Models in Finance - Slides 23

Master in Actuarial Science

João Guerra

ISEG

The Cox-Ingersoll-Ross (CIR) model

- The CIR model SDE for $r(t)$ under Q:

$$dr(t) = \alpha(\mu - r(t))\,dt + \sigma\sqrt{r(t)}\,d\widetilde{W}_t, \tag{1}$$

  where $\widetilde{W}_t$ is a standard Bm under Q, and the parameter α is positive.
- The drift is the same as for the Vasicek model.
- The difference between the two models occurs in the volatility, which increases with the square-root of $r(t)$ for the CIR model.
- Since this diminishes to zero as $r(t)$ approaches zero, if σ^2 is not too large ($\sigma^2 \leq 2\alpha\mu$), we can guarantee that $r(t)$ will not hit zero and all other interest rates will remain strictly positive.
- This dynamics ensures mean reversion of the interest rate towards the long run value μ, with speed of adjustment governed by the strictly positive parameter α.

The Cox-Ingersoll-Ross (CIR) model

- The power $(1/2)$ in $\sqrt{r(t)}$ was chosen because is the value that "just" prevents the process $r(t)$ from reaching zero.
- Under the CIR model, the zero-coupon bond prices are:

$$B(t,T) = e^{a(\tau)-b(\tau)r(t)}, \tag{2}$$

where

$$\begin{aligned}
\tau &= T - t, \\
b(\tau) &= \frac{2\left(e^{\theta\tau}-1\right)}{(\theta+\alpha)\left(e^{\theta\tau}-1\right)+2\theta}, \\
a(\tau) &= \frac{2\alpha\mu}{\sigma^2}\log\left(\frac{2\theta e^{(\theta+\alpha)\tau/2}}{(\theta+\alpha)\left(e^{\theta\tau}-1\right)+2\theta}\right), \\
\theta &= \sqrt{\alpha^2+2\sigma^2}.
\end{aligned}$$

The Cox-Ingersoll-Ross (CIR) model

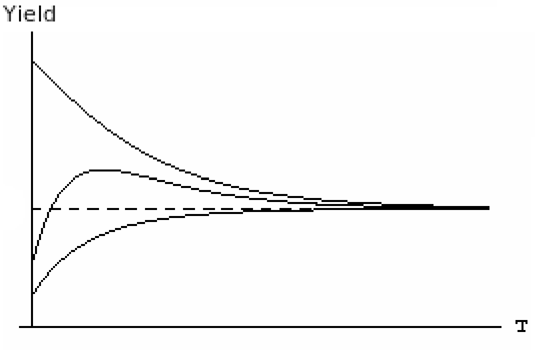
- Pricing formulae for European call and put options on zero-coupon bonds look similar to those for the Vasicek model and to the Black-Scholes formulae for equity options.
- However, where the latter models use the cumulative distribution function of the Normal distribution, the CIR formulae use the cumulative distribution function of the non-central chi-squared distribution.
- If $X_i \sim N(d_i, 1)$ then $Y_d = X_1^2 + X_2^2 + \cdots + X_n^2$ is said to have a non-central chi–squared distribution with n degrees of freedom and non-centrality parameter $d = \sum_{i=1}^{n} d_i^2$.
- From the point of view of implementation, the CIR model is slightly more tricky than the Vasicek model.

Yield curves

- For Vasicek and CIR models, the yield curves (for spot rates)

$$R(t,T) = \frac{-\ln B(t,T)}{T-t} = -\frac{a(\tau) - b(\tau) r_t}{T-t}$$

are of three relates types:



The Hull and White (HW) model

- The SDEs for both the Vasicek and CIR models gave us time-homogeneous models (bond prices at t depend only on $r(t)$ and on the term to maturity $\tau = T - t$). $\Longrightarrow$ lack of flexibility for pricing related contracts.
- For example, on any given date theoretical bond prices will probably not match exactly observed market prices.
- We can re-estimate $r(t)$ to improve the match and even re-estimate the constant parameters α,μ and σ but we will still, normally, be unable to get a precise match.
- A simple way to get theoretical prices to match observed market prices is to introduce some elements of time-inhomogeneity into the model.
- The Hull and White (HW) model does this by extending the Vasicek model in a simple way.

The Hull and White (HW) model

- The HW model SDE for $r(t)$ under Q:

$$dr(t) = \alpha\left(\mu(t) - r(t)\right)dt + \sigma d\widetilde{W}_t, \tag{3}$$

  where $\widetilde{W}_t$ is a standard Bm under Q, the parameter α is positive and $\mu(t)$ is a deterministic function.
- $\mu(t)$ has the natural interpretation of being the local mean-reversion level for $r(t)$.
- This is similar to Vasicek model but now $\mu(t)$ is no longer a constant.
- HW model can be extended to include a time-varying deterministic $\sigma(t)$. This allows us to calibrate the model to traded option prices as well as zero-coupon bond prices.

The Hull and White (HW) model

- Since $\mu(t)$ is deterministic the HW model is as tractable as the Vasicek model.
- An advantage of the HW model is that it allows us to price interest-rate linked contracts more accurately.
- This is important for a variety of reasons:
  1. In insurance, the fair value of fixed liabilities must accurately reflect the current observed term-structure of interest rates. The use of the Vasicek (or CIR) model, even after fitting the model to the current term structure, might introduce some bias into the fair value.
  2. Bond and interest-rate-derivatives traders want to be able to quote prices which are in line with prices being quoted by other traders. This is facilitated by the use of models like the HW model and other, more sophisticated, no-arbitrage models.
  3. The HW model suffers from the same drawback as the Vasicek model: interest rates might become negative.

Summary of short rate models

- The properties of these models are summarised below (the last two columns indicate if there are analytic solutions for $B(t, T)$ and option prices and $\mathcal{N}$, $NC\chi^2$ denote, respectively, normal and non-central χ^2 distributions).

Model	Dynamics	$r > 0$	$r \sim$	$P(t,T)$	c_t, B_t
Vasicek (1977)	$dr_t = \kappa(\theta - r_t)dt + \sigma d\widehat{W}_t$	N	$\mathcal{N}$	Y	Y
CIR (1985)	$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}d\widehat{W}_t$	Y	$NC\chi^2$	Y	Y
HW–V (1990)	$dr_t = \kappa(t)(\theta(t) - r_t)dt + \sigma(t)d\widehat{W}_t$	N	$\mathcal{N}$	Y	Y
HW–CIR (1990)	$dr_t = \kappa(t)(\theta(t) - r_t)dt + \sigma(t)\sqrt{r_t}d\widehat{W}_t$	Y	$NC\chi^2$	Y	Y

Limitations of one-factor models

1. Historical interest rate data $\Longrightarrow$ changes in the prices of bonds with different terms to maturity are not perfectly correlated as expected to see if a one-factor model was correct.
2. We can see sometimes that short-dated bonds fall in price while long-dated bonds go up.
3. Around three random factors are required to capture most of the randomness in bonds of different durations.
4. If we look at the long run of historical data $\Longrightarrow$ sustained periods of both high and low interest rates with periods of both high and low volatility.
5. This is difficult to capture without more random factors. This is important for: (i) the pricing and hedging of long-dated insurance contracts with interest-rate guarantees; (ii) asset-liability modelling and long-term risk-management.
6. We need more complex models to deal effectively with derivative contracts more complex than standard European call options.

Limitations of Equilibrium models and one-factor models

- The so-called Equilibrium Models start with a theory about the economy, such that interest rates revert to some long run average, are positive or their volatility is constant or geometric. Examples of Equilibrium models are Vasicek and Cox-Ingersoll-Ross.
- These Equilibrium Models rarely reproduce observed term structures.
- The so-called No-arbitrage Models use the term structure as an input and are formulated to adhere to the no-arbitrage principle. An example of a No-arbitrage Model is the Hull-White (one-factor and two-factor).
- We have also seen limitations of one-factor models. One-factor models do, nevertheless have their place as tools for: valuation of simple liabilities with no option characteristics; or short-term, straightforward derivatives contracts.
- For other problems it is appropriate to make use of models which have more than one source of randomness: so-called multifactor models.

Multifactor models

- Example of a multifactor model: the 2-factor Vasicek model, which models 2 processes: $r(t)$ and $m(t)$, the local mean reversion level:

$$dr(t) = \alpha_r \left(m\left(t\right) - r(t)\right) dt + \sigma_{r1} d\widetilde{W_1}\left(t\right) + \sigma_{r2} d\widetilde{W_2}\left(t\right), \tag{4}$$

$$dm(t) = \alpha_m \left(\mu - m(t)\right) dt + \sigma_{m1} d\widetilde{W_1}\left(t\right), \tag{5}$$

  where $\widetilde{W_1}\left(t\right)$ and $\widetilde{W_2}\left(t\right)$ are independent, standard Brownian motions under the risk neutral measure Q.

- One big difference of this model with respect to the HW model is that here, the mean-reversion level $m(t)$ is stochastic.

Multifactor models

- In general, zero-coupon bond prices can be calculated by the risk-neutral approach formula:

$$B(t, T) = E_Q\left[e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t \right]. \tag{6}$$

- For the 2-factor Vasicek model (it is a Markov process):

$$B(t, T) = E_Q\left[e^{-\int_t^T r(u)du} \middle| r(t), m(t) \right]. \tag{7}$$

PDE's for Bond prices

- One can approach the Bond pricing problem under the equivalent martingale measure also by using a PDE approach, by assuming that the price of a bond is a function of t and r_t:

$$B(t, T) = g(t, r_t),$$

where

$$dr_t = \mu(t, r_t)\, dt + \sigma(t, r_t)\, d\widetilde{W}_t.$$

By using the Itmula and requiring that the discounted price of $B(t, T)$ to be a martingale under the equivalent martingale measure Q, one can derive the following PDE (with boundary condition) for the price of a bond:

$$\frac{\partial g}{\partial t}(t, r) + \frac{\partial g}{\partial r}(t, r)\mu(t, r) + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}(t, r)\sigma^2(t, r) - rg(t, r) = 0,$$
$$g(T, r) = B(T, T) = 1 \text{ for all } r.$$

- More details about this PDE approach can be found in the Core Reading (2019), Section 5.3 (Unit 4, pages 46-48).

Market models

- When is important to take into account the correlation between different maturity rates (for example in the pricing of some financial derivatives or options like swaptions), one can use the so-called market models. In these models, each maturity instrument (such as forward interest rate) is treated as a distinct object, correlated to other similar objects using a multi-dimensional Brownian motion.
- In these models, one considers the bond price $B(s, T)$ as the numeraire and assumes that the price of a traded asset divided by this numeraire is a martingale under the measure associated with the numeraire, which is labeled $\mathbb{Q}^T$. The forward rates $F(s, t, T)$ under this measure are martingales.
- For more details about these models and this approach, see the Core Reading (2019), Section 5.9 (Unit 4, pages 53-54) or see the book of Bjork, Chapter 26.