Equivalent martingale measures in Lévy markets

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From the previous lecture

- If the process $e^Y = \left(e^{Y(t)}, t \geq 0\right)$ is a martingale then

$$e^{Y(t)} = 1 + \int_0^t e^{Y(s-)} F(s)\, dB(s) + \int_0^t \int_{|x|<1} e^{Y(s-)} \left(e^{H(s,x)} - 1\right) \widetilde{N}(ds, dx)$$
$$+ \int_0^t \int_{|x|\geq 1} e^{Y(s-)} \left(e^{K(s,x)} - 1\right) \widetilde{N}(ds, dx) \qquad (1)$$

- $\widetilde{S}$ is a Q-martingale if and only if

$$m\sigma + \mu - r + k\sigma F(t) + \sigma \int_{\mathbb{R}-\{0\}} x \left(e^{H(t,x)} - 1\right) \nu(dx) = 0 \text{ a.s.} \qquad (2)$$

Incomplete markets and Esscher transform

- Equivalent measures Q exist with respect to which $\widetilde{S}$ will be a martingale, but these will no longer be unique in general
- We must follow a selection principle to reduce the class of all possible measures Q to a subclass, within which a unique measure can be found.
- Aditional assumption:

$$\int_{|x|\geq 1} e^{ux} \nu(dx) < \infty$$

  for all $u \in \mathbb{R}$.
- In this case, we can analytically continue the Lévy- Khintchine formula to obtain

$$\mathbb{E}\left[e^{-uX(t)}\right] = e^{-t\psi(u)}$$

  where

$$\psi(u) = -\eta(iu) = bu - \frac{1}{2}k^2u^2 + \int_c^{\infty} \left(1 - e^{-ux} - ux\mathbf{1}_{\{|x|<1\}}(x)\right)\nu(dx).$$

Incomplete markets and Esscher transform

- The processes

$$M_u(t) = \exp(iuX(t) - t\eta(u)),$$
$$N_u(t) = M_{iu}(t) = \exp(-uX(t) + t\psi(u))$$

  are martingales and N_u is strictly positive.
- Define a new probability measure by

$$\frac{dQ_u}{dP}|_{\mathcal{F}_t} = N_u(t).$$

- Q_u is called the Esscher transform of P by N_u.

Incomplete markets and Esscher transform

- Applying the Itô formula to N_u, we have

$$dN_u(t) = N_u(t-)\left(-kuB(t) + \left(e^{-ux} - 1\right)\widetilde{N}(dt, dx)\right).$$

- Comparing this with (1) for exponential martingales e^Y, we have that

$$F(t) = -ku,$$
$$H(t, x) = -ux$$

and the condition for Q_u to be a martingale (2) is

$$m\sigma + \mu - r - k^2u\sigma + \sigma\int_c^\infty x\left(e^{-ux} - 1\right)\nu(dx) = 0 \text{ a.s.}$$

Incomplete markets and Esscher transform

- Let $z(u) = \int_c^\infty x\left(e^{-ux} - 1\right)\nu(dx) - k^2u$. Then the martingale condition is:

$$z(u) = \frac{r - \mu - m\sigma}{\sigma}. \quad (3)$$

- Since $z'(u) < 0$, z is strictly decrerasing, and therefore there is a unique u (a unique measure Q_u) that satisfies (3).
- The Esscher transform is such that this measure Q_u minimizes the relative entropy $H(Q|P)$ between the measures Q and P (a measure of "distance" between two measures), where

$$H(Q|P) = \mathbb{E}_Q\left[\ln\left(\frac{dQ}{dP}\right)\right] = \mathbb{E}_P\left[\frac{dQ}{dP}\ln\left(\frac{dQ}{dP}\right)\right].$$

Absence of arbitrage in exponential Lévy models

- Let X be a Lévy process and consider a market model where $S_t = S_0 \exp(rt + X_t)$.
- Theorem (see Cont and Tankov, pages 310-311): If the trajectories of X are neither increasing (a.s.) nor decreasing (a.s.), then the exponential Lévy market model given by $S_t = S_0 \exp(rt + X_t)$ is arbitrage free: there exists a probability measure Q equivalent to P (equivalent martingale measure) such that $\widetilde{S}_t = e^{-rt} S_t$ is a Q-martingale.
- In other words, the exponential-Lévy model is arbitrage free in the following cases (not mutually exclusive):
  1) X has a nonzero Gaussian component (or diffusion coeff.): $\sigma > 0$.
  2) X has both positive and negative jumps.
  3) X has infinite variation paths: $\int_{|x|<1} |x| \, \nu(dx) = \infty$.
  4) X has positive jumps and negative drift or negative jumps and positive drift.

The mean-correcting martingale measure

- A practical way to obtain an equivalent martingale measure Q in a exponential Lévy model of type $S_t = S_0 \exp(X_t)$, is by mean correcting the exponential of a Lévy process (see Schoutens, pages 79-80).
- We can correct the exponential of the Lévy process X, by adding a new drift term mt (with new parameter m):

$$\overline{X}_t = mt + X_t.$$

- When comparing the characteristics triplet of $\overline{X}$ with those of X, the only parameter that changes is the drift: $\overline{b} = b + m$.

The mean-correcting martingale measure

- We can change the m parameter of the process X such that $\widetilde{S}_t = e^{-rt} S_t$ is a martingale. This is equivalent to choose an equivalent martingale measure Q.
- Example: in the Black-Scholes model, we change the mean of the normal distribution $\mu - \frac{1}{2}\sigma^2 = m_{old}$ (the m_{old}) into the new m parameter:

$$m_{new} = r - \frac{1}{2}\sigma^2,$$

or

$$m_{new} = m_{old} + r - \ln\left[\phi\left(-i\right)\right],$$

where $\phi\left(x\right)$ is the characteristic function of the log-returns involving the m_{old} parameter.
In the Black-Scholes model, $\ln\left[\phi\left(-i\right)\right] = \mu$.
- This choice of m_{new} will imply that the discounted price $\widetilde{S}_t = e^{-rt} S_t$ is a martingale.

The mean-correcting martingale measure

- Procedure:
1) Estimate in some way the parameters involved in the process.
2) Then change the m parameter in a way that

$$m_{new} = m_{old} + r - \ln\left[\phi\left(-i\right)\right],$$

where $\phi\left(x\right)$ is the characteristic function of the log-returns involving the m_{old} parameter.
3) Then, with this new m_{new} parameter in the Lévy process, the discounted price $\widetilde{S}_t = e^{-rt} S_t$ is a martingale and we have chosen the mean-correcting equivlent martingale measure.
- In page 78 of Schoutens, the author lists what is the value of the m parameter for several Lévy processes (CGMY, VG, NIG, etc...)

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