	1									
2Y	0.214	0.165	0.42	0.901	1								
3Y	0.278	0.246	0.484	0.79	0.946	1							
4Y	0.26	0.225	0.444	0.754	0.913	0.983	1						
5Y	0.224	0.179	0.381	0.737	0.879	0.935	0.981	1					
6Y	0.216	0.168	0.352	0.704	0.837	0.892	0.953	0.991	1				
7Y	0.228	0.182	0.35	0.661	0.792	0.859	0.924	0.969	0.991	1			
8Y	0.241	0.199	0.351	0.614	0.745	0.826	0.892	0.936	0.968	0.992	1		
9Y	0.238	0.198	0.339	0.58	0.712	0.798	0.866	0.913	0.95	0.981	0.996	1	
10Y	0.202	0.158	0.296	0.576	0.705	0.779	0.856	0.915	0.952	0.976	0.985	0.99	1

Daily changes in French swap markets in 1998

EQUILIBRIUM ONE-FACTOR CT MODELS OF THE SHORT RATE

➢ **Equilibrium vs No-Arbitrage Models**

- Equilibrium models don't automatically fit today's TSIR, even though they can provide an approximate fit to many observed TSIRs, if the parameters are properly chosen.
- Many traders, when valuing derivatives, argue that they can these models provide unsatisfactory estimates to bond prices, that may lead to errors of 25% in option pricing.
- Conversely, no-arbitrage models are designed to be exactly consistent with the current TSIR.

EQUILIBRIUM ONE-FACTOR CT MODELS OF THE SHORT RATE

➢ **Equilibrium vs No-Arbitrage Models**

- Equilibrium models:

(i) the current TSIR is an output

(ii) the drift of the short rate is not usually a function of time

- No-arbitrage models:

(i) the current TSIR is an input

(ii) the drift is, in general, dependent on time, as the shape of the initial zero curve governs the average path taken by the short rate in the future – positively sloped zero curve => positive drift for the short rate.

⬇

Equilibrium models can be transformed into no-arbitrage models by including a function of time in the drift of the short rate.

EQUILIBRIUM ONE-FACTOR CT MODELS OF THE SHORT RATE

- The short-term rate is the single factor => endogenous models:

$$dr = m(r)\,dt + s(r)\,dz$$

1. Rendleman and Bartter $dr = \mu r\,dt + \sigma r\,dz$

Rendleman, R. and B. Bartter (1980). "The Pricing of Options on Debt Securities". *Journal of Financial and Quantitative Analysis*. **15**: 11–24).

Pros:

- More tractable model, as it follows a GMB.

Cons:

- Assumes that interest rates follow a stochastic process similar to stocks, while they usually exhibit a mean-reversion behavior.

EQUILIBRIUM ONE-FACTOR CT MODELS OF THE SHORT RATE

2. Vasicek (1977) $m(r) = a(b - r);\ s(r) = \sigma$

Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177–188.

Pros:

- More tractable model, due to constant volatility.
- Interest rates are mean-reverting (to *b*), at a reversion rate (pace) *a*.

Cons:

- Gaussian distributions for interest rates are not compatible with market implied distributions.
- Interest rate volatility is often variable, namely during periods of higher uncertainty, when the estimation of interest rates becomes more complex but also more useful.

EQUILIBRIUM ONE-FACTOR CT MODELS OF THE SHORT RATE

3. Cox, Ingersoll and Ross (CIR)

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dz$$

Stochastic volatility model => higher volatility with higher interest rates.

Cox, Ingersoll, and Ross. 1985, "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol 53, March.

Pros:

- Model closer to reality, as interest rates have stochastic volatilities (higher volatilities with higher interest rates).

Cons:

- Model becomes less tractable, as it requires the single factor to be positive.

NO-ARBITRAGE SHORT RATE CT MODELS

1. **Ho-Lee (1986)**

Ho, T.S.Y., and S.-B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41 (December 1986): 1011–29.

$$dr = \theta(t)\,dt + \sigma\,dz$$

$\theta(t)$ defines the average direction that *r* moves at time *t*:

2. **Hull-White One-Factor Model (1990)**

Hull, J.C., and A. White, "Pricing Interest Rate Derivative Securities," *The Review of Financial Studies*, 3, 4 (1990): 573–92.

Extended version of Vasicek, to provide an exact fit to the initial TSIR: $dr = [\theta(t) - ar]\,dt + \sigma\,dz$ or $dr = a\left[\frac{\theta(t)}{a} - r\right]dt + \sigma\,dz$

Corresponds to Ho-Lee model, with mean reversion at rate *a*.

NO-ARBITRAGE SHORT RATE CT MODELS

3. Black-Derman-Toy (1990)

Black, F., E. Derman, and W. Toy, "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Prices," *Financial Analysts Journal*, January/February 1990: 33–39.

$$d \ln r = [\theta(t) - a(t) \ln r]\, dt + \sigma(t)\, dz$$

with $a(t) = -\dfrac{\sigma'(t)}{\sigma(t)}$ and $\sigma'(t)$ is the derivative of σ with respect to *t*.

- It is similar to Hull-White One-Factor Model, but in logs and with mean reversion rate *a* being time-dependent.
- It doesn't allow negative interest rates.

Constant volatility => $\sigma'(t)$ = 0 => a(t)=0 => BDT model: $d \ln r = \theta(t)\, dt + \sigma\, dz$

Log-normal version of Ho-Lee model

NO-ARBITRAGE SHORT RATE CT MODELS

4. Black-Karasinski (1991)

Black, F., and P. Karasinski, "Bond and Option Pricing When Short Rates Are Lognormal," *Financial Analysts Journal*, July/August (1991): 52–59.

Extended version of BDT (1990) model, where the reversion rate and volatility are determined independently of each other:

$$d \ln r = [\theta(t) - a(t) \ln r]\, dt + \sigma(t)\, dz$$

The model is the same as BDT (1990), but with no relation between *a(t)* and $\sigma(t)$.

2.2.3. CT MULTI FACTOR MODELS

1. **Fong and Vasicek (1991) model -** short rate and its volatility (*v*) as two state variables

H. G. Fong and O. A. Vasicek: Fixed-income volatility management. Journal of Portfolio Management, 41-56, 1991.

$$dr = \alpha(\bar{r} - r)dt + \sqrt{v}dz_1$$

$$dv = \gamma(\bar{v} - v)dt + \xi\sqrt{v}dz_2$$

2.2.3. CT Multi Factor Models

2. Longstaff and Schwartz (1992) model

Longstaff, F. A. and E. S. Schwartz, "Interest Rate Volatility and the Term Structure: A Two Factor General Equilibrium Model," *Journal of Finance*, 47, 4 (September 1992): 1259–82.

- Longstaff and Schwartz (1992) use the same two state variables (the short rate and its volatility) , but with a different specification.
- The starting point is a two-factor model, where the drift is governed by the two factors or state variables, while the variance is a function of only one of them:

$$\frac{dQ}{Q} = (\mu X + \theta Y)\, dt + \sigma \sqrt{Y}\, dZ_1$$

- With this specification, it is ensured that the drift and the variance are not perfectly correlated.
- The dynamics of the state variables are as follows:

$$dX = (a - bX)\, dt + c\sqrt{X}\, dZ_2$$

$$dY = (d - eY)\, dt + f\sqrt{Y}\, dZ_3 :$$

2.2.3. CT MULTI FACTOR MODELS

2. Longstaff and Schwartz (1992) model

- With the rescaling of the state variables to $x = X/c^2$ and $y = Y/f^2$, the dynamics of state variables are as follows:

$$dr = \left(\alpha\gamma + \beta\eta - \frac{\beta\delta - \alpha\xi}{\beta - \alpha} r - \frac{\xi - \delta}{\beta - \alpha} V \right) dt$$

$$+ \alpha \sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}}\, dZ_2 + \beta \sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}}\, dZ_3,$$

$$dV = \left(\alpha^2\gamma + \beta^2\eta - \frac{\alpha\beta(\delta - \xi)}{\beta - \alpha} r - \frac{\beta\xi - \alpha\delta}{\beta - \alpha} V \right) dt$$

$$+ \alpha^2 \sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}}\, dZ_2 + \beta^2 \sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}}\, dZ_3$$

where $\gamma = a/c^2$, $\delta = b$, $\eta = d/f^2$, $\xi = e$, r is the instantaneous riskless rate,

where $\alpha = \mu c^2$ and $\beta = (\theta - \sigma^2) f^2$

2.2.3. CT Multi Factor Models

2. Longstaff and Schwartz (1992) model

- Relevant features:

(i) All parameters are positive;

(ii) *r* is non-negative, since both state variables follow square root processes;

(iii) *r* has a long-run stationary distribution with mean and variance:

$$E[r] = \frac{\alpha\gamma}{\delta} + \frac{\beta\eta}{\xi} \qquad Var[r] = \frac{\alpha^2\gamma}{2\delta^2} + \frac{\beta^2\eta}{2\xi^2}$$

(iv) Volatility also has a stationary distribution with mean

$$E[V] = \frac{\alpha^2\gamma}{\delta} + \frac{\beta^2\eta}{\xi} \qquad Var[V] = \frac{\alpha^4\gamma}{2\delta^2} + \frac{\beta^4\eta}{2\xi^2}$$

(v) *r* depends on volatility, but volatility also depends on *r;*

2.2.3. CT Multi Factor Models

Most Popular Models

2. Longstaff and Schwartz (1992) model

- Closed-form expressions for riskless discount bond prices with τ maturity ($\tau = 0 \Rightarrow F = 1$)

$$F(r, V, \tau) = A^{2\gamma}(\tau) B^{2\eta}(\tau) \exp(\kappa\tau + C(\tau) r + D(\tau) V),$$

where

$$A(\tau) = \frac{2\phi}{(\delta + \phi)(\exp(\phi\tau) - 1) + 2\phi},$$

$$B(\tau) = \frac{2\psi}{(\nu + \psi)(\exp(\psi\tau) - 1) + 2\psi},$$

$$C(\tau) = \frac{\alpha\phi(\exp(\psi\tau) - 1)B(\tau) - \beta\psi(\exp(\phi\tau) - 1)A(\tau)}{\phi\psi(\beta - \alpha)}$$

$$D(\tau) = \frac{\psi(\exp(\phi\tau) - 1)A(\tau) - \phi(\exp(\psi\tau) - 1)B(\tau)}{\phi\psi(\beta - \alpha)},$$

and

$$\nu = \xi + \lambda,$$

$$\phi = \sqrt{2\alpha + \delta^2},$$

$$\psi = \sqrt{2\beta + \nu^2},$$

$$\kappa = \gamma(\delta + \phi) + \eta(\nu + \psi).$$

2.2.3. CT MULTI FACTOR MODELS

2. Longstaff and Schwartz (1992) model

- YTM of riskless discount bonds with τ maturity:

$$Y_\tau = -(\kappa\tau + 2\gamma \ln A(\tau) + 2\eta \ln B(\tau) + C(\tau)r + D(\tau)V)/\tau$$

- For a given τ maturity, the yield is a linear function of *r* and *V*.

2.2.3. CT MULTI FACTOR MODELS

2. Longstaff and Schwartz (1992) model

- It can be shown that:

$\tau \rightarrow 0 \Rightarrow Y_t \rightarrow r$

$\tau \rightarrow \infty \Rightarrow Y_t$ tends to a constant $\gamma(\phi - \delta) + \eta(\psi - \nu)$

↓

- The current values of *r* and *V* become less relevant for very distant cash-flows

↓

- The current term structure is irrelevant for the determination of very long interest rates.

2.2.3 – CT MULTI FACTOR MODELS

2. Longstaff and Schwartz (1992) model

- This model offers a much larger variety of shapes than single factor models, with one inflexion point for the slope and the convexity.
- Instantaneous expected return for a discount bond:

$$r + \lambda \frac{(\exp(\psi\tau) - 1)B(\tau)}{\psi(\beta - \alpha)}(\alpha r - V)$$

- Subtracting r from the previous result, one obtains the risk premium.
- For a given τ maturity, the term premium is a linear function of *r* and *V*, depending on λ (market price of risk):
  - λ <0 => term premium > 0.
  - λ =0 => term premium = 0 => Expectations theory holds.
- For small τ, the term premium is an increasing function of *r*.

3. Balduzzi et al. (1996) models

Balduzzi, P., S. R. Das, S. Foresi, and R. Sundaran, 1996, "A Simple Approach to Three-Factor Affine Term Structure Models," *The Journal of Fixed Income,* 6, 14–31.

- Balduzzi et al. (1996) suggest the use of a three-factor model by adding the mean of the short-term rate (θ) to a 2-factor model.

$$dr = \mu_r(r, \theta, t)dt + \sigma_r(r, V, t)dz$$

$$d\theta = \mu_\theta(\theta, t)dt + \sigma_\theta(\theta, t)dw$$

$$dV = \mu_V(V, t)dt + \sigma_V(V, t)dy$$

$$dr = \kappa(\theta - r)dt + \sqrt{V}\,dz$$

$$d\theta = \alpha(\beta - \theta)dt + \eta\,dw$$

$$dV = a(b - V)dt + \phi\sqrt{V}\,dy$$