Microeconomics

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Fall 2019

Chapter 7:
Game theory

Strategic form games

A **strategic form game** is a tuple $G = (S_i, u_i)_{i=1}^N$, where for each player $i = 1, \ldots, N$, S_i is the set of strategies available to player i, and $u_i : \times_{j=1}^N S_j \to \mathbb{R}$ describes player i's payoff as a function of the strategies chosen by all players. A strategic form game is finite if each player's strategy set contains finitely many elements.

Dominant strategies

A strategy $\hat{s}_i$ for player i is **strictly dominant** if $u_i(\hat{s}_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $(s_i, s_{-i}) \in S$ with $s_i \neq \hat{s}_i$.

Player i' s strategy $\hat{s}_i$ **strictly dominates** strategy $\bar{s}_i$, if $u_i(\hat{s}_i, s_{-i}) > u_i(\bar{s}_i, s_{-i})$ for all $s_{-i} \in S_i$. In this case, we also say that $\bar{s}_i$ is strictly dominated in S.

Dominant strategies

A strategy s_i for player i is **iteratively strictly undominated** in S (or survives iterative elimination of strictly dominated strategies) if $s_i \in S_i^n$, for all $n \geq 1$.

Player i' s strategy $\hat{s}_i$ **weakly dominates** strategy $\bar{s}_i$, if $u_i(\hat{s}_i, s_{-i}) \geq u_i(\bar{s}_i, s_{-i})$ for all $s_{-i} \in S_i$, with at least one strict inequality. In this case, we also say that $\bar{s}_i$ is weakly dominated in S.

A strategy s_i for player i is **iteratively weakly undominated** in S (or survives iterative elimination of weakly dominated strategies) if $s_i \in W_i^n$, for all $n \geq 1$.

Nash equilibrium

Given a strategic form game $G = (S_i, u_i)_{i=1}^N$, the joint strategy $\hat{s} \in S$ is a **pure strategy Nash equilibrium** of G if for each player i, $u_i(\hat{s}) \geq u_i(s_i, \hat{s}_{-i})$ for all $s_i \in S_i$.

Mixed strategies

Fix a finite strategic form game $G = (S_i, u_i)_{i=1}^N$. A **mixed strategy** m_i for player i is a probability distribution over S_i. That is, $m_i : S_i \to [0,1]$ assigns to each $s_i \in S_i$ the probability, $m_i(s_i)$, that s_i will be played. We shall denote the set of mixed strategies for player i by M_i. Consequently, $M_i = \{m_i : S_i \to [0,1] | \sum_{s_i \in S_i} m_i(s_i) = 1\}$.
From now on, we shall call S_i player i's set of pure strategies.

Nash equilibrium

Given a finite strategic form game $G = (S_i, u_i)_{i=1}^N$, a joint strategy $\hat{m} \in M$ is a **Nash equilibrium** of G if for each player i, $u_i(\hat{m}) \geq u_i(m_i, \hat{m}_{-i})$ for all $m_i \in M_i$.

Characterization of Nash equilibrium

Theorem 7.1: The following statements are equivalent:

1. $\hat{m} \in M$ is a Nash equilibrium.
2. For every player i, $u_i(\hat{m}) = u_i(s_i, \hat{m}_{-i})$ for all $s_i \in S_i$ with positive weight in $\hat{m}_i$ and $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$ for all $s_i \in S_i$ with zero weight in $\hat{m}_i$.
3. For every player i, $u_i(\hat{m}) \geq u_i(s_i, \hat{m}_{-i})$ for all $s_i \in S_i$.

Existence of Nash equilibrium

Theorem 7.2:
Every finite strategic form game possesses at least one Nash equilibrium.

Game of incomplete information (Bayesian game)

A **game of incomplete information** is a tuple $G = (p_i, T_i, S_i, u_i)_{i=1}^{N}$, where for each player $i = 1, \ldots, N$, the set T_i is finite, $u_i : S \times T \to \mathbb{R}$, and for each $t_i \in T_i$, $p_i(\cdot|t_i)$ is a probability distribution on T_{-i}. If, in addition, for each player i, the strategy set S_i is finite, then G is called a **finite game of incomplete information**. A game of incomplete information is also called a **Bayesian game**.

Bayesian-Nash equilibrium

A **Bayesian-Nash equilibrium** of a game of incomplete information is a Nash equilibrium of the associated strategic form game.

Existence of Bayesian-Nash equilibrium

Theorem 7.3:
Every finite game of incomplete information possesses at least one Bayesian-Nash equilibrium.

Extensive form games

An **extensive form game**, denoted by Γ, is composed of the following elements:

1. A finite set of players N.
2. A set of actions A which includes all possible actions that might potentially be taken at some point in the game. A need not be finite.
3. A set of nodes, or histories, X where:
   0.1 X contains a distinguished element x_0, called the initial node, or empty history,
   0.2 each $x \in X \setminus \{x_0\}$ takes the form $x = (a_1, a_2, \ldots, a_k)$ for some finitely many actions $a_i \in A$, and
   0.3 if $(a_1, a_2, \ldots, a_k) \in X \setminus \{x_0\}$ for some $k > 1$, then $(a_1, a_2, \ldots, a_{k-1}) \in X \setminus \{x_0\}$.

A node, or history, is then simply a complete description of the actions taken so far in the game.

Extensive form games

We shall use the terms history and node interchangeably. Let $A(x) = \{a \in A : (x, a) \in X\}$ denote the set of actions available to the player whose turn it is to move after the history $x \in X \setminus \{x_0\}$.

4 A set of actions $A(x_0) \subseteq A$ and a probability distribution π on $A(x_0)$ to describe the role of chance in the game. Chance always moves first, and just once, by randomly selecting an action from $A(x_0)$ using the probability distribution π. Thus, $(a_1, a_2, \ldots, a_k) \in X \setminus \{x_0\}$ implies that $a_i \in A(x_0)$ for $i = 1$ and only $i = 1$.

5 A set of end nodes, $E = \{x \in X : (x, a) \notin X \text{ for all } a \in A\}$. Each end node describes one particular complete play of the game from beginning to end.

Extensive form games

6 A function $\iota : X \setminus (E \cup \{x_0\}) \to N$ that indicates whose turn it is at each decision node in X. Let $X_i = \{x \in X \setminus (E \cup \{x_0\}) : \iota(x) = i\}$ denote the set of decision nodes belonging to player i.

7 A partition $\mathcal{I}$ of the set of decision nodes, $X \setminus (E \cup \{x_0\})$, such that if x and x' are in the same element of the partition, then (i) $\iota(x) = \iota(x')$, and (ii) $A(x) = A(x')$. $\mathcal{I}$ partitions the set of decision nodes into information sets. The information set containing x is denoted by $\mathcal{I}(x)$.

8 For each $i \in N$, a von Neumann-Morgenstern payoff function whose domain is the set of end nodes, $u_i : E \to R$. This describes the payoff to each player for every possible complete play of the game.

Extensive form games

We write $\Gamma =< N, A, X, E, \iota, \pi, \mathcal{I}, (u_i)_{i \in N} >$. If the sets of actions, A, and nodes, X, are finite, then Γ is called a **finite extensive form game**.

Extensive form game strategy

Consider an extensive form game Γ. Formally, a **pure strategy** for player i in Γ is a function $s_i : \mathcal{I}_i \to A$, satisfying $s_i(\mathcal{I}(x)) \in A(x)$ for all x with $\iota(x) = i$. Let S_i denote the set of pure strategies for player i in Γ.

(Kuhn) Backward induction and Nash equilibrium

Theorem 7.4: If s is a **backward induction strategy** for the perfect information finite extensive form game Γ, then s is a Nash equilibrium of Γ.

Existence of pure strategy Nash equilibrium

Every finite extensive form game of perfect information possesses a pure strategy Nash equilibrium.

Subgames

A node x is said to define a **subgame of an extensive form game** if $\mathcal{I}(x) = \{x\}$ and whenever y is a decision node following x, and z is in the information set containing y, then z also follows x.

Pure strategy subgame perfect equilibrium

A joint pure strategy s is a **pure strategy subgame perfect equilibrium** of the extensive form game Γ if s induces a Nash equilibrium in every subgame of Γ.

Pure strategy subgame perfect equilibrium

Theorem 7.5: For every finite extensive form game of perfect information, the set of backward induction strategies coincides with the set of pure strategy subgame perfect equilibria.

Perfect recall

An extensive form game has **perfect recall** if whenever two nodes x and $y = (x, a, a_1, \ldots, a_k)$ belong to a single player, then every node in the same information set as y is of the form $w = (z, a, a'_1, \ldots, a'_l)$ for some node z in the same information set as x.

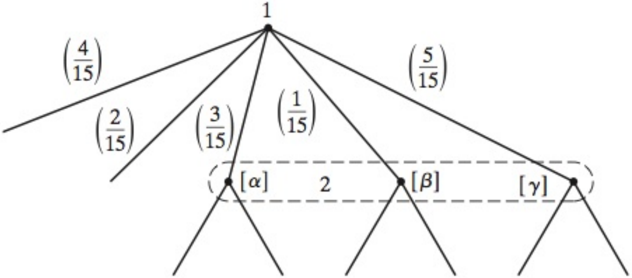
Subgame perfect equilibrium

A joint behavioural strategy b is a **subgame perfect equilibrium** of the finite extensive form game Γ if it induces a Nash equilibrium in every subgame of Γ.

(Selten) Existence of subgame perfect equilibrium

Theorem 7.6: Every finite extensive form game with perfect recall possesses a subgame perfect equilibrium.

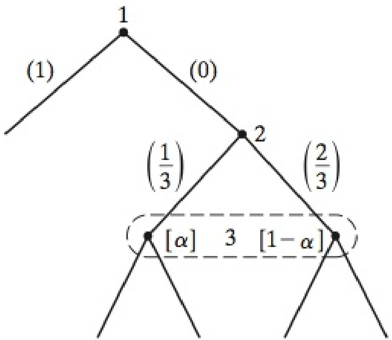
Example 1



Bayes' rule

Beliefs must be derived from behavioral strategies using Bayes' rule whenever possible.

Example 2



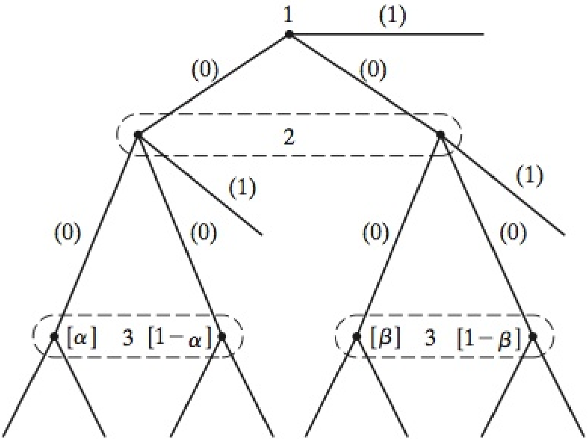
Independence

Beliefs must reflect that players choose their strategies independently.

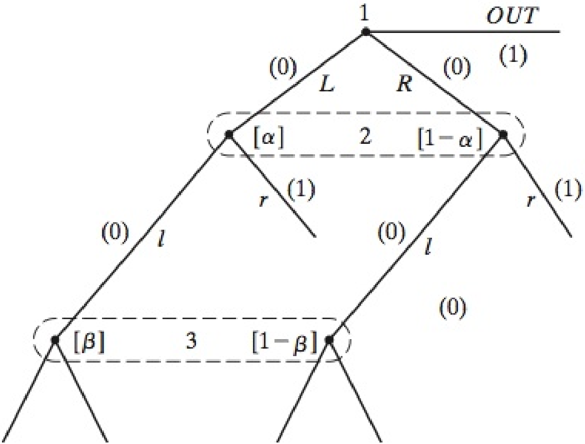
Common beliefs

Players with identical information have identical beliefs.

Example 3



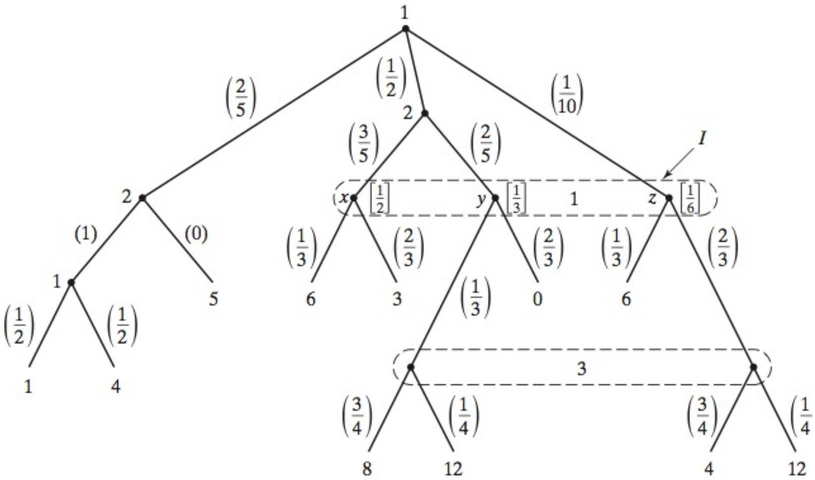
Example 4



Consistent assessments

An assessment (p, b) for a finite extensive form game Γ is **consistent** if there is a sequence of completely mixed behavioural strategies b_n, converging to b, such that the associated sequence of Bayes' rule induced systems of beliefs p_n, converges to p.

Example 5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Sequential rationality

An assessment (p, b) for a finite extensive form game is **sequentially rational** if for every player i, every information set I belonging to player i, and every behavioural strategy b_i' of player i,

$$v_i(p, b|I) \geq v_i(p, (b_i', b_{-i})|I).$$

We also call a joint behavioural strategy b sequentially rational if for some system of beliefs p the assessment (p, b) is sequentially rational as above.

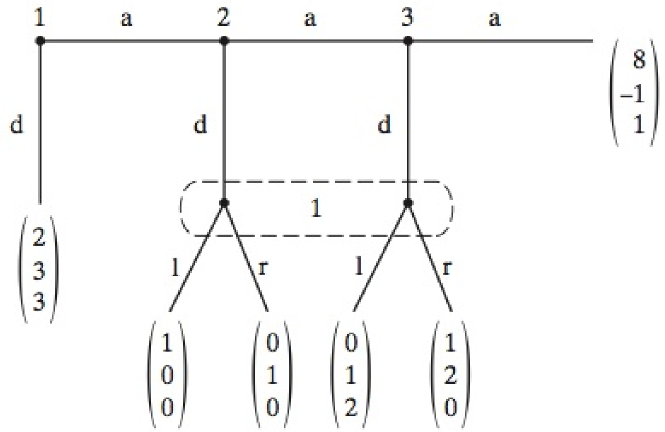
Sequential equilibrium

An assessment for a finite extensive form game is a **sequential equilibrium** if it is both consistent and sequentially rational.

(Kreps and Wilson) Existence of sequential equilibrium

Theorem 7.7: Every finite extensive form game with perfect recall possesses at least one sequential equilibrium. Moreover, if an assessment (p, b) is a sequential equilibrium, then the behavioural strategy b is a subgame perfect equilibrium.

Example 6



Consider the extensive form game above. Each of players 1, 2, and 3 can play down (d) or across (a), and player 1 can also play left (l) or right (r).

1. Identify all subgames.
2. Find a pure strategy subgame perfect equilibrium, b, such that (p, b) is not sequentially rational for any system of beliefs p.
3. Find an assessment, (p, b), that is sequentially rational and satisfies Bayes' rule in every subgame.

Weak perfect Bayesian equilibrium

An assessment (p,b) for a finite extensive form game is a **weak perfect Bayesian equilibrium** if beliefs p are derived using Bayes' rule when possible and if the assessment (p,b) is sequentially rational.

Signalling game