Models in Finance - Class 16

Master in Actuarial Science

João Guerra

ISEG

Continuous time models: preliminary concepts

- Model for prices: stochastic process S_t adapted to a filtration $\{\mathcal{F}_t, t \geq 0\}$ (that is, given $\mathcal{F}_t$ we know the value of S_u for all $u \leq t$).
- $\mathcal{F}_t$ is a subset of a larger possible range of past and future events, $\mathcal{F}$.
- Let A be some event contained in $\mathcal{F}$. Then $P(A)$ is the actual "real world" probability that the event A will occur.
- Suppose also that we have a risk-free cash account which has a value at time t of B_t (we assume that the risk-free rate of interest is constant: that is, $B_t = B_0 e^{rt}$).

Self-financing strategies

- Suppose that at time t we hold the portfolio (φ_t, ψ_t) where φ_t represents the number of units of S_t held at time t and ψ_t is the number of units of the cash bond held at time t.
- Assume that (φ_t, ψ_t) are previsible (φ_t, ψ_t are known based upon the information up to but not including time t).
- Value of the portfolio at time t: $V(t) = \varphi_t S_t + \psi_t B_t$.
- Consider the instantaneous pure investment gain in the value of this portfolio over the period t up to $t + dt$ (assuming that there is no inflow or outflow of cash during the period $[t, t + dt]$). This is equal to $\varphi_t dS_t + \psi_t dB_t$.
- The portfolio strategy is self-financing if $dV(t) = \varphi_t dS_t + \psi_t dB_t$.

Complete markets

- Let X be any derivative payment (payoff) contingent upon $\mathcal{F}_U$ where U is the time of payment of X (i.e. X is $\mathcal{F}_U$-measurable and, therefore, depends upon the path of S_t up to time U.
- A replicating strategy is a self-financing strategy (φ_t, ψ_t), defined for $0 \leq t \leq U$, such that $V(U) = \varphi_U S_U + \psi_U B_U = X$.
- In other words, for an initial investment of $V(0)$ at time 0, if we follow the self-financing portfolio strategy (φ_t, ψ_t) we will be able to reproduce the
- derivative payment without risk.
- The market is said to be complete if for any such contingent claim X there is a replicating strategy (φ_t, ψ_t).

Continuous time models: preliminary concepts

- Example of a complete market: the binomial model (we could replicate any derivative payment contingent on the history of the underlying asset price).
- Another example of a complete market is the continuous-time lognormal model for share prices:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma Z_t\right),$$

where Z_t is a standard Brownian motion.

Continuous time models: preliminary concepts

- Two measures P and Q which apply to the same sigma-algebra $\mathcal{F}$ are said to be equivalent if for any event $E \in \mathcal{F} : P(E) > 0$ if and only if $Q(E) > 0$, where $P(E)$ and $Q(E)$ are the probabilities of E under P and Q respectively.
- For the binomial model, for the equivalence of P and Q the only constraint on the real-world measure P is that at any point in the binomial tree the probability of an up move lies strictly between 0 and 1. The only constraint on Q is the same.

Continuous time models: preliminary concepts

- Suppose that Z_t is a standard Brownian motion under P and let $X_t = \gamma t + \sigma Z_t$ be a Brownian motion with drift under P.
- Is there a measure Q under which X_t is a standard Brownian motion and which is equivalent to P ?
- Yes if $\sigma = 1$ but no if $\sigma \neq 1$
- In other words: we can change the drift of the Brownian motion but not the volatility.
- Theorem (Cameron-Martin-Girsanov or Girsanov): Suppose that Z_t is a standard Brownian motion under P and that γ_t is a previsible process. Then there exists a measure Q equivalent to P and where $\widetilde{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q. Conversely, if Z_t is a standard Brownian motion under P and if Q is equivalent to P then there exists a previsible process γ_t such that $\widetilde{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q.

Continuous time models: preliminary concepts

- Assume that under P (geometric Bm):$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma Z_t\right)$. Then ($e^{-rt} S_t$ is the discounted price):

$$E_P\left[e^{-rt} S_t\right] = e^{(\mu - r)t}$$

and $e^{-rt} S_t$ is not a martingale under P (unless $\mu = r$).

- Take $\gamma_t = \gamma = \frac{\mu - r}{\sigma}$ and define $\widetilde{Z}_t = Z_t + \int_0^t \gamma_s ds = Z_t + \frac{(\mu - r)}{\sigma} t$. Then:

$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma \widetilde{Z}_t - (\mu - r)\, t\right) \\ &= S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma \widetilde{Z}_t\right). \end{aligned}$$

By the Cameron-Martin-Girsanov theorem, exists Q equivalent to P such that $\widetilde{Z}_t$ is a Q-standard Bm.

Continuous time models: preliminary concepts

- And clearly, we have (for $u < t$):

$$
\begin{aligned}
&E_Q\left[e^{-rt}S_t|\mathcal{F}_u\right] = \\
&= e^{-rt}S_u E_Q\left[\exp\left(\left(r-\frac{1}{2}\sigma^2\right)(t-u)+\sigma\left(\widetilde{Z}_t-\widetilde{Z}_u\right)\right)\right] \\
&= e^{-ru}S_u E_Q\left[\exp\left(\left(-\frac{1}{2}\sigma^2\right)(t-u)+\sigma\left(\widetilde{Z}_t-\widetilde{Z}_u\right)\right)\right] \\
&= e^{-ru}S_u e^{\left(-\frac{1}{2}\sigma^2\right)(t-u)+\frac{1}{2}\sigma^2(t-u)} = e^{-ru}S_u
\end{aligned}
$$

- Therefore, the discounted price $e^{-rt}S_t$ is a Q-martingale.

Continuous time models: preliminary concepts

- Suppose that X_t is a P-martingale and Y_t is another P-martingale.
- Martingale Representation Theorem (MRT): Exists a unique previsible process ϕ_t such that

$$Y_t = Y_0 + \int_0^t \phi_s dX_s \quad (\text{or: } dY_t = \phi_t dX_t)$$

if and only if there is no other measure equivalent to P under which X_t is a martingale.

5 step method or martingale method

1. Establish the equivalent martingale measure Q.
2. Propose a fair price for the derivative V_t and its discounted value $F_t = e^{-rt} V_t$.
3. Use the MRT to construct a hedging strategy (portfolio) (ϕ_t, ψ_t).
4. Show that the hedging strategy (ϕ_t, ψ_t) replicates the derivative payoff at time n.
5. Therefore V_t is the fair price of the derivative at time t.