

# Models in Finance - Class 19

Master in Actuarial Science

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## B-S model: replication of European call

- Discounted values:  $E_t = e^{-rt} V_t$  and  $D_t = e^{-rt} S_t$  are both martingales under  $Q$ .
- Moreover, we know that under  $Q$ ,

$$dS_t = S_t \left( rdt + \sigma d\tilde{Z}_t \right), \quad (1)$$

$$dD_t = \sigma D_t d\tilde{Z}_t, \quad (2)$$

$$\begin{aligned} dE_t &= -re^{-rt} V_t dt + e^{-rt} dV_t \\ &= e^{-rt} (-rV_t dt + dV_t). \end{aligned} \quad (3)$$

## B-S model: replication of European call

- By Ito's formula:

$$\begin{aligned}
 dV_t &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial s} dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial s^2} (dS_t)^2 \\
 &= \left[ \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial s} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial s^2} \right] dt + \\
 &\quad + \sigma \frac{\partial V}{\partial s} S_t d\tilde{Z}_t \\
 &= \left[ \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial s} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial s^2} \right] dt + \\
 &\quad + \frac{\partial V}{\partial s} e^{rt} dD_t.
 \end{aligned}$$

## B-S model: replication of European call

- Using Eq. (3) we obtain:

$$dE_t = e^{-rt} \left( -rV_t + \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial s} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial s^2} \right) dt \quad (4)$$

$$+ \frac{\partial V}{\partial s} dD_t. \quad (5)$$

- Since  $E_t$  and  $D_t$  are both martingales under  $Q$ , by the MRT, exists previsible process  $\phi_t$  such that

$$dE_t = \phi_t dD_t = \sigma \phi_t D_t d\tilde{Z}_t \quad (6)$$

- Comparing Eqs (4)-(5) with Eq. (6), we have that:

$$\phi_t = \frac{\partial V}{\partial s}, \quad (7)$$

$$rV_t = \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial s} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial s^2}. \quad (8)$$

# B-S model: replication of European call

- The last PDE equation is the Black-Scholes PDE.
- We can show that:

$$\phi_t = \frac{\partial V}{\partial s} = \Phi(d_1) \quad (9)$$

- The martingale approach has provided an alternative derivation of the B.-S. PDE and Eq. (9) gives explicit formula for  $\phi_t$  of the replicating portfolio (is equal to the Delta  $\Delta$ ).

## Advantages of the martingale approach

- The martingale approach is much more clear in the process of pricing derivatives, comparing to the PDE approach.
- Under the PDE approach we derived a PDE and had to “guess” the solution for a given set of boundary conditions.
- Under the martingale approach we have an expectation which can be evaluated explicitly in some cases and in a straightforward numerical way in other cases.
- The martingale approach also gives us the replicating strategy for the derivative.
- The martingale approach can be applied to any  $\mathcal{F}_T$ -measurable derivative payment, including path-dependent options (for example, Asian options), whereas the PDE approach, in general, cannot.

# Risk neutral pricing

- Exercise: You are trying to replicate a 6-month European call option with strike price 500, which you purchased 4 months ago. If  $r = 0.05$ ,  $\sigma = 0.2$ , and the current share price is 475, what portfolio should you be holding (assuming no dividends) ?
- The martingale approach is also known as risk-neutral pricing. The measure  $Q$  is commonly called the risk-neutral measure. However,  $Q$  is also referred to as the equivalent martingale measure because the discounted prices  $D_t$  and  $E_t$  are martingales under  $Q$ .

## State price deflator approach

- Recall that:

$$dS_t = S_t (\mu dt + \sigma dZ_t), \text{ under } P, \quad (10)$$

$$dS_t = S_t \left( r dt + \sigma d\tilde{Z}_t \right), \text{ under } Q, \quad (11)$$

where

$$d\tilde{Z}_t = dZ_t + \gamma dt \quad (12)$$

and

$$\gamma = \frac{\mu - r}{\sigma}. \quad (13)$$

- A corollary to the Cameron-Martin-Girsanov theorem states that there exists a process  $\eta_t$  such that for a payoff  $X$  we have:

$$E_Q [X | \mathcal{F}_t] = E_P \left[ \frac{\eta_T}{\eta_t} X | \mathcal{F}_t \right],$$

where (in this case):

$$\eta_t = e^{-\gamma Z_t - \frac{1}{2} \gamma^2 t} \quad (14)$$

## State price deflator approach

- Define

$$A_t = e^{-rt} \eta_t. \quad (15)$$

- The price of the derivative is:

$$\begin{aligned} V_t &= e^{-r(T-t)} E_Q [X | \mathcal{F}_t] = e^{-r(T-t)} E_P \left[ \frac{\eta_T}{\eta_t} X | \mathcal{F}_t \right] \\ &= \frac{E_P [A_T X | \mathcal{F}_t]}{A_t}. \end{aligned} \quad (16)$$

- $A_t$  is called a state-price deflator (also deflator; state-price density; pricing kernel; or stochastic discount factor).

## The B-S model with dividends

- Suppose that dividends are payable continuously at the constant rate of  $q$  p.a.: that is, the dividend payable over the interval  $[t, t + dt]$  is  $qS_t dt$ .
- Suppose that  $S_t$  is subject to the same SDE:

$$dS_t = S_t (\mu dt + \sigma dZ_t), \text{ under } P.$$

- Let  $\tilde{S}_t$  be the value of an investment of  $\tilde{S}_0 = S_0$  at time 0 in the underlying asset assuming that all dividends are reinvested in the same asset at the time of payment of the dividend.
- $\frac{\tilde{S}_t}{\tilde{S}_0}$  described as the total return on the asset from time 0 to time  $t$ .

## The B-S model with dividends

- $\tilde{S}_t$  is the tradable asset and not  $S_t$  in the following sense: If we pay  $S_0$  at time 0 for the asset then we are buying the right to future dividends as well as future growth of the capital.
- It is straightforward to see that the SDE for  $\tilde{S}_t$  is:

$$d\tilde{S}_t = \tilde{S}_t [(\mu + q) dt + \sigma dZ_t], \text{ under } P. \quad (17)$$

Solving the SDE we have (geometric Bm):

$$\tilde{S}_t = \tilde{S}_0 \exp \left[ \left( \mu + q - \frac{1}{2} \sigma^2 \right) t + \sigma Z_t \right]. \quad (18)$$

## The B-S model with dividends

- Denote the value at time  $t$  of an European call option on the dividend paying share by  $f(t, S_t)$ . Then we have:

**Proposition:** (Garman-Kohlhagen formula):

$$f(t, S_t) = S_t e^{-q(T-t)} \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (19)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad (20)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \quad (21)$$

For a put option, we have

$$f(t, S_t) = K e^{-r(T-t)} \Phi(-d_2) - S_t e^{-q(T-t)} \Phi(-d_1). \quad (22)$$

## The B-S model with dividends

- These formulas can be derived by the PDE approach and by the martingale approach (as in the non-dividend-paying case) - see core reading (homework).
- The B.-S. PDE is (substituting  $S_t = s$ )

$$\frac{\partial f}{\partial t} + (r - q) s \frac{\partial f}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} = rf(t, s). \quad (23)$$

- In the martingale approach, the basic SDE for  $\tilde{S}_t$  under  $Q$  is (see core reading)

$$d\tilde{S}_t = \tilde{S}_t \left[ rdt + \sigma d\tilde{Z}_t \right] \quad \text{under } Q. \quad (24)$$

## The B-S model with dividends

- In the martingale approach, we have (see the 5 step method for this case in the core reading)

$$V_t = e^{-r(T-t)} E_Q [X | \mathcal{F}_t].$$

- By the MRT there exists a previsible process  $\tilde{\phi}_t$  such that:

$$dE_t = \tilde{\phi}_t d\tilde{D}_t,$$

and  $\psi_t = E_t - \tilde{\phi}_t \tilde{D}_t$ .

- $\tilde{\phi}_t$  units of  $\tilde{S}_t$  is equivalent to  $\phi_t = e^{qt} \tilde{\phi}_t$  units of  $S_t$ . At time  $t$ , the replicating portfolio is  $(\tilde{\phi}_t, \psi_t)$  or  $(\phi_t, \psi_t)$  if we consider  $S_t$  instead of  $\tilde{S}_t$ .
- See more detail on the dividend-paying case of the Black-Scholes model in the Core Reading (homework).