# 2. STRUCTURAL MODELS OF CREDIT RISK

o Intensity-based models do not provide any fundamental explanation for the arrival of default, providing only a consistent description of the distribution of arrival times.

o Structural models provide this theoretical background, explaining the arrival of default by using company's balance sheet and stock market data.

o **In structural models, the default time is determined endogenously by the behavior of the company's asset** ⇔ default occurs when the market value of assets falls below the debt face value - **1st passage of assets by a default boundary**.

o These models became quite popular in the last decades, also due to the drawbacks of traditional credit risk models and rating updates by agencies.

o The rationale of structural models is that **market prices are the best assessment available on the companies' capital or debt value**, notwithstanding the higher volatility of market prices, namely for stocks, leading to false alarms of defaults.

# Altman Z-score (1968)

○ The first attempt to incorporate market prices in a credit risk model was done in the Z-Score model, by Altman (1968), which is an ad-hoc specification for credit risk as a function of several financial ratios, being one of them dependent on stock market capitalization.

○ 22 financial ratios from 66 companies between 1946 and 1965 were used, evenly split between defaulting and non-defaulting companies.

# Altman Z-score (1968)

○ For defaulting companies, financial statements one year before the default were used, having been obtained the following model (PD decrease along with the Z-score):

$$Z = 1.2 \, X_1 + 1.4 \, X_2 + 3.3 \, X_3 + 0.6 \, X_4 + 1.0 \, X_5$$

where:

$X_1$ = working capital (net) / total assets;

$X_2$ = retained earnings/ total assets

$X_3$ = EBIT / total assets;

$X_4$ = market capitalization/book value of long-term liabilities

$X_5$ = sales/total assets

○ Z<1,81 – defaulting companies
○ Z>2,99 – non-defaulting companies

# Merton Model

- Later, Black and Scholes (1973) and Merton (1974) developed a corporate valuation approach based on financial options.

  Black, F. and Scholes, M. (1973) The pricing of options and corporate liabilities. *Journal of Political Economy*, 81: 637–654.

  Merton, R.C. (1974) On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance*, 29: 449–470.

- **This approach became known in the literature as the Merton Model.**

- The model is based on the assumption that, when the company issues debt, shareholders transfer the control of the company to creditors.

- However, they retain an option of recovering that control if the company reimburses the debt.

# Merton Model

○ Therefore, the value of capital may be seen as the price of a call-option on the company assets, with a strike equal to the debt value:

$$V_E = Max(V_A - X, 0)$$

where

$V_E$ = market value of the company's own funds

$V_A$ = market value of company's assets

$X$ = nominal value of the company's total debt payable in maturity T.

# Merton Model

○ The value of the debt can also be seen as a derivative, as its payoff correspond to:

(i) face value – if there is no default

(ii) the market value of assets – if there is a default (in this case, the recovery will provide bondholders a payment that will stem from the asset liquidation)
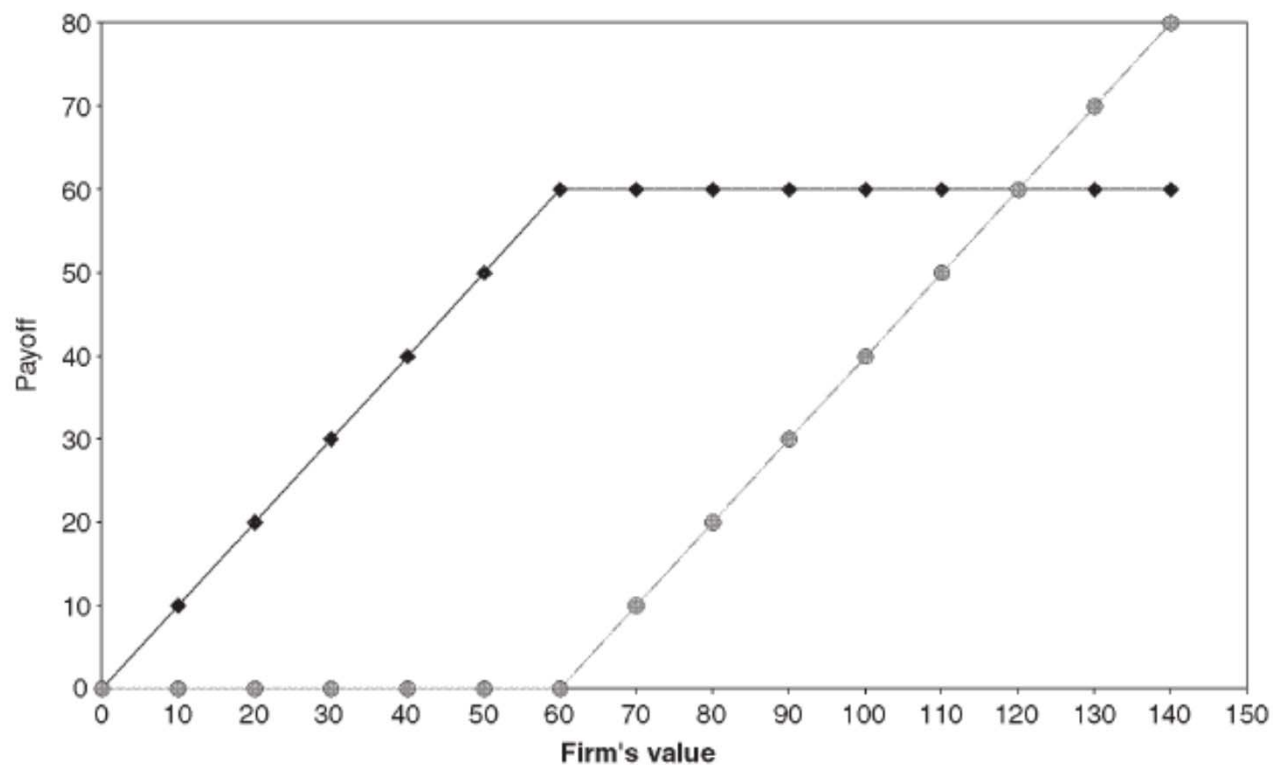
$$V_B = Min(X, V_A)$$

where

$V_B$ = market value of the company's debt

# Merton Model

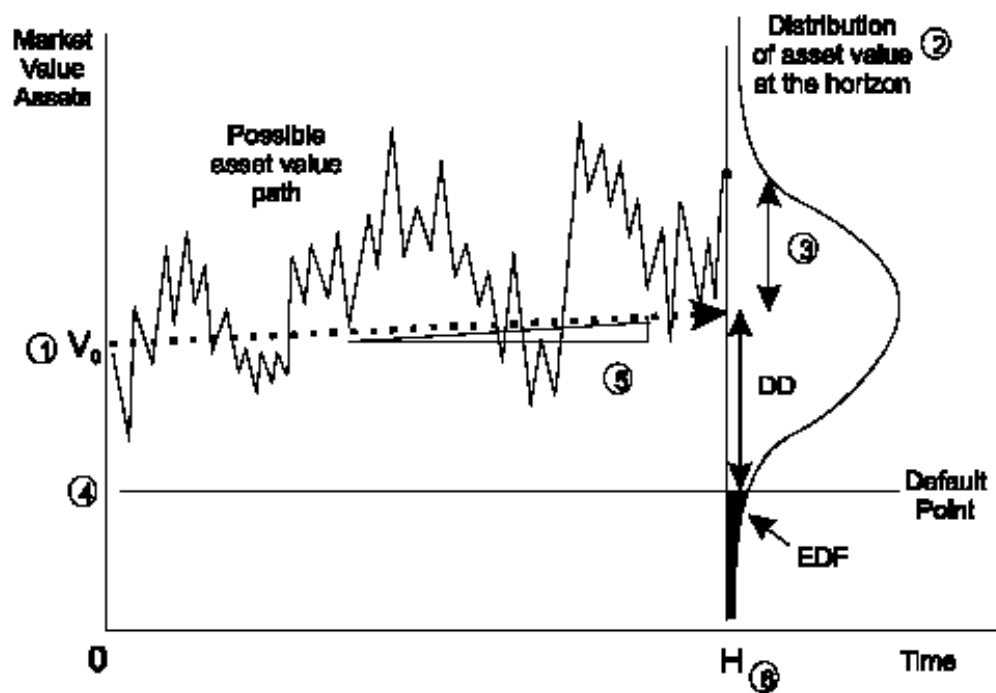○ Payoffs of shares and bonds for *X* = 60:



Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

# Merton Model

○ If on the redemption date, the market value of assets is lower than the debt value, the shareholders don't exercise the call option, i.e. the debt is not repaid =>

PD= P[Market Value of Assets < Debt value].



Source: Crosbie and Bohn (2002), "Modeling Default Risk", KMV.

# Merton Model

○ Consequently, if the call-option can be valued, **the PD will be obtained from the distribution function resulting from the stochastic process of the company's asset market value**.

○ Assuming that the option is European and the asset market value may be taken as the price of non-paying dividend asset, **one can use the Black-Scholes formula and calculate the PD from the implied volatility of the company's asset value** and an estimate for the respective growth rate.

○ The Merton model is based on the assumption of the growth rates of the company's market value of assets ($V_A$) being normally distributed:

$$dV_A = \mu V_A dt + \sigma_A V_A dz \Leftrightarrow dV_A / V_A = \mu dt + \sigma_A dz$$

where $V_A$ is the company's market value of assets, $\mu$ and $\sigma_A$ the respective trend and instantaneous volatility and $dz$ is a Wiener process (random shocks normally distributed).

# Merton Model

○ Given that this is exactly the stochastic process of the underlying asset of an European option under the assumptions taken in the Black-Scholes pricing formula, the pricing formula for the European *call-option* on the company's market value of assets that corresponds to the stock price is:

$$V_E = V_A N(d1) - e^{-rT} X N(d2)$$

where

$V_E$ is the market value of the company's own funds

$N$ is the cumulative normal distribution function

$r$ is the risk-free interest rate for the maturity $T$

$X$ is the nominal value of the company's total debt payable in maturity T.

$$d1 = \frac{\ln\left(\frac{V_A}{X}\right) + \left(r + \frac{\sigma_A^2}{2}\right)T}{\sigma_A \sqrt{T}} \qquad d2 = d1 - \sigma_A \sqrt{T}.$$

# Merton Model

- Therefore, the option valuation involves the calculation of the market value of assets and equity.

- Accounting identity => $V_A = V_B + V_E$

- However, most debt in $V_B$ is not observable for most firms (most firms don't even have traded debt).

- $V_A$ is not observable neither.

- In the option pricing formula, there are 2 unknowns: $V_A$ and $\sigma_A$.

# Merton Model

o Consequently, an additional equation is required, in order to determine the values for those 2 variables.

o This equation will result from the relationship between the volatility of assets and the volatility of capital:

(1) $\qquad \sigma_E = \dfrac{V_A}{V_E} N(d1)\sigma_A$ (from $\sigma_E = \dfrac{\partial V_E}{\partial V_A}\dfrac{V_A}{V_E}\sigma_A$)

o In Jarrow and Rudd (1983), it is shown that the stock volatility is a multiple of the volatility of the market value of assets:
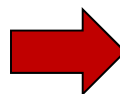
(2) $\qquad \sigma_E = \eta\sigma_A$

# Merton Model

- Given (1) and that

$$(3) \quad \delta = \frac{\partial C(X)}{\partial V_A} = N(d_1)$$

one gets (from (1) and (2):

$$\frac{V_A}{V_E} N(d1)\sigma_A = \eta\sigma_A \Leftrightarrow$$

$$\Leftrightarrow \frac{V_A}{V_E}\delta = \eta \Leftrightarrow$$

$$(4)$$

$$\Leftrightarrow \delta = \frac{\eta}{V_A/V_E} \Leftrightarrow \delta = \frac{\sigma_E/\sigma_A}{V_A/V_E} \Leftrightarrow$$

$$\Leftrightarrow \delta = \frac{\sigma_E}{\sigma_A}\frac{V_E}{V_A} \Leftrightarrow V_A = \frac{\sigma_E}{\sigma_A}\frac{V_E}{\delta}$$

From inputs $V_E$, $\sigma_E$ , X, r and T, the equation system including the option pricing formula and (4) allows to estimate $V_A$ and $\sigma_A$.

# Merton Model

○ The PD is thus the probability of the market prices of assets falling below the nominal value of debt at the expiry date:

$$p_t = \Pr\left[V_A^t \le X_t \mid V_A^0 = V_A\right] = \Pr\left[\ln V_A^t \le \ln X_t \mid V_A^0 = V_A\right]$$

○ Given that the market value of assets follows a log-normal distribution, one gets (with $\mu$ = expected asset returns):

$$\ln V_A^t = \ln V_A + \left(\mu - \frac{\sigma_A^2}{2}\right)t + \sigma_A\sqrt{t}\,\varepsilon$$

○ Therefore, the PD is:

$$p_t = \Pr\left[\ln V_A + \left(\mu - \frac{\sigma_A^2}{2}\right)t + \sigma_A\sqrt{t}\,\varepsilon \le \ln X_t\right] = \Pr\left[-\frac{\ln\dfrac{V_A}{X_t} + \left(\mu - \dfrac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}} \ge \varepsilon\right] \Leftrightarrow p_t = N\left[-\frac{\ln\dfrac{V_A}{X_t} + \left(\mu - \dfrac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}}\right]$$

○ Risk-neutral PD ($\mu$ = r ):   $p_t = N\left[-d_2\right]$

# Merton Model

- Open issues:

  - How to obtain values for $\mu$ and $\sigma_E$?

  - How to deal with complex debt structures, with different maturities, seniority degrees and installments?

  - How to deal with the sensitivity of PDs to the leverage ratio?

  - How to solve the kurtosis problem in the market value of assets?

  - How to use the PD estimates as a leading indicator of rating changes?

- Estimation – non-linear least squares, minimizing the sum of the squared differences between the market value and the estimated value of the stocks (through the option pricing formula) and the assets.

# Moody's KMV Model

○ Moody's KMV overcomes the distribution problems motivated by the normality assumption through a database of loans providing empirical PDs as a function of the distance-to-default measure:

$$DD = \left[ \frac{\ln \dfrac{V_A}{X_t} + \left( \mu - \dfrac{\sigma_A^2}{2} \right) t}{\sigma_A \sqrt{t}} \right]$$
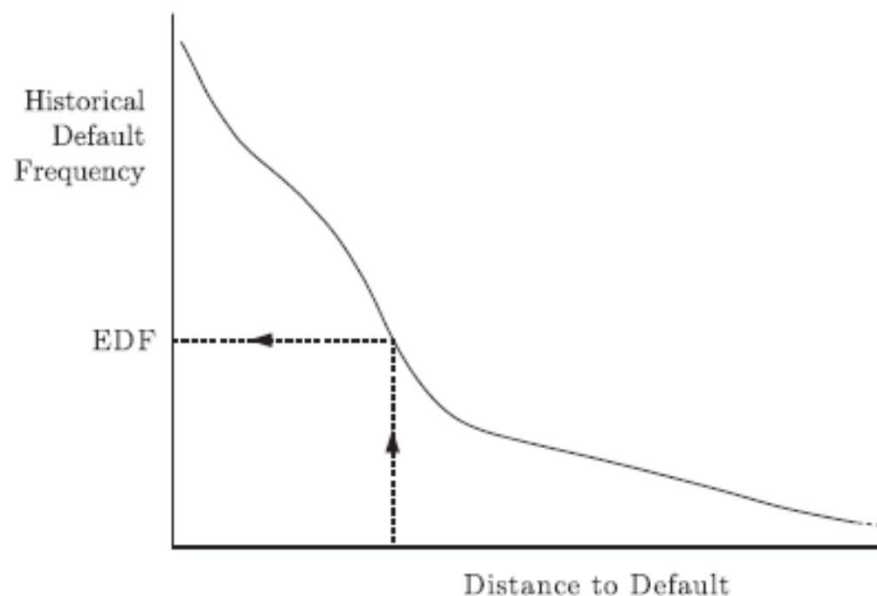


Source: Duffie, Darrell and Kenneth J. Singleton (2003), "Credit Risk", Princeton University Press.

# Moody's KMV Model

- In this model, $\sigma_A$ **is a linear combination of a modeled and an empirical volatility** (the latter weighting 70%, 80% for Financial Institutions).

- **Empirical vols -** calculated as the annualized standard deviation of the growth rates of the nominal value of assets, using 3 years of weekly observations for US companies (5 years of monthly data for European companies), excluding extreme values and adjusting for effects of M&A.

- **Modeled vols -** obtained from a regression between the observed vol and size, revenues, profitability, sector and region variables.

# Moody's KMV Model

○ For banks, the PD is harder to estimate, given the diversity and uncertainty of the liabilities' maturities.

○ Moreover, by definition, banks are highly leveraged companies.
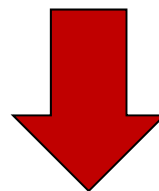


○ Moody's KMV proposes the *default point* (the value of the payable liabilities in the maturity considered) to be calculated as a % of the total liabilities, being that % differentiated according to the type of institution.

# Moody's KMV Model

○ In Chan-Lau and Sy (2006), it is proposed an adjustment to the Moody's KMV model, in order to accommodate the possibility of a bail-out.

○ Consequently, the "*Distance-Risk measure*" concept is created, with $L_t$ being the bank's liabilities ($\lambda$=1 => DR=DD) and PCAR the planned capital ratio:

$$DR_T = \frac{\ln\left(\frac{V_t}{\lambda L_t}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \qquad \lambda = \frac{1}{1 - PCAR_i}$$

○ With a very low PCAR, $\lambda$ gets higher and the DR lower ⇔ with a lower capital target, the bank gets closer to default and can reach this stage at a lower level of liabilities.

○ According to Oderda et al. (2002), Moody's KMV model anticipates defaults with a lead of around 15 months, but also produces false alarms in 88% of the cases.

# Fitch EIR

○ In order to smooth the excessive volatility of PDs obtained from equity prices, hybrid models were developed, being the PD obtained from corporate financial, market and macroeconomic information.

○ One of these models was developed by Fitch, the **Equity Implied Rating (EIR),** relating the DD to a set of financial ratios and macroeconomic variables:



Figure 8: Modeling Step of EIR
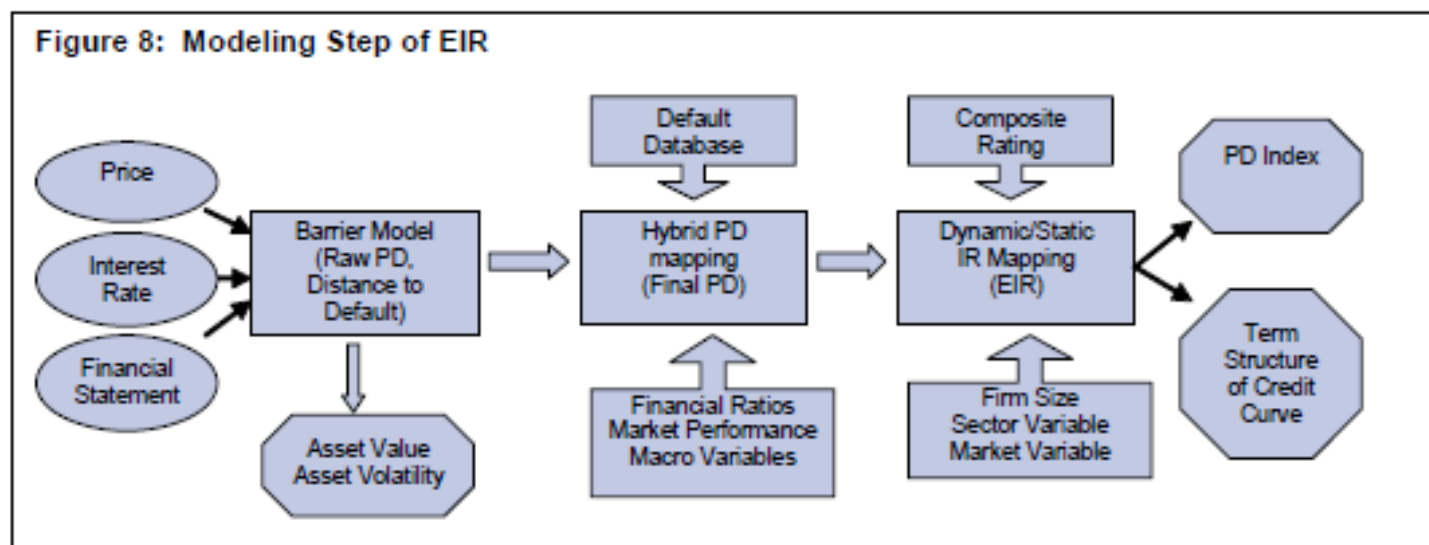
Source: Fitch (2007).

# Bondscore

○ Another model is the Bondscore, developed by CreditSights:

$$p = -9.593 + 7.366X_1 - 3.989X_2 - 5.308X_3$$
$$- 6.333X_4 - 2.501X_5 + 3.807X_6 + 5.469X_7$$

being:
$X_1$ = Total Liabilities/Market Value of Capital
$X_2$ = EBITDA/Sales
$X_3$ = Sales/ Total Assets
$X_4$ = Working Capital / Total Assets
$X_5$ = log(Assets)
$X_6$ = Vol of EBITDA/Sales
$X_7$ = Vol of Market Value of Capital

# 3. REDUCED FORM MODELS

o A structural model of credit risk provides a link between the prices of equity and debt instruments issued by a given firm.

o **A reduced-form model does not give any fundamental reason for the arrival of the defaults**, assuming that hazard rates for the different companies are stochastic processes correlated with macroeconomic variables.

o Given that *credit spreads* can be decomposed in default risk (PD, or $\lambda$) and recovery risk (LGD, or $\phi$), the **PD can be modeled from the credit spreads and LGDs**.

o Taking several maturities, one can obtain a term structure of PDs.

o However, we must have in mind that spreads are not only a function of PDs and LGDs, but also of liquidity, taxation and risk premia charged by investors => **PDs implied by spreads are risk neutral**.

# Credit spreads

○ 2 equivalent ways to calculate the price of a risky zero coupon bond (assuming one-period maturity and redemption value of one monetary unit):

(i) Expected value of the future cash-flows, discounted at the risk-free rate:

$$P = \frac{E_0(X_1)}{1+r} = \frac{\lambda\phi + (1-\lambda)\cdot 1}{1+r}$$

(ii) Future cash-flows, discounted at the risk-free rate plus the credit spread:

$$P = \frac{1}{1+r+s}$$

# Credit spreads

○ Equalizing both expressions =>

$$s = \frac{\lambda(1-\phi)(1+r)}{1-\lambda(1-\phi)} \cong \lambda(1-\phi)$$

=> Credit spread:

○ Increases with the probability of default $\lambda$;

○ Decreases with the recovery rate $\phi$;

○ Increases with the risk-free rate $r$;

○ In reality, these spreads may also be impacted by risk premium due to uncertainty about risk-free interest rates, PDs and LGDs.

# Credit spreads

o This relationship can be generalized for any maturity:

(i) Expected value of the future cash-flows, discounted at the risk-free rate:

$$P_0 = \sum_{i=1}^{T} \frac{E_0(X_i)}{(1+r)^i} = \sum_{i=1}^{T} \frac{[\lambda\phi + (1-\lambda)]C_i}{(1+r)^i} + \frac{[\lambda\phi + (1-\lambda)]NV}{(1+r)^T}$$

(ii) Future cash-flows, discounted at the risk-free rate plus the credit spread:

$$P_0 = \sum_{i=1}^{T} \frac{E_0(X_i)}{(1+r+s)^i} = \sum_{i=1}^{T} \frac{C_i}{(1+r+s)^i} + \frac{NV}{(1+r+s)^T}$$

# Credit spreads

- Consequently, the (risk-neutral) PD can be obtained by modeling the risk-free and the recovery rate, instead of the market value of the company's assets.

- From the spreads of similar bonds for different maturities, one can obtain the PD term structure, that can be compared to the statistics of rating agencies (the "true" PDs).

- The initial and most popular reduced form models were presented in Artzner and Delbaen (1995), Jarrow and Turnbull (1995) and Duffie and Singleton (1995).

# Reduced Form Models

**Advantages:**

✔ Credit spreads are directly modelled: the intensity of the default process *is* the credit spread.

✔ Can be fitted to observed credit spreads.

✔ Realistic credit spreads through discontinuous dynamics.

✔ Suitable for the pricing of credit derivatives.

**Disadvantages:**

X There is no explicit link to the company's fundamentals

# Reduced Form Models

- **Zero recovery defaultable bond price:**

$$d_0(t, T) = E_t^* \left[ e^{- \int_t^T (r(u) + \lambda^*(u))\, du} \right]$$

**being $\lambda^*$ the risk-neutral hazard rate**

- **Risk-free interest and hazard rates depend on a set of macroeconomic variables ($X(t)$):**

$$r(t) = a_r(t) + b_r(t) \cdot X(t)$$

$$\lambda^*(t) = a_{\lambda^*}(t) + b_{\lambda^*}(t) \cdot X(t)$$

As both depend on X($t$), the hazard rate becomes correlated with the interest rates.

# REDUCED FORM MODELS

- From the equations in the previous slide, we get **prices for the defaultable and the risk-free bond**, respectively:

$$d_0(t, T) = e^{\alpha_d(t, T) + \beta_d(t, T) \cdot X(t)} \qquad \delta(t, T) = e^{\alpha_\delta(t, T) + \beta_\delta(t, T) \cdot X(t)}$$

- **Credit risk spread**:

$$s(t, T) = -\frac{\log d_0(t, T) - \log \delta(t, T)}{T - t} \qquad s(t, T) = -\frac{\alpha_s(t, T) + \beta_s(t, T) \cdot X(t)}{T - t}$$

for $\alpha_s$ and $\beta_s$ obtained by subtraction of the respective $\alpha$'s and $\beta$'s

# 4. CREDIT RATING MODELS

○ Credit risk may be assessed for different levels of credit ratings, by modelling rating changes.

○ Stylized facts:

(i) Frequencies for low-probability events are usually based on a very small number of observations

(ii) Ratings momentum – rating changes tend to be more frequent for entities whose ratings have been revised recently

(iii) Ratings delay – rating changes tend to lag market prices for several months

(iv) Credit spreads are often misaligned with PDs, as the latter are just historical information.

(v) Credit spreads change along time.

# 4. CREDIT RATING MODELS

○ An important source about rating frequencies corresponds to the regular reports published by rating agencies.

○ These reports include information about cumulative PDs, 1y PDs along time and transition matrices, namely for 1y.

|   | A | B | D |
|---|---|---|---|
| A | $p_{AA} = 0.80$ | $p_{AB} = 0.15$ | $p_{AD} = 0.05$ |
| B | $p_{BA} = 0.10$ | $p_{BB} = 0.80$ | $p_{BD} = 0.10$ |
| D | $p_{DA} = 0.00$ | $p_{DB} = 0.00$ | $p_{DD} = 1.00$ |

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

○ From these 1y rating transition matrices, it is possible to calculate frequencies of default for larger periods.

# 4. CREDIT RATING MODELS

○ 2y PD for rating A?

○ As this default may occur either in the 1st or in the 2nd year, the most straightforward answer would be to calculate the Cumulative Probability of Default as 1 - Cumulative Probability of Survival, being the latter the joint probability of surviving in both years:

○ $(1 - p_{AD})(1 - p_{AD}) = 0,95^2 = 0,9025 => PD_{2y} = 1 - 0,9025 = 0,0975$

○ However, this answer would be valid only with no ratings (or no rating transitions besides defaults, or a single non-default rating).

○ We need to take into account all rating transitions during the whole period before default, not only the transitions to default, but also the rating changes before default (i.e. during the 1st year).

# 4. CREDIT RATING MODELS

○ Therefore, if a default occurs in the 2$^{nd}$ year, the rating transition to default may be either from rating A or from B, as defaults are often preceded by rating downgrades.

○ Actually, during the 1$^{st}$ year , a rating A may be kept or may be downgraded to B, or even move straight to default:

$$A \to A \to D \Rightarrow p_{AA}p_{AD} = 0,80 \cdot 0,05 = 0,04$$

$$A \to B \to D \Rightarrow p_{AB}p_{AD} = 0,15 \cdot 0,10 = 0,015$$

$$A \to D (\to D) \Rightarrow p_{AD}p_{DD} = 0,05 \cdot 1 = 0,05$$

○ $PD_{2y} = p_{AA}p_{AD} + p_{AB}p_{AD} + p_{AD}p_{DD} = 0,04 + 0,015 + 0,05 = 0,105$ (which compares to 0,0975 when rating transitions before default were discarded).

○ All 2y transition frequencies will result from the squared rating transition matrix.

# TYPES OF MODELS

○ Default mode (DM) – take into consideration only the changes in the value of bonds due to defaults.

○ Marked-to-market (MTM) – allows to assess the impact on the credit value of any change in its risk.

○ Individual models – focus on the changes of a credit value, regardless the correlations with other credits in the portfolio.

○ Portfolio models – incorporate the correlations between the several assets of a credit portfolio.

# Challenges in Estimating Portfolio Credit Risk

○ Non-normal returns - credit returns are highly skewed and fat-tailed.

○ Difficulty in modeling correlations - lack of data, contrary to equities.



Comparison of distribution of credit returns and market returns
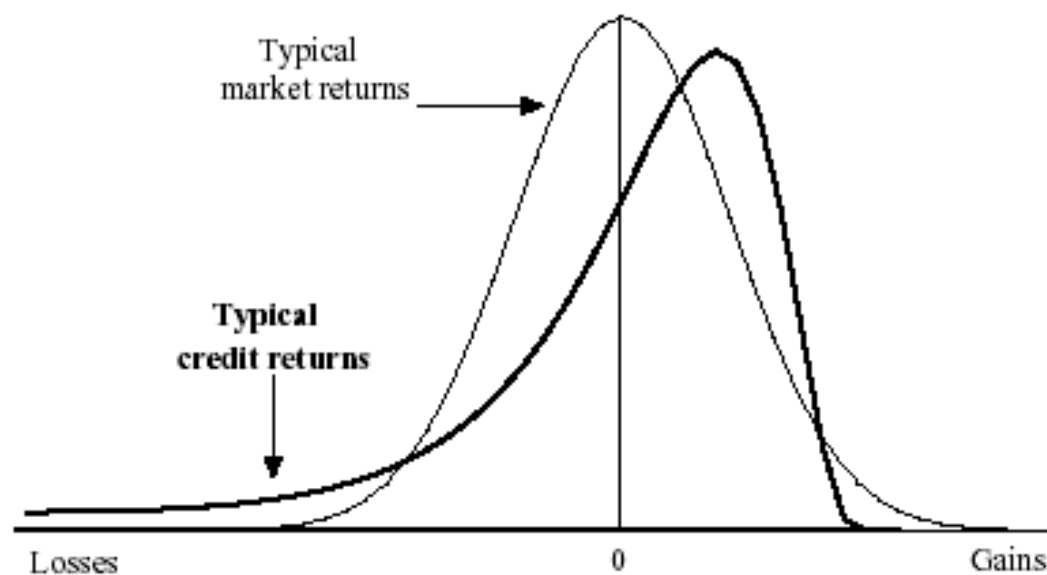
Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

# Credit-VaR

○ CreditMetrics is a portfolio model that estimates the Credit-Var, taking into account the current ratings of bonds, the transition frequencies and the correlation between bond prices.

○ We will start by considering a very simple portfolio, comprising just one BBB 5y-bond at par (redemption value of 100), with an annual coupon rate of 6%, paid annually.

○ To calculate the potential losses 1 year ahead, due to rating downgrades, we need to obtain the value of the bond after rating migrations in 1 year.

○ This can be done by using forward interest rates for each rating level to discount the remaining cash-flows 1 year ahead (the 4 remaining coupons and the redemption value).

# Credit-VaR

○ Credit-Var = Unexpected loss due to credit risk increase = Difference between the price after a very unlikely unfavorable event and the expected value of the future price at the risk horizon.

○ Credit-Var = Difference between the mean of the distribution and a value at the left tail

**Distribution of value of a BBB par bond in one year**

| Year-end rating | Value ($) | Probability (%) |
|---|---|---|
| AAA | 109.37 | 0.02 |
| AA | 109.19 | 0.33 |
| A | 108.66 | 5.95 |
| BBB | 107.55 | 86.93 |
| BB | 102.02 | 5.30 |
| B | 98.10 | 1.17 |
| CCC | 83.64 | 0.12 |
| Default | 51.13 | 0.18 |



Distribution of value for a 5-year BBB bond in one year
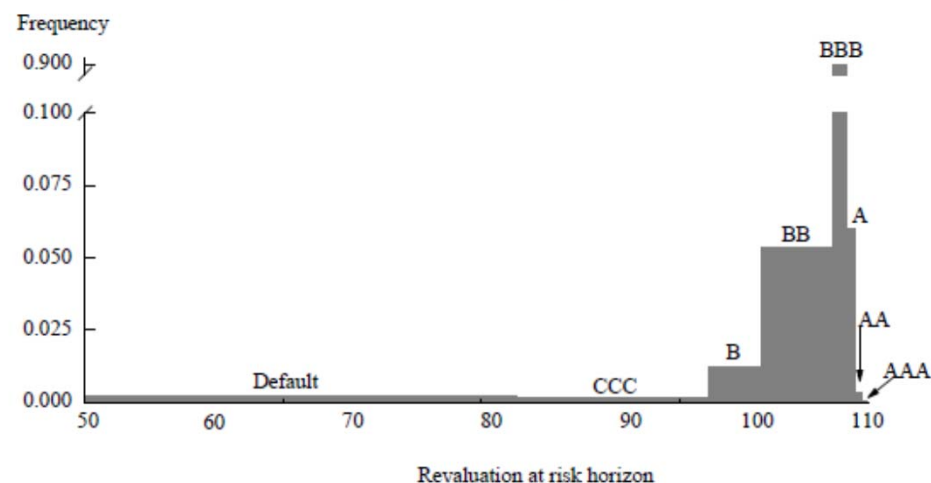
Note: The default price is the expected recovery rate.

Source: JPMorgan (1997), "CreditMetrics - Technical document"

# Credit-VaR

○ Calculation of prices for the risk horizon requires:

(i) Obtain the forward interest rate curves for each rating (m= risk horizon; n = maturity at the risk horizon of the remain cash-flows

(ii) Calculate the NPV at the risk horizon of the remaining cash-flows until maturity.

| Category | Year 1 | Year 2 | Year 3 | Year 4 |
|----------|--------|--------|--------|--------|
| AAA | 3.60 | 4.17 | 4.73 | 5.12 |
| AA | 3.65 | 4.22 | 4.78 | 5.17 |
| A | 3.72 | 4.32 | 4.93 | 5.32 |
| BBB | 4.10 | 4.67 | 5.25 | 5.63 |
| BB | 5.55 | 6.02 | 6.78 | 7.27 |
| B | 6.05 | 7.02 | 8.03 | 8.52 |
| CCC | 15.05 | 15.02 | 14.03 | 13.52 |

○ Forward Price if the upgrade from BBB to rating A occurs:

$$V = 6 + \frac{6}{(1 + 3.72\%)} + \frac{6}{(1 + 4.32\%)^2} + \frac{6}{(1 + 4.93\%)^3} + \frac{106}{(1 + 5.32\%)^4} = 108.66$$

Source: JPMorgan (1997), "CreditMetrics - Technical document"

# Credit-VaR

○ **1-year 99% Credit-Var** = Mean-$P_{1,B}$ (as the probability of having 1 year ahead a rating not above B = P(B) + P(CCC) + P(D)) = 1,17 + 0,12 + 0,18 ≈ 1% = 107,1-98,1 = **9**.

$$Mean = p_1 \cdot V_1 + p_2 \cdot V_2 + \ldots + p_{64} \cdot V_{64}$$

| Ratings | Probability of Transition (%) (1) | Loan Value at year-end (2) | Difference to the mean (3)=(2)μ | Contribution to the variance (4)=(1)x(3)^2 |
|---|---|---|---|---|
| AAA | 0.02 | 109.37 | 2.27 | 0.00 |
| AA | 0.33 | 109.19 | 2.09 | 0.0 |
| A | 5.96 | 108.66 | 1.56 | 0.15 |
| BBB | 86.93 | 107.55 | 0.45 | 0.18 |
| BB | 5.3 | 102.02 | -5.08 | 1.37 |
| B | 1.17 | 98.10 | -9.00 | 0.95 |
| CCC | 0.12 | 83.64 | -23.46 | 0.66 |
| Default | 0.18 | 51.13 | -55.97 | 5.64 |
| | | | | |
| Mean (μ) | | 107.09 | | |
| Variance (Σ(4)) | | 8.95 | | |
| Standard-dev. | | 2.99 | | |

# Credit-VaR

- Now we add a single-A 3y bond, with annual coupon rate of 5%.
- Year-end price of the single-A 3y bond, after the several potential rating migrations 1 year ahead:

| | Year-end Bond Price | Probability of Transition (%) |
|---|---|---|
| AAA | 106.59 | 0.09 |
| AA | 106.49 | 2.27 |
| A | 106.30 | 91.05 |
| BBB | 105.64 | 5.52 |
| BB | 103.15 | 0.74 |
| B | 101.39 | 0.60 |
| CCC | 88.71 | 0.01 |
| Default | 51.13 | 0.06 |

The rating where the cumulative probability of 1% is crossed is BB (0,74+0,60+0,01+0,06=1,41), where the price is 103,15.

**1-year 99% Credit-Var of the standalone A-bond** = Mean - $P_{1,BB}$ = 106,54 – 103,15 = 3,39 (lower than the B-bond, as the credit risk is lower).

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

# Credit-VaR

○ All potential values of the portfolio will result from the combination of the 8 potential values for each bond (8x8):

**All possible 64 year-end values for a two-bond portfolio ($)**

| Obligor #1 (BBB) | | Obligor #2 (single-A) | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | AAA | AA | A | BBB | BB | B | CCC | Default |
| | | 106.59 | 106.49 | 106.30 | 105.64 | 103.15 | 101.39 | 88.71 | 51.13 |
| AAA | 109.37 | 215.96 | 215.86 | 215.67 | 215.01 | 212.52 | 210.76 | 198.08 | 160.50 |
| AA | 109.19 | 215.78 | 215.68 | 215.49 | 214.83 | 212.34 | 210.58 | 197.90 | 160.32 |
| A | 108.66 | 215.25 | 215.15 | 214.96 | 214.30 | 211.81 | 210.05 | 197.37 | 159.79 |
| BBB | 107.55 | 214.14 | 214.04 | 213.85 | 213.19 | 210.70 | 208.94 | 196.26 | 158.68 |
| BB | 102.02 | 208.61 | 208.51 | 208.33 | 207.66 | 205.17 | 203.41 | 190.73 | 153.15 |
| B | 98.10 | 204.69 | 204.59 | 204.40 | 203.74 | 201.25 | 199.49 | 186.81 | 149.23 |
| CCC | 83.64 | 190.23 | 190.13 | 189.94 | 189.28 | 186.79 | 185.03 | 172.35 | 134.77 |
| Default | 51.13 | 157.72 | 157.62 | 157.43 | 156.77 | 154.28 | 152.52 | 139.84 | 102.26 |

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

# Credit-VaR

o The joint probabilities would just be product of the rating migration probability for each bond, <u>if these ratings were independent</u>.

**Joint migration probabilities with <u>zero correlation</u> (%)**

| Obligor #1 (BBB) | | Obligor #2 (single-A) | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | AAA | AA | A | BBB | BB | B | CCC | Default |
| | | 0.09 | 2.27 | 91.05 | 5.52 | 0.74 | 0.26 | 0.01 | 0.06 |
| AAA | 0.02 | 0.00 | 0.00 | 0.02 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| AA | 0.33 | 0.00 | 0.01 | 0.30 | 0.02 | 0.00 | 0.00 | 0.00 | 0.00 |
| A | 5.95 | 0.01 | 0.14 | 5.42 | 0.33 | 0.04 | 0.02 | 0.00 | 0.00 |
| BBB | 86.93 | 0.08 | 1.98 | 79.15 | 4.80 | 0.64 | 0.23 | 0.01 | 0.05 |
| BB | 5.30 | 0.00 | 0.12 | 4.83 | 0.29 | 0.04 | 0.01 | 0.00 | 0.00 |
| B | 1.17 | 0.00 | 0.03 | 1.06 | 0.06 | 0.01 | 0.00 | 0.00 | 0.00 |
| CCC | 0.12 | 0.00 | 0.00 | 0.11 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 |
| Default | 0.18 | 0.00 | 0.00 | 0.16 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 |

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

# Credit-VaR

- However, ratings do not tend to be independent, as they may be moved by the same macroeconomic factors.

- **Joint rating migration probabilities with correlated bonds:**

Joint migration probabilities with **0.30 asset correlation** (%)

| Obligor #1 (BBB) | | Obligor #2 (single-A) | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | AAA | AA | A | BBB | BB | B | CCC | Default |
| | | 0.09 | 2.27 | 91.05 | 5.52 | 0.74 | 0.26 | 0.01 | 0.06 |
| AAA | 0.02 | 0.00 | 0.00 | 0.02 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| AA | 0.33 | 0.00 | 0.04 | 0.29 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| A | 5.95 | 0.02 | 0.39 | 5.44 | 0.08 | 0.01 | 0.00 | 0.00 | 0.00 |
| BBB | 86.93 | 0.07 | 1.81 | 79.69 | 4.55 | 0.57 | 0.19 | 0.01 | 0.04 |
| BB | 5.30 | 0.00 | 0.02 | 4.47 | 0.64 | 0.11 | 0.04 | 0.00 | 0.01 |
| B | 1.17 | 0.00 | 0.00 | 0.92 | 0.18 | 0.04 | 0.02 | 0.00 | 0.00 |
| CCC | 0.12 | 0.00 | 0.00 | 0.09 | 0.02 | 0.00 | 0.00 | 0.00 | 0.00 |
| Default | 0.18 | 0.00 | 0.00 | 0.13 | 0.04 | 0.01 | 0.00 | 0.00 | 0.00 |

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

# Credit-VaR

**Distribution of value for a portfolio of two bonds**
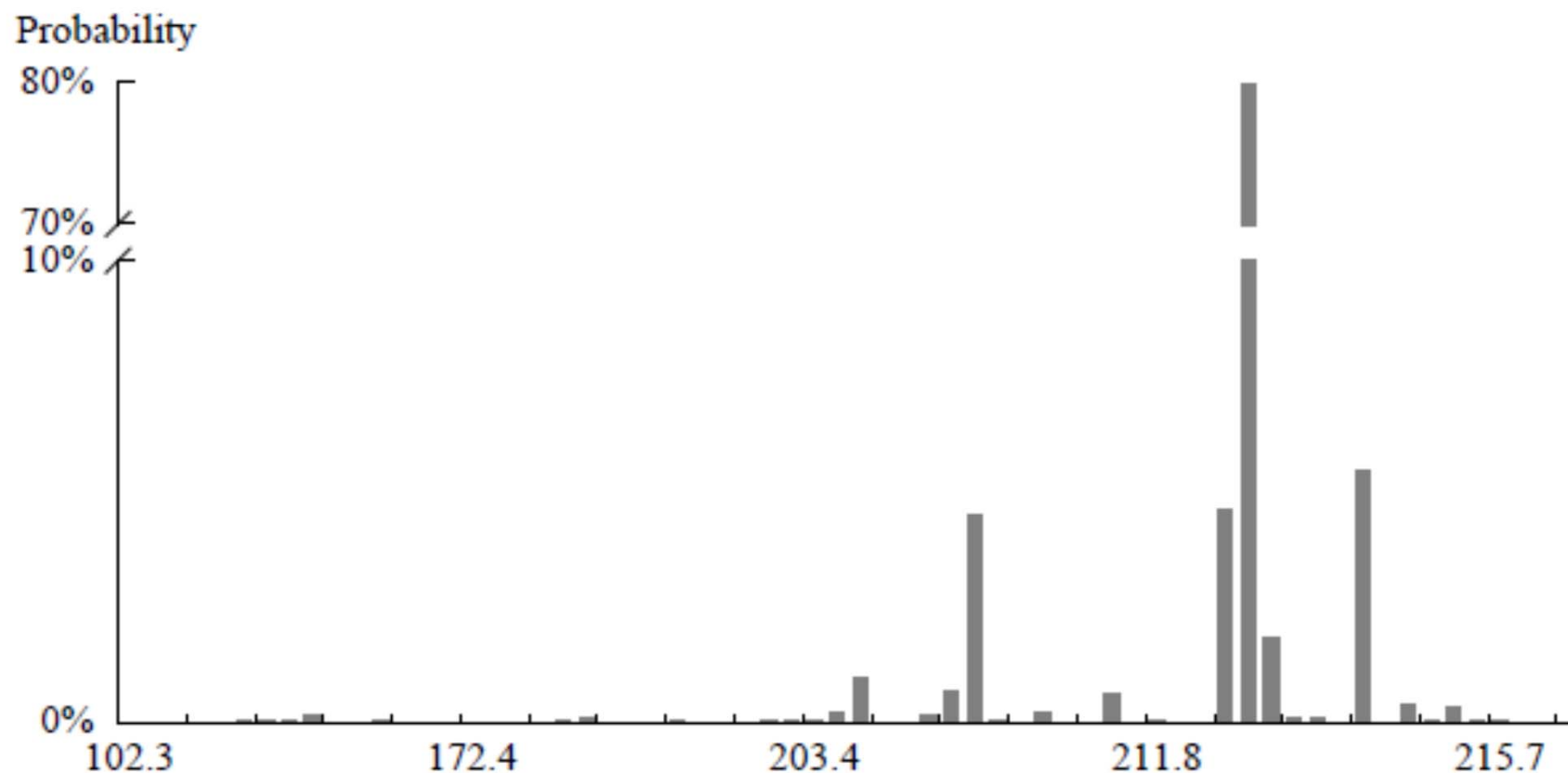


Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

# Credit-VaR

There are at least four interesting features in the joint likelihood table above:

1. The probabilities across the table necessarily sum to 100%.

2. The most likely outcome is that both obligors simply remain at their current credit ratings. In fact, the likelihoods of joint migration become rapidly smaller as the migration distance grows.

3. The effect of correlation is generally to increase the joint probabilities along the diagonal drawn through their current joint standing (in this case, through BBB-A).

4. The sum of each column or each row must equal the chance of migration for that obligor standing alone. For instance, the sum of the last row must be 0.18%, which is the default likelihood for Obligor #1 (BBB) in isolation.

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

# Credit-VaR

$$S = 64$$

Mean: $\mu_{Total} = \sum_{i=1}^{S=64} p_i \mu_i = 213.63$

Variance: $\sigma_{Total}^2 = 11.22$

| | BBB Bond | A Bond | Portfolio |
|---|---|---|---|
| Mean | 107,10 | 106,53 | 213,63 |
| St.-Dev. | 2,99 | 1,49 | 3,35 |

sq. root of 11.22

**Conclusion: The means of the BBB and the A bonds sum directly, but the risk (standard deviations) is much less than the summed individual risks due to diversification.**

**1-year 99% Credit-VaR** = Mean-P$^P$$_{1,(B,A)}$ (as the probability of having 1 year after a rating not above B in the 1st bond and A in the 2nd bond = P(B,A)+P(B,BBB)+…+P(D,D) = 0,92+0,18+…+0 =1,45 ≈ 1%) = 213,63-204,4 = 9,23.

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

# Credit-VaR

With zero correlation, the ratings considered to calculate the 99% Credit-VaR would be the same, as the cumulative probability =1% is only achieved at the rating combination (B,A), but the actual degree of freedom would be slightly smaller:

**1-year 99% Credit-VaR** = Mean-P$^P_{1,(B,A)}$ (as the probability of having 1 year after a rating not above B in the 1$^{st}$ bond and A in the 2$^{nd}$ bond = P(B,A)+P(B,BBB)+…+P(D,D) = 0,92+0,18+…+0 =1,45 ≈ 1%) = 213,63-204,4 = **9,23**.

○ Assuming a normal distribution, **the VaR would be**:

$$VaR(\alpha\%) = N(1-\alpha\%)*\sigma =>$$

$$VaR(5\%) = 1.65*\sigma = 1.65*3.35=5.53$$
$$VaR(1\%) = 2.33*\sigma = 2.33*3.35=7.81 \text{ (lower than the observed}$$
$$\text{value - 9,23 - due to fat tails)}$$

# Credit-VaR

○ The decision to hold a bond or not is likely to be made within the context of some existing portfolio.

○ Thus, the more relevant calculation is the **marginal increase to the portfolio risk** that would be created by adding a new bond to it = **0,36** in standard-deviation and **0,24** in Credit-VaR.

○ This increase in Credit-VaR is much smaller than the A-Bond 99% Credit-Var (3,39) due to the diversification effect.
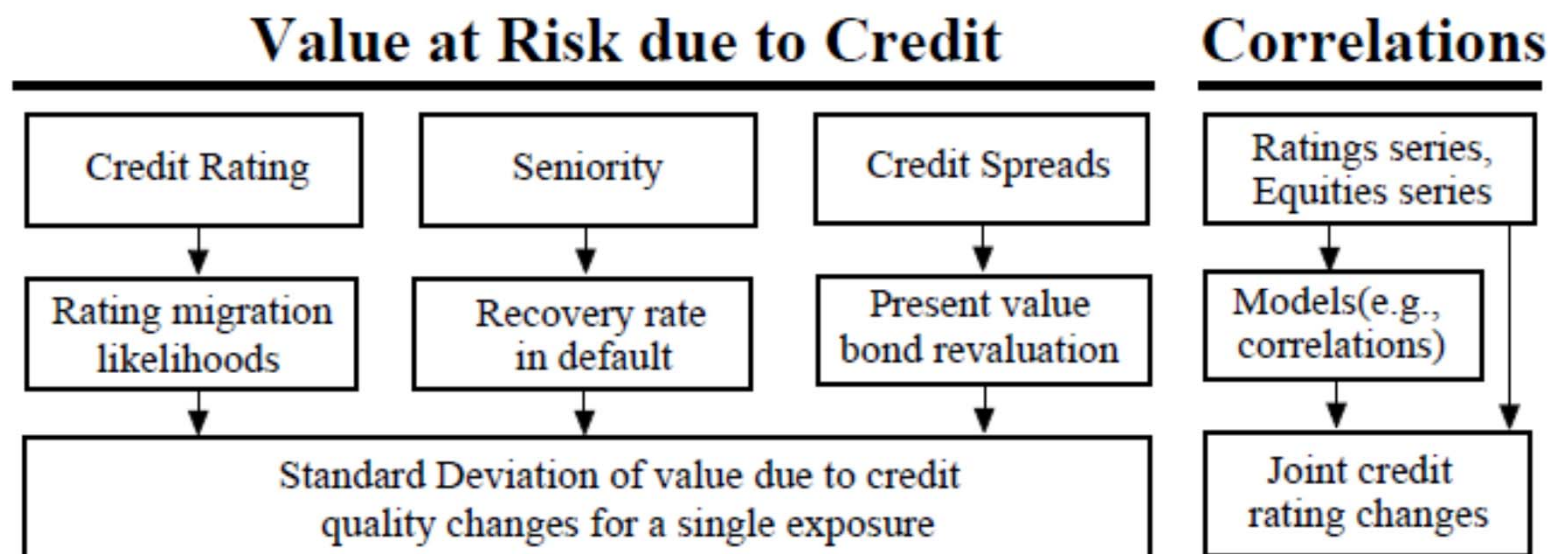
| | BBB-Bond (1) | A-Bond (2) | Portfolio (3) | A-Bond Marginal Risk (4) = (3)-(1) |
|---|---|---|---|---|
| Standard-deviation | 2,99 | 1,49 | 3,35 | **0,36** |
| **99% Credit-Var** | 8,99 | 3,39 | 9,23 | **0,24** |

Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".

# Creditmetrics

In our examples of one and two bond portfolios, we have been able to specify the entire distribution of values for the portfolio. We remark that this becomes inconvenient, and finally impossible, to do this in practice as the size of the portfolio grows. Noting that for a three asset portfolio, there are 512 (that is, 8 times 8 times 8) possible joint rating states. For a five asset portfolio, this number jumps to 32,768, and in general, for a portfolio with N assets, there are $8^N$ possible joint rating states.

**Information required:**



Source: Riskmetrics Group (2007), "CreditMetrics – Technical Document".