

CHAPTER 7: GAME THEORY

Exercise 1

Solve the following games using (iterative) elimination of (weakly) dominated strategies.

1.

	<i>L</i>	<i>CL</i>	<i>CR</i>	<i>R</i>
<i>U</i>	5, 10	0, 11	1, 10	10, 20
<i>MU</i>	4, 0	1, 0	2, 0	20, 1
<i>MD</i>	3, 2	0, 4	4, 3	50, 1
<i>D</i>	2, 93	0, 92	0, 91	100, 90

2.

	<i>L</i>	<i>C</i>	<i>R</i>
<i>U</i>	3, 3	0, 3	0, 0
<i>M</i>	3, 0	2, 2	0, 2
<i>D</i>	0, 0	2, 0	1, 1

Exercise 2

Consider a second-price sealed-bid auction with two bidders denoted by $i = 1, 2$, with valuations $v_1 > v_2$. Assume that, in case of a tie, i.e., if both bidders submit the same bid, bidder 1 wins the auction. Formalize this auction as a strategic-form game and find the equilibrium in weakly dominant strategies.

Exercise 3

Players 1 and 2 simultaneously choose a positive integer smaller or equal to K . If both players choose the same number, player 2 pays 1€ to player 1; otherwise no payment occurs. Determine the unique Nash equilibrium of this game.

Exercise 4

Determine the set of Nash equilibria of the following games:

1.

	<i>L</i>	<i>R</i>
<i>U</i>	0, 1	0, 2
<i>D</i>	2, 2	0, 1

2.

	<i>L</i>	<i>R</i>
<i>U</i>	6, 0	0, 6
<i>D</i>	3, 2	6, 0

3.

	<i>M1</i>			<i>M2</i>		
		<i>L</i>	<i>R</i>		<i>L</i>	<i>R</i>
<i>U</i>	1, 1, 1	0, 0, 0		<i>U</i>	0, 0, 0	0, 0, 0
<i>D</i>	0, 0, 0	0, 0, 0		<i>D</i>	0, 0, 0	2, 2, 2

Exercise 5

Consider a first-price sealed-bid auction with two bidders denoted by $i = 1, 2$, with valuations $v_1 > v_2$. Assume that, in case of a tie, i.e., if both bidders submit the same bid, bidder 1 wins the auction. Also assume that valuations are common knowledge. Formalize this auction as a strategic-form game and determine the set of Nash equilibria.

Exercise 6

Consider the Cournot model with n firms, which simultaneously choose how much to produce. Let q_i be the quantity produced by firm i and let $Q = q_1 + \dots + q_n$ be total quantity produced. Let p be the equilibrium price and assume that the inverse market

demand is: $p(Q) = \max\{0, a - Q\}$. Total cost of producing q_i by firm i is $c_i(q_i) = c_i q_i$, with $c_i < a$ for all $i=1, \dots, n$. All of this is common knowledge.

- i. Assume $c_i = c$ for all $i=1, \dots, n$. Determine, as a function of n , the quantities produced, the price, and the profits in Nash equilibrium (Cournot equilibrium).
- ii. Determine the limits of the functions obtained in i. when n goes to infinity. Explain.
- iii. Assume $n = 2$. Determine the Nash equilibrium when $0 < c_i < \frac{a}{2}$ for each firm? What if $c_1 < c_2 < a$, but $2c_2 > a + c_1$?

Exercise 7

Find the Nash equilibrium of a Bertrand duopoly where the two firms in the market have the same cost structure.

Exercise 8

Find all Bayesian-Nash equilibria of the following game with incomplete information:

- (a) Nature chooses J_1 and J_2 with 50% probability.
- (b) Player 1 observes Nature's choice, but player 2 does not.
- (c) Player 1 chooses C or B ; simultaneously, player 2 chooses E or D .

J_1	E	D
C	1, 1	0, 0
B	0, 0	0, 0

J_2	E	D
C	0, 0	0, 0
B	0, 0	2, 2

Exercise 9

Consider a Cournot duopoly with market demand given by $P(Q) = a - Q$, where $Q = q_1 + q_2$. Firm 1's cost function, given the quantity produced, is $C_1(q_1) = c q_1$ and firm 2's cost function is $C_2(q_2) = c_H q_2$ with probability a and $C_2(q_2) = c_L q_2$ with probability

1 – a. All of this is common knowledge. However, information is asymmetric: firm 2 knows its cost function, but firm 1 does not.

- i. Formulate this situation as game in strategic form.
- ii. Compute a Bayesian-Nash equilibrium.

Exercise 10

Consider the Battle of Sexes:

	<i>Bach</i>	<i>Stravinski</i>
<i>Bach</i>	3, 1	0, 0
<i>Stravinski</i>	0, 0	1, 3

- i. Find all Nash equilibria of this game.
- ii. Now assume that this game has incomplete information:

	<i>Bach</i>	<i>Stravinski</i>
<i>Bach</i>	$3 + t_1, 1$	0, 0
<i>Stravinski</i>	0, 0	$1, 3 + t_2$

Where t_1 and t_2 follow a uniform distribution in $[0, x]$. Determine the Bayesian-Nash equilibrium in pure strategies and show that as x goes to 0, the Bayesian-Nash equilibrium tends to the mixed strategies equilibrium of the complete information game.

Exercise 11

Two players, 1 and 2, share 1€ using the following procedure: each player i chooses a number s_i , $s_i \in [0, 1]$, $i=1,2$. The choices are simultaneous. If $s_1 + s_2 \leq 1$, each player gets the amount chosen; if $s_1 + s_2 > 1$ both get 0.

- i. Determine the set of pure Nash equilibria.

Suppose that player 2, before choosing s_2 , observes the number chosen by player 1 and this fact is common knowledge.

- ii. Find a few examples of pure Nash equilibria of the modified game.
- iii. Determine the set of pure subgame perfect Nash equilibria.

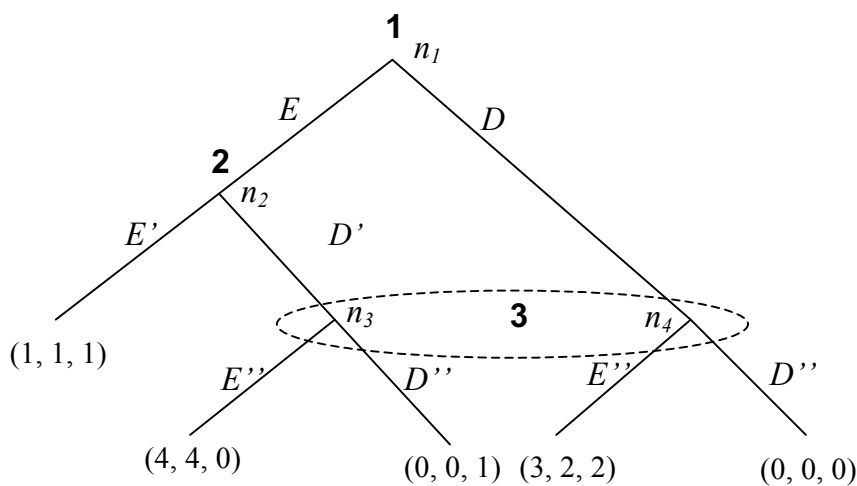
Exercise 12

Player 1 may choose Stop or Continue. If he chooses Stop, the game ends and each player gets 1€. If he chooses Continue, both players simultaneously choose non-negative integers and each player gets the product of the chosen numbers.

- i. Formulate this situation as an extensive-form game with imperfect information.
- ii. Determine the set of pure subgame perfect Nash equilibria.
- iii. How does this set change if the non-negative integers are at most equal to $M > 1$?

Exercise 13

Consider the following extensive-form game with imperfect information Γ :



This game has two types of Nash equilibria:

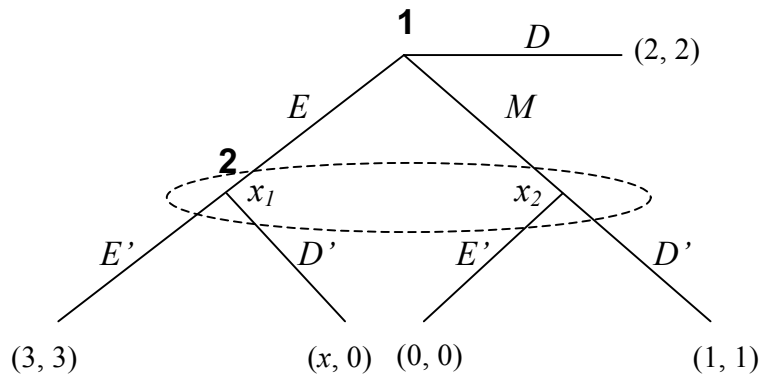
Type 1: $x_1(E)=1, x_2(E')=1$ and $x_3(E'') \in [0, 1/4]$.

Type 2: $x_1(E)=0, x_2(E') \in [1/3, 1]$ and $x_3(E'')=1$.

- i. Show that equilibria of Type 1 are perfect Bayesian equilibria of Γ .
- ii. Show that no equilibrium of Type 2 is a perfect Bayesian equilibria of Γ .

Exercise 14

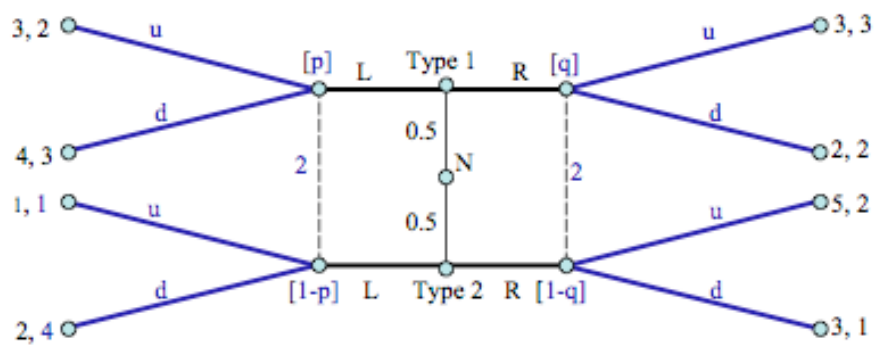
Consider the following extensive-form game with imperfect information Γ :



Show that the strategy $x = (x_1, x_2)$, with $x_1(D)=1$ and $x_2(D')=1$, is a perfect Bayesian equilibrium for $x < 2$.

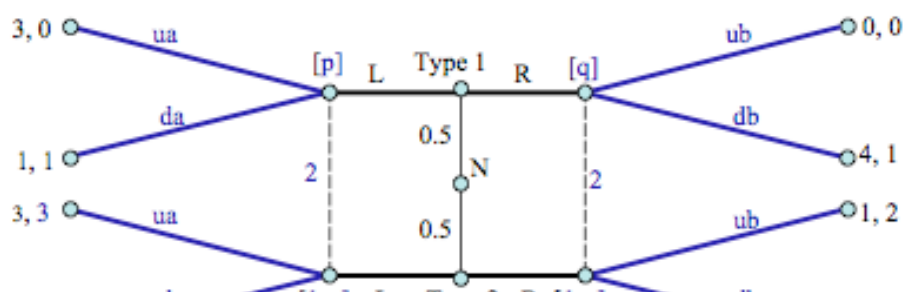
Exercise 15

Check whether Player 1's strategies (L,R), (R,L), (R,R) and (L,L) are part of perfect Bayesian equilibria of the following game:



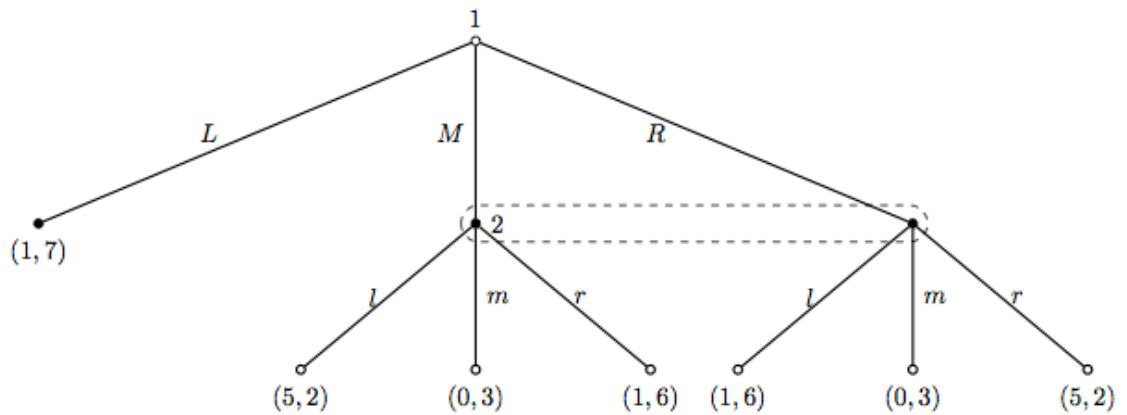
Exercise 16

Show that Player 1's strategy (L,R) is part of a perfect Bayesian equilibrium of the following game:



Exercise 17

Consider the following extensive-form game with imperfect information:



- Write the game in normal form.
- Determine the set of pure strategy Nash equilibria of the game.
- How many subgames does this game have?
- Determine the set of pure strategy subgame perfect Nash equilibria of the game.
- Check whether the equilibria found in d) are perfect Bayesian equilibria.