5. DEFAULT CORRELATION MODELS

- Credit spreads of different issuers are correlated through time.
- However, a good model for the default correlations across firms is still an open challenge for credit risk researchers.
- Correlations across equities are considerably higher than observed default correlations.
- **Two patterns** are found in time series of spreads:

1) Spreads vary smoothly with general macroeconomic factors in a correlated fashion.

↓

Cyclical correlation between defaults

2) Jumps are common on several firm credit spreads. This suggests that the sudden variation in the credit risk of one issuer, which causes the jump in first place, can propagate to other issuers as well.

5. DEFAULT CORRELATION MODELS

Historically, defaults tended to cluster as the following examples from the USA show.

- Oil industry: 22 companies defaulted in 1982–1986.
- Railroad conglomerates: 1 default each year 1970–1977.
- Airlines: 3 defaults in 1970–1971, 5 defaults in 1989–1990.
- Thrifts (savings and loan crisis): 19 defaults in 1989–1990.
- Casinos/hotel chains: 10 defaults in 1990.
- Retailers: >20 defaults in 1990–1992.
- Construction/real estate: 4 defaults in 1992.

If defaults were indeed independent, such clusters of defaults should not occur.

Secondly, there also seems to be serial dependence in the default rates of subsequent years. A year with high default rates is more likely to be followed by another year with an above average default rate than to be followed by a low default rate. The same applies to low default rates.

5. DEFAULT CORRELATION MODELS

- Conditional probabilities:

$$p_{A|B} = \frac{p_{AB}}{p_B}, \qquad p_{B|A} = \frac{p_{AB}}{p_A}$$

- Correlation coefficient:

$$\varrho_{AB} = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}}.$$

$COV(A,B)/(\sigma_A \sigma_B)$

The joint default probability is given by:

$$p_{AB} = p_A p_B + \varrho_{AB}\sqrt{p_A(1-p_A)p_B(1-p_B)}$$

$P_{AB} = P_A P_B + COV(A,B)$

and the conditional default probabilities are:

Dividing $p_{A|B}$ by p_B

$$p_{A|B} = p_A + \varrho_{AB}\sqrt{\frac{p_A}{p_B}(1-p_A)(1-p_B)}.$$

5. DEFAULT CORRELATION MODELS

- Calculation of default correlation:

  - Historically observed joint rating and default events: The obvious source of information on default correlation is the historical incidence of joint defaults of similar firms in a similar time frame. We used such data in Section 10.1.1 when we analysed the evidence for default dependency in aggregated historical US default rate data. Such data is objective and directly addresses the modelling problem. Unfortunately, because joint defaults are rare events, historical data on joint defaults is very sparse. To gain a statistically useful number of observations, long time ranges (several decades) have to be considered and the data must be aggregated across industries and countries. In the majority of cases direct data will therefore not be available.

  - Credit spreads: Credit spreads contain much information about the default risk of traded bonds, and changes in credit spreads reflect changes in the markets' assessment of the riskiness of these investments. If the credit spreads of two obligors are strongly correlated it is likely that the defaults of these obligors are also correlated. Credit spreads have the further advantage that they reflect market information (therefore they already contain risk premia) and that they can be observed far more frequently than defaults. Disadvantages are problems with data availability, data quality (liquidity), and the fact that there is no theoretical justification for the size and strength of the link between credit spread correlation and default correlation.[2]

5. DEFAULT CORRELATION MODELS

- Calculation of default correlation:

  - Equity correlations: Equity price data is much more readily available and typically of better quality than credit spread data. Unfortunately, the connection between equity prices and credit risk is not obvious. This link can only be established by using a theoretical model, and we saw that these models have difficulties in explaining the credit spreads observed in the market. Consequently, a lot of pre-processing of the data is necessary until a statement about default correlations can be made.

Independent Defaults

If defaults are independent and happen with probability p over the time horizon T, then the loss distribution of a portfolio of N loans is described by the binomial distribution function.

Definition 10.1 (binomial distribution) *Consider a random experiment with success probability p which is repeated N times and let X be the number of successes observed. All repetitions are independent from each other. The binomial frequency function $b(n; N, p)$ gives the probability of observing $n \leq N$ successes. The binomial distribution function $B(n; N, p)$ gives the probability of observing less than or equal to n successes:*

$$b(n; N, p) := \mathbf{P}[X = n] = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$$B(n; N, p) := \mathbf{P}[X \leq n] = \sum_{m=0}^{n} \binom{N}{m} p^m (1-p)^{N-m}.$$

In our credit setting, the probability of exactly $X = n$ (with $n \leq N$) defaults until time T is $b(n; N, p)$ and the probability of up to n defaults is $B(n; N, p)$.

Independent Defaults

Distribution of default losses under independence
(number of obligors =100 and p =0,05)

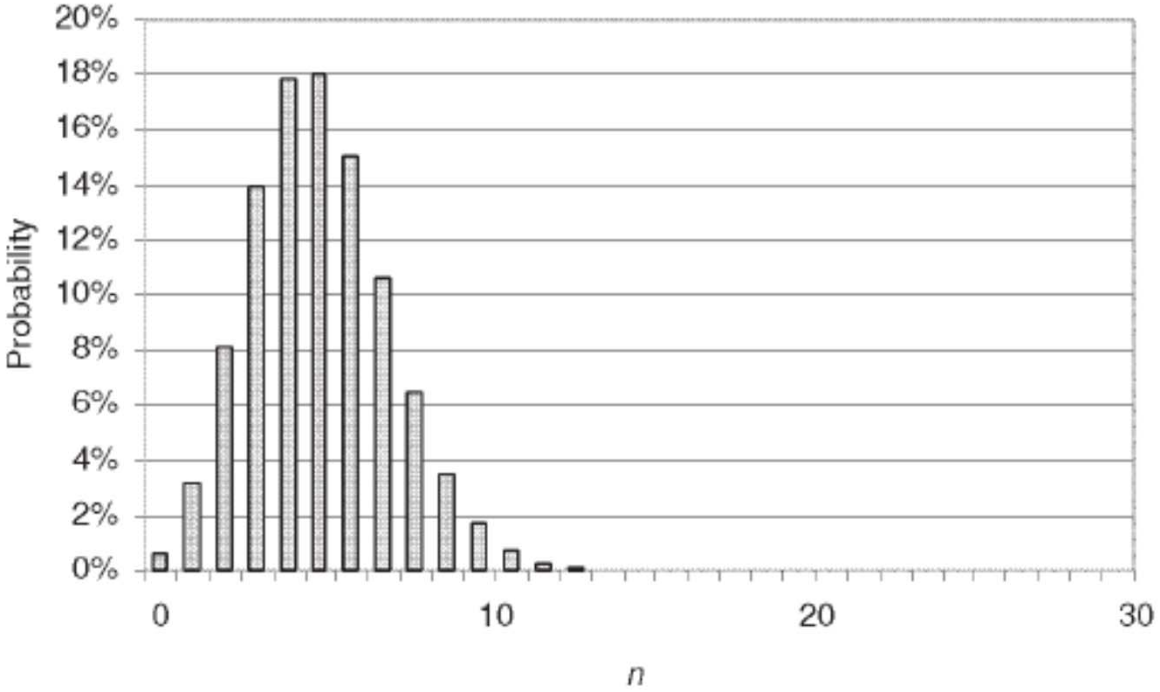


Figure 10.5 Distribution of default losses under independence. Parameters: number of obligors $N = 100$, individual default probability $p = 5\%$

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

Independent Defaults

- This example shows that the tail of the distribution is very thin

↓

- The Credit-VaR for very high degrees of confidence is achieved at a low number of defaults:

  - 99% VaR = 11 defaults
  - 99,9% VaR = 13 defaults
  - 99,99% VaR = 15 defaults

Independent Defaults

- The number of defaults corresponding to a Credit-VaR for very high degrees of confidence increases with the PD:

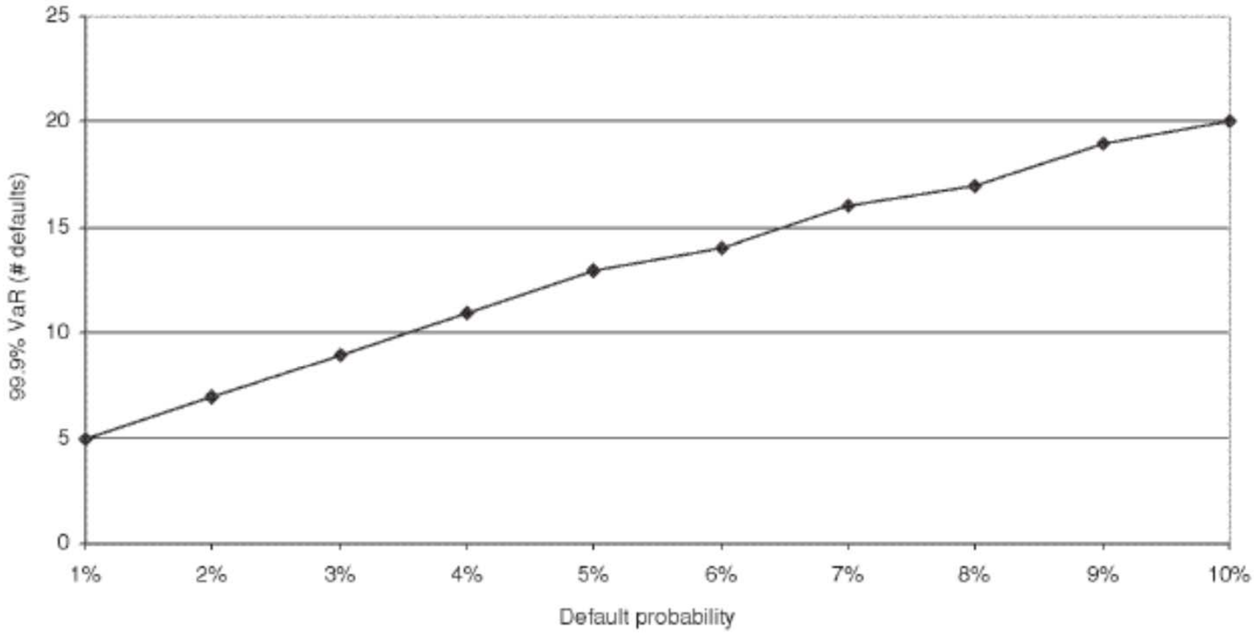
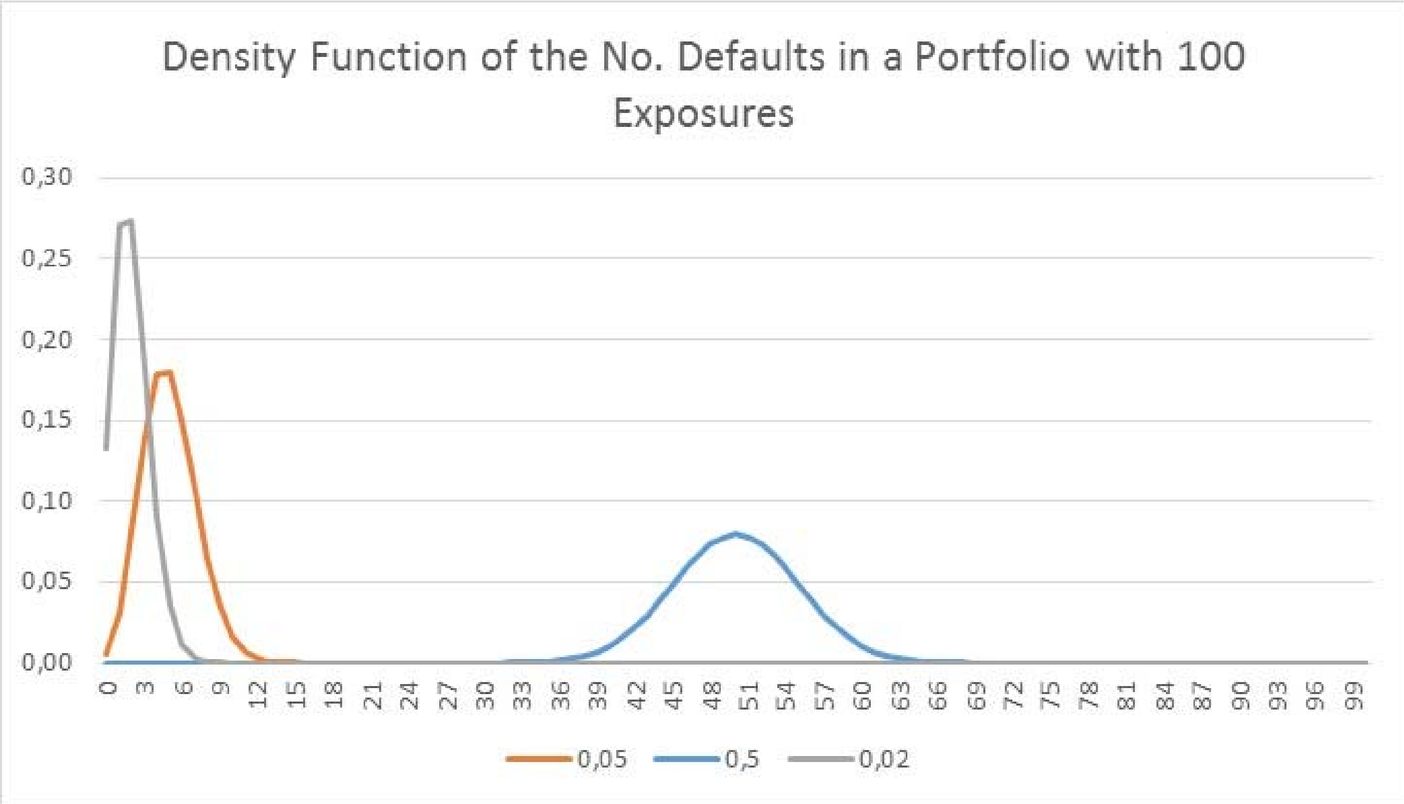


Figure 10.6 99.9% VaR levels of a portfolio of 100 independent obligors for different individual default probabilities

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

Independent Defaults

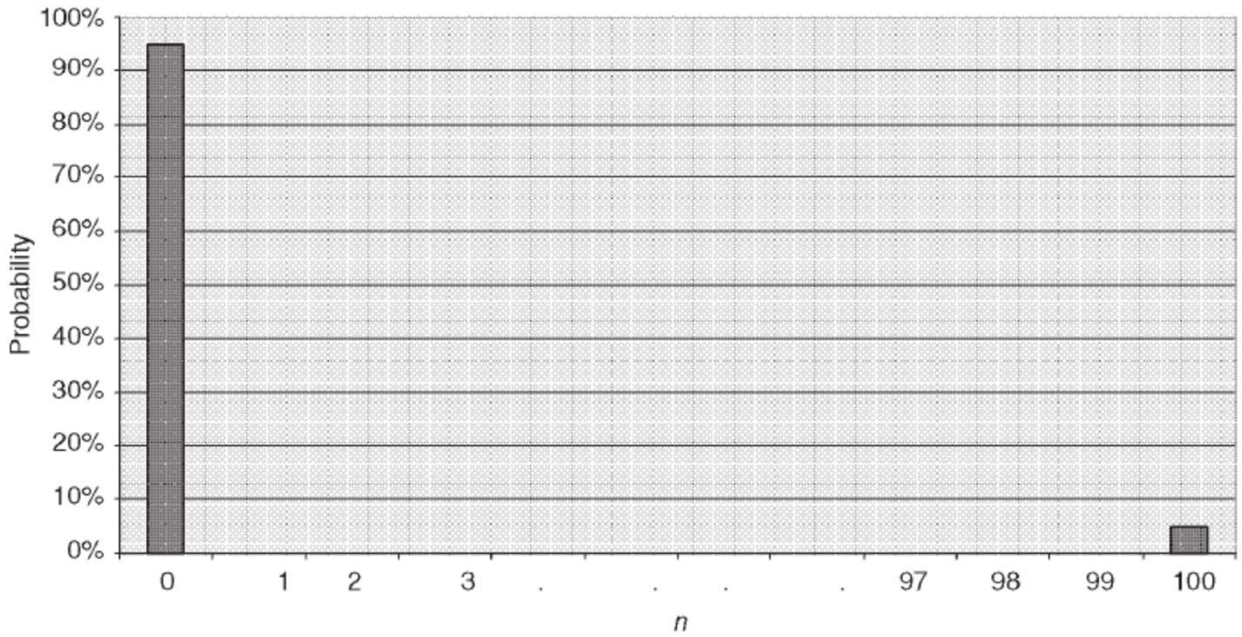
- The dispersion of the No. defaults distribution also increases with the PD:



Perfectly Correlated Defaults

- *Either all* obligors default (with 5% probability),
- *Or none* of the obligors defaults (with 95% probability).

Perfectly dependent defaults



Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

BINOMIAL EXPANSION TECHNIQUE (BET)

- Independent defaults:

The binomial expansion technique (BET) is a method used by the ratings agency Moody's to assess the default risk in bond and loan portfolios. It was one of the first attempts to quantify the risk of a portfolio of defaultable bonds. The method is not based upon a formal portfolio default risk model, it can be inaccurate and it is generally unsuitable for pricing, yet it has become something of a market standard in risk assessment and portfolio credit risk concentration terminology.[5]

The BET is based upon the following observation. Assume we analyse a loan portfolio of $N = 100$ loans of the same size, with the same loss L in default and the same default probability $p = 5\%$. If the defaults of these obligors are independent, we know from the previous section that the loss distribution function is given by the binomial distribution function. The probability of a loss of exactly $X = nL$ (with $n \leq N$) until time T is (10.8):

$$\mathbf{P}[X = nL] = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} =: b(n; N, p).$$

BINOMIAL EXPANSION TECHNIQUE (BET)

- Perfectly dependent defaults:

Let us now consider the other extreme. If all defaults are perfectly dependent (i.e. either *all* or *none* of the obligors default), we have:

$$\mathbf{P}[X > 0] = p = 5\% = \mathbf{P}[X = NL],$$
$$\mathbf{P}[X = 0] = 1 - p = 95\% = \mathbf{P}[X = 0].$$

The key point to note here is that this can also be represented as a binomial distribution function with probability $p = 5\%$, but this time only *one* binomial draw is taken and the stakes are much higher: a loss of NL if the 5% event occurs.

BINOMIAL EXPANSION TECHNIQUE (BET)

Thus we have the following results.

- Perfect independence is $N = 100$ obligors with loss L and loss probability $p = 5\%$ each. The probability of a loss X of less than x is

$$\mathbf{P}[X \leq x] = B(n; N, p),$$

where the parameters are:
– $N = 100$;
– $n = \lfloor x/L \rfloor$ ("rounding down", the largest integer less than or equal to x/L);
– $p = 5\%$.

$\leftrightarrow x = nL$ → Loss (x) = No. of defaults x Loss with each default

BINOMIAL EXPANSION TECHNIQUE (BET)

- Perfect dependence is equivalent to $N' = 1$ obligors with loss $L' = NL$ and loss probability $p = 5\%$. The probability of a loss X of less than x is

$$\mathbf{P}[X \leq x] = B(n'; N', p),$$

where:
– $N' = 1$, an adjusted number of obligors;
– $n' = \lfloor x/L' \rfloor$
– $p = 5\%$.

- Independence between defaults => N' = N with loss amount from each default event L' = L
- Perfect dependence between defaults => N' = 1 with loss amount from each default event L' = N x L
- It is convenient to calculate intermediate degrees of dependence assuming that we have N' = D < N independent debtors with losses L' = LN/D each.

BINOMIAL EXPANSION TECHNIQUE (BET)

- In a portfolio of *N* debtors with a total exposure of *K* and *p* as the average individual PD, the BET loss distribution with a diversity score *D* is:

$$P^{BET}(x; N, K, D) := B(\lfloor x/L' \rfloor; D, p) \longrightarrow$$

from $\mathbf{P}[X \leq x] = B(n'; N', p)$

with

– $N' = 1$, an adjusted number of obligors;

– $n' = \lfloor x/L' \rfloor$

– $p = 5\%$.

- With N = 100:
  - D = 100 <-> we have 100 independent debtors with potential losses of L each
  - D = 50 <-> we have 50 independent debtors with potential losses of 2L each
  - D = 1 <-> we have perfect dependency between all exposures, with a potential losses of N x L

BINOMIAL EXPANSION TECHNIQUE (BET)

- Lower diversity scores => less continuous distributions and higher probability attached to higher losses => more concentrated portfolios are riskier.

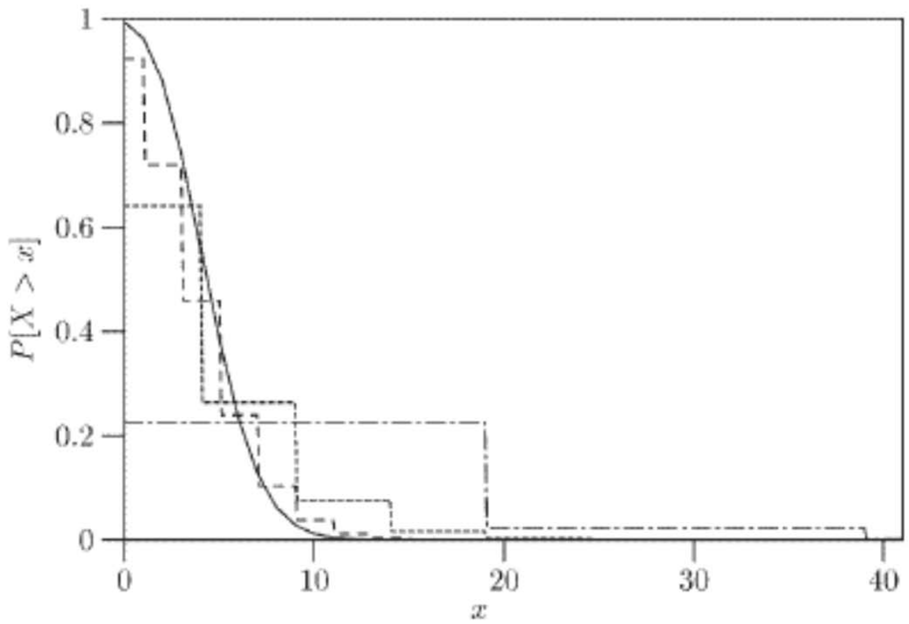


Figure 10.8 Loss exceedance probabilities for different diversity scores. Parameters: $N = 100$, $p = 5\%$, $D = 100, 50, 20, 5$ (solid, dashed, short dashed, dot dashed)

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.

BINOMIAL EXPANSION TECHNIQUE (BET)

- Moody's starts by assuming perfect diversification ($D = N$).

- D is adjusted downwards according to:

(i) Exposure sizes:

- Large exposures are penalized (portfolios with identical exposures are not adjusted).

(ii) Industry diversification

(iii) Regional diversification

GAUSSIAN COPULA

- Defining t_1 and t_2 as the time to default of companies 1 and 2, respectively, if these variables are not normally distributed, we may transform them into new variables x_1 and x_2:

where $x_1 = N^{-1}[Q_1(t_1)], \qquad x_2 = N^{-1}[Q_2(t_2)]$

Q_1 and Q_2 – cumulative probability distributions for t_1 and t_2

N^{-1} – inverse of the cumulative normal distribution

- x_1 and x_2 - default threshold of each company, determined by the balance sheet structure, normally distributed, with zero mean and unit standard deviation

- The joint distribution of x_1 and x_2 is a bivariate normal.

GAUSSIAN COPULA

GAUSSIAN COPULA

- This assumption is convenient as it allows to characterize the joint distribution of t_1 and t_2 by the cumulative default probability distributions Q_1 and Q_2, with a single correlation parameter between x_1 and x_2 – the copula correlation.
- To avoid defining a different correlation between x_i and x_j for each pair of companies *i* and *j*, a one-factor model is often used:

$$x_i = a_i F + \sqrt{1 - a_i^2}\, Z_i$$

with

F = the common or systematic risk factor – can be viewed as a business cycle indicator

$Z_i \sim N(0,1)$, i.i.d. – can be viewed as an idiosyncratic or firm-specific factor (e.g. quality of the management, level of innovation)

a_i = constant parameters between -1 and +1

$a_i * a_j$ = correlation between x_i and x_j

GAUSSIAN COPULA

- A default occurs until *T* when $N(x_i) < Q_i(T) \Leftrightarrow x_i < N^{-1}[Q_i(T)]$

$$x_i = a_i F + \sqrt{1 - a_i^2}\, Z_i \quad \Leftrightarrow \quad Z_i < \frac{N^{-1}[Q_i(T)] - a_i F}{\sqrt{1 - a_i^2}}$$

$$a_i F + \sqrt{1 - a_i^2}\, Z_i < N^{-1}[Q_i(T)]$$

- As $x_1 = N^{-1}[Q_1(t_1)]$, $\quad x_2 = N^{-1}[Q_2(t_2)]$, PD conditional on the factor F is:

$$Q_i(T \mid F) = N\left(\frac{N^{-1}[Q_i(T)] - a_i F}{\sqrt{1 - a_i^2}}\right)$$

GAUSSIAN COPULA

- With all loans having the same probability distributions of default and the same correlation => $a_i = \sqrt{\rho}$ ➡ $Q(T \mid F) = N\left(\frac{N^{-1}[Q(T)] - \sqrt{\rho}F}{\sqrt{1-\rho}}\right)$

corresponds to the % defaults in a homogeneous portfolio by time T as a function of F.

- ρ drives the weight of the idiosyncratic and systematic components:

– ρ = 0 => the business cycle is irrelevant to explain credit risk, i.e. the PD will not fluctuate.

– ρ = 1 => the business cycle is the only driver of defaults.

GAUSSIAN COPULA

- As F ~ N(0,1), we are X% sure that F > $N^{-1}(1 - X) = -N^{-1}(X)$

↓

- We are X% sure that the percentage of losses over *T* years on a large homogeneous portfolio will be less than *V(X,T)* – Worst Case Default Rate (WCDR):

$$V(X, T) = N\left(\frac{N^{-1}[Q(T)] + \sqrt{\rho}\, N^{-1}(X)}{\sqrt{1-\rho}}\right)$$

↓

- Vasicek (1987) (published in Risk Magazine, in 2002, as "Loan Portfolio Value"):

O. Vasicek, "Probability of Loss on a Loan Portfolio," Working Paper, KMV, 1987.

- **Credit-VaR = *V(X,T)* x *LGD x EAD***

GAUSSIAN COPULA

- As F ~ N(0,1), we are X% sure that F > $N^{-1}(1-X) = -N^{-1}(X)$

- We are X% sure that the percentage of losses over *T* years on a large homogeneous portfolio will be less than *V(X,T)*:

$$V(X,T) = N\left(\frac{N^{-1}[Q(T)] + \sqrt{\rho}\,N^{-1}(X)}{\sqrt{1-\rho}}\right)$$

- Vasicek (1987) (published in Risk Magazine, in 2002, as "Loan Portfolio Value"):

O. Vasicek, "Probability of Loss on a Loan Portfolio," Working Paper, KMV, 1987.

- X very close to 1 => Worst Case Default Rate (WCDR)

- **Credit-VaR = *WCDR x LGD x EAD*** → M. B. Gordy, "A Risk-Factor Model Foundation for Ratings-Based Bank Capital Ratios," *Journal of Financial Intermediation* 12 (2003): 199–232.

GAUSSIAN COPULA

- Example - Retail loan portfolio:

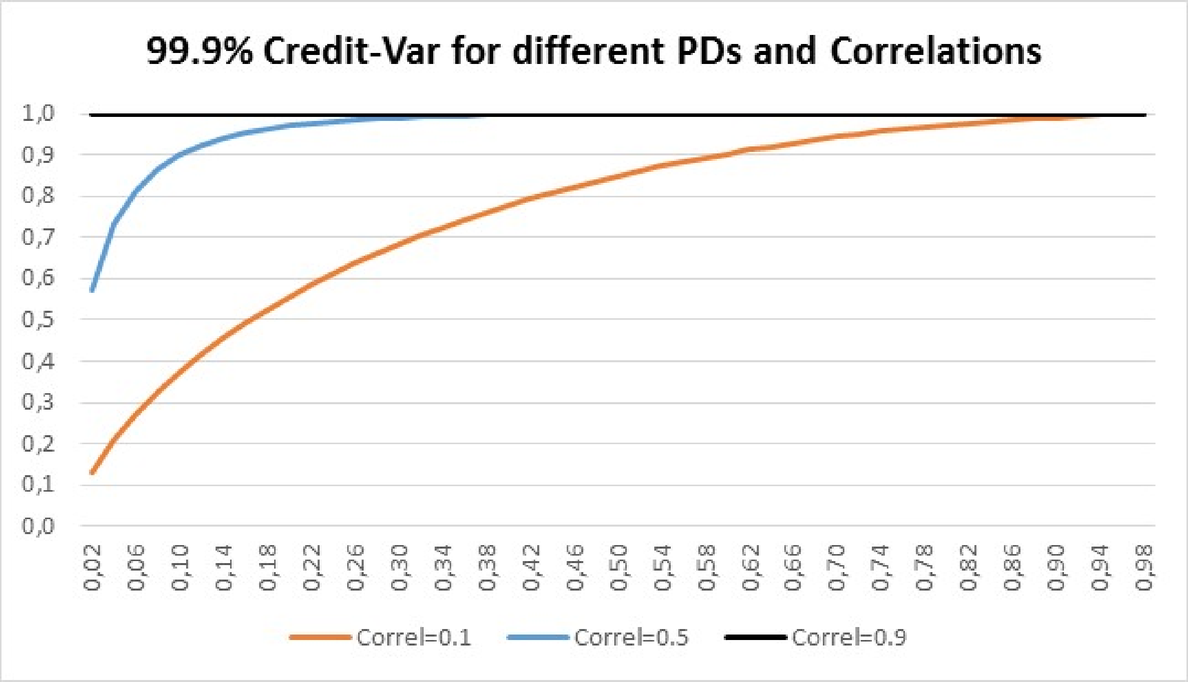
- Value = 100 M€
- 1y PD = 2%
- ρ = 0.1
- RR = 60%

$$V(X,T) = N\left(\frac{N^{-1}[Q(T)] + \sqrt{\rho}\, N^{-1}(X)}{\sqrt{1-\rho}}\right) \Rightarrow V(0.999, 1) = N\left(\frac{N^{-1}(0.02) + \sqrt{0.1}\, N^{-1}(0.999)}{\sqrt{1-0.1}}\right) = 0.128$$

- 1y Credit-VaR = *WCDR* x *LGD* x *EAD* = 0.128 x (1- 0.6) x 100 M€ = 5.13 M€
- 1y EL = PD x LGD x EAD = 0.02 x (1- 0.6) x 100 M€ = 0.8 M€
- ρ = 0 => $V(X,T) = N(N^{-1}[Q(T)])$ = Q(T) = PD => Credit-VaR = EL
- ρ -> 1=> $V(X,T)$ -> $N(\propto)$ -> 1 => Credit-VaR -> LGD x EAD

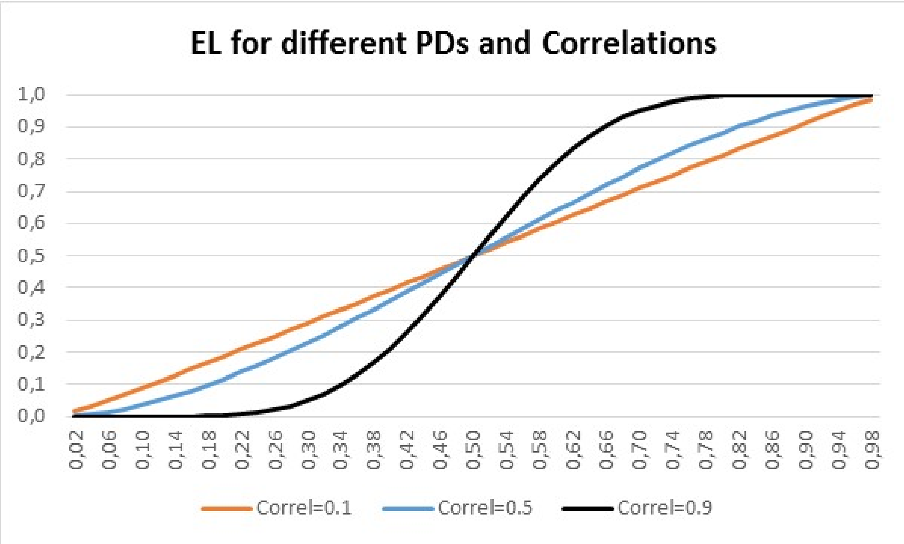
GAUSSIAN COPULA

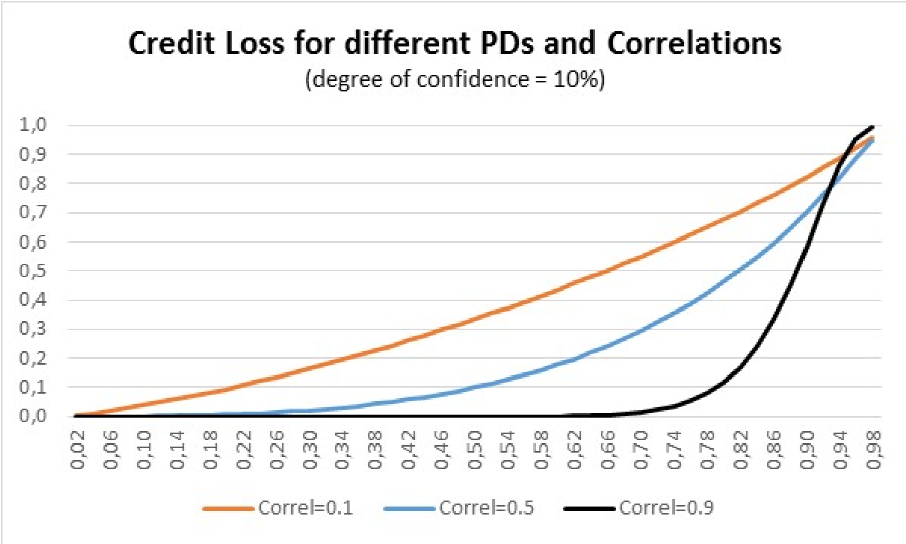
- Portfolio credit loss tends to 1 with the 1y PD, regardless the level of confidence and the correlation coefficient.
- For very high degrees of confidence (X), credit loss converges to 1 following a concave curve, at a faster speed and from a higher value with higher correlations.



GAUSSIAN COPULA

- For lower degrees of confidence, credit loss converges slower to 1, at a faster speed with higher correlations and degrees of freedom.





GAUSSIAN COPULA

- Basel II allows banks to calculate their capital requirements for the different portfolios using internal estimates for PD and LGD, using regulatory formulas based on the WCDR and assuming different functions for the correlation coefficient.

- Corporate, Sovereign and Bank Exposures:

$$\rho = 0.12\frac{1-\exp(-50\times PD)}{1-\exp(-50)} + 0.24\left[1-\frac{1-\exp(-50\times PD)}{1-\exp(-50)}\right]$$

- Capital Requirement: $\text{EAD}\times\text{LGD}\times(\text{WCDR}-\text{PD})\times\text{MA}$

- Maturity Adjustment (MA): $\text{MA} = \dfrac{1+(\text{M}-2.5)\times b}{1-1.5\times b}$

- Where $b = [0.11852 - 0.05478\times\ln(\text{PD})]^2$ and M is the maturity.

GAUSSIAN COPULA

- Probability of having exactly *n* defaults ($X=n$) – average of the conditional probabilities of *n* defaults, averaged over the possible realizations of *F* and weighted by the probability density function $Q(T|f)$:

$$P[X = n] = \int_{-\infty}^{+\infty} P[X = n|F = f]Q(T|f)df$$

- Probability of *n* defaults, conditional on the realization $F = f$ of the systematic factor:

$$P[X = n|F = f] = \binom{N}{n}\left(p(f)\right)^n\left(1 - p(f)\right)^{N-n}$$

GAUSSIAN COPULA

- From $Q(T \mid F) = N\left(\frac{N^{-1}[Q(T)] - \sqrt{\rho}\, F}{\sqrt{1-\rho}}\right)$, we get the **probability density function of defaults**:

$$P[X = n] = \int_{-\infty}^{+\infty} \binom{N}{n} \left(N\left(\frac{N^{-1}[Q(T)] - \sqrt{\rho} f}{\sqrt{1-\rho}} \right) \right)^{n} \left(1 - N\left(\frac{N^{-1}[Q(T)] - \sqrt{\rho} f}{\sqrt{1-\rho}} \right) \right)^{N-n} Q(T|f) df$$

- **Probability distribution function:**

$$P[X \leq m] = \sum_{n=0}^{m} \binom{N}{n} \int_{-\infty}^{+\infty} \left(N\left(\frac{N^{-1}[Q(T)] - \sqrt{\rho} f}{\sqrt{1-\rho}} \right) \right)^{n} \left(1 - N\left(\frac{N^{-1}[Q(T)] - \sqrt{\rho} f}{\sqrt{1-\rho}} \right) \right)^{N-n} Q(T|f) df$$

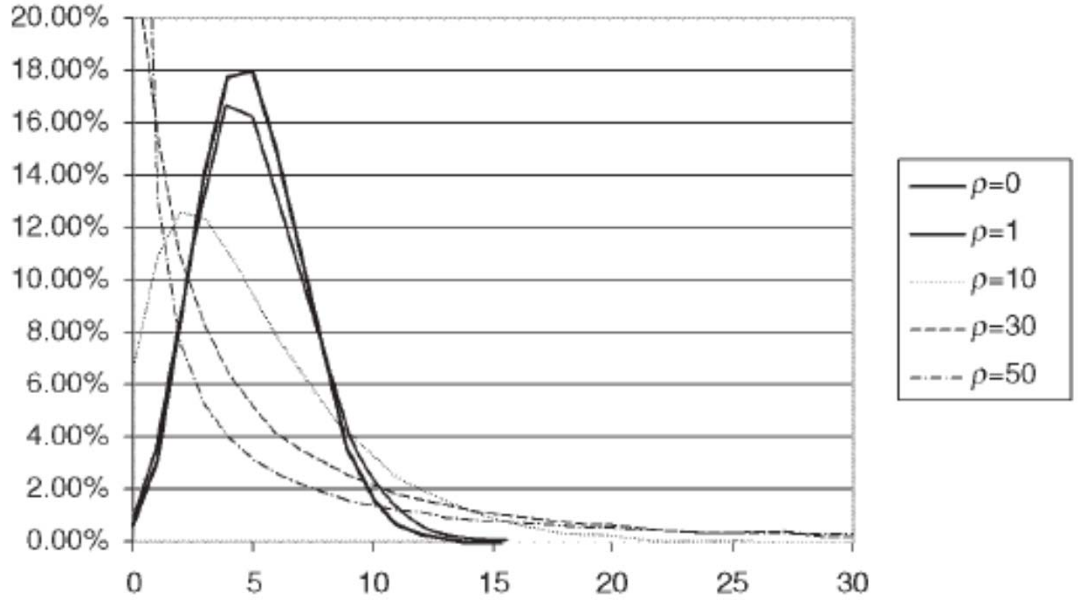
- These functions are in line with the previous result for independent loans, but incorporating the copula and the single factor:

$$b(n; N, p) := \mathbf{P}[X = n] = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$$B(n; N, p) := \mathbf{P}[X \leq n] = \sum_{m=0}^{n} \binom{N}{m} p^m (1-p)^{N-m}.$$

GAUSSIAN COPULA

Default losses with one-factor model (N=100, N[Q(T)] = 5% for different levels of ρ:



- Increasing asset and default correlation => pdf shift to the left and increase of the right tail, as very good events (no or very few defaults) become equally more likely than very bad events (many defaults) => VaR increases with asset correlation.

Source: Schonbucher, Philipp J. (2003), "Credit Derivatives Pricing Models – Models, Pricing and Implementation", Wiley.