Master in Actuarial Science

Models in Finance

06-01-2020

Time allowed: Two hours (120 minutes)

Solutions

1. .

(a) By Itô's lemma (or Itô's formula) applied to $g(t,x)$ (it is a $C^{1,2}$ function):

$$\begin{aligned} dg(t,S_t) &= \frac{\partial g}{\partial t}(t,S_t)dt + \frac{\partial g}{\partial x}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,S_t)\,(dS_t)^2 \\ &= \left[\frac{\partial g}{\partial t}(t,S_t) + \mu S_t \frac{\partial g}{\partial x}(t,S_t) + \frac{1}{2}\,(h(t,S_t))^2\,\frac{\partial^2 g}{\partial x^2}(t,S_t)\right]dt \\ &+ h(t,S_t)\frac{\partial g}{\partial x}(t,S_t)dB_t \\ &= 0 + h(t,S_t)\frac{\partial g}{\partial x}(t,S_t)dB_t, \end{aligned}$$

where we have used $(dB_t)^2 = dt$. Therefore,

$$Y_t = Y_0 + \int_0^t h(u,S_u)\frac{\partial g}{\partial x}(u,S_u)dB_u,$$

and since h and $\frac{\partial g}{\partial x}$ are continuous and bounded, the process $h(u,S_u)\frac{\partial g}{\partial x}(u,S_u)$ is adapted and the integral of the expected value of the squared process is finite. Hence, the process belongs to the space $L^2_{a,T}$ and therefore Y_t is a martingale (it is a well defined stochastic integral).

(b) We have

$$dS_t = 0.1 S_t dt + 0.25 S_t dB_t,$$

which is the SDE of a geometric Brownian motion with $\mu = 0.1$ and $\sigma = 0.25$. The solution is (it can be obtained by applying the Itô formula to $f(x) = \log(x)$)

$$\begin{aligned} S_t &= S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right] \\ &= S_0 \exp\left[\left(0.1 - \frac{1}{2}\,(0.25)^2\right)t + 0.25 B_t\right] \end{aligned}$$

Therefore

$$S_t = S_0 \exp\left[0.06875t + 0.25B_t\right].$$

Since $B_t \sim N(0;t)$, then $\log(S_t) \sim N(\log(S_0) + 0.06875t; 0.0625t)$.

$$\begin{aligned} P\left(\frac{S_2}{S_0} \leq 1.15\right) &= P\left(\exp\left[0.06875 \times 2 + 0.25B_2\right] \leq 1.15\right) \\ &= P\left(Z \leq \ln(1.15)\right),. \end{aligned}$$

where $Z = 0.1375 + 0.25B_2 \sim N(0.1375; 0,125)$.
Therefore: $P\left(\frac{S_2}{S_0} \leq 1.15\right) = 0.5026$.

2. .

(a) $1+i$ has lognormal distribution with parameters (μ, σ^2). We also know that $E[1+i] = 1.05$ and $Var[1+i] = 0.004$. Therefore

$$1.05 = exp(\mu + \sigma^2/2)$$

and

$$0.004 = exp\left(2\mu + \sigma^2\right)\left(exp\left(\sigma^2\right) - 1\right).$$

From these equations, we get

$$2\mu + \sigma^2 = 2ln(1.05)$$

and

$$\sigma^2 = ln\left(1 + \frac{0.004}{(1.05)^2}\right).$$

Then $\sigma^2 = 0.003622$ and $\mu = 0.04698$.

(b) We know that in this case we have $ln(S_{10})$ has a normal distribution with mean $10 \times \mu = 0.4698$ and variance $10 \times \sigma^2 = 0.03622$. Therefore, the $P[S_{10} > 2] = P[ln(S_{10}) > ln(2)]$ and this can be calculated as

$$1 - P\left[Z \leq \frac{ln(2) - 0.4698}{\sqrt{0.03622}}\right] = 1 - P[Z \leq 1.17356] = 0.12.$$

The probability that $ln(1+i) < ln(1.04)$ can be calculated using the normal distribution of $ln(1+i)$ and therefore

$$P[ln(1+i) < ln(1.04)] = P\left[Z < \frac{ln(1.04) - 0.04698}{\sqrt{0.003622}}\right] = 0.4487$$

Since the rates of return are independent, the probability that $1 + i < 1.04$ in all the 10 years is simply

$$(0.4487)^{10} = 0.00033.$$

3. .

(a) Let us consider two portfolios. Portfolio A: one European call option + cash $D_1e^{-r(T_1-t)} + D_2e^{-r(T_2-t)} + Ke^{-r(T-t)}$

Portfolio B: one European put option + one dividend paying share.

At time T, the value of portfolio A is $S_T - K + D_1e^{r(T-T_1)} + D_2e^{r(T-T_2)} + K = S_T + D_1e^{r(T-T_1)} + D_2e^{r(T-T_2)}$ if $S_T > K$ and $D_1e^{r(T-T_1)} + D_2e^{r(T-T_2)} + K$ if $S_T \leq K$.

At time T, the value of portfolio B is $0 + S_T + D_1e^{r(T-T_1)} + D_2e^{r(T-T_2)}$ if $S_T > K$ and $K - S_T + S_T + D_1e^{r(T-T_1)} + D_2e^{r(T-T_2)} = D_1e^{r(T-T_1)} + D_2e^{r(T-T_2)} + K$ if $S_T \leq K$.

Therefore, the portfolios have the same value at maturity. Then, by the no-arbitrage principle, the porfolios have the same value for any time $t < T$, i.e.,

$$c_t + D_1e^{-r(T_1-t)} + D_2e^{-r(T_2-t)} + Ke^{-r(T-t)} = p_t + S_t.$$

(b) For the price of the put option we use the Black-Scholes formula with dividend yield

$$f(t, S_t) = Ke^{-r(T-t)}\Phi(-d_2) - S_te^{-q(T-t)}\Phi(-d_1). \qquad (1)$$

and use the data given in the problem with $q = 0.2$,$r = 0.05$,$T - t = 1.5$, $\sigma = 0.2$ and $S_t = 18$, $K = 20$,$d_1 = -0.124$ and $d_2 = -0.369$. Using the formula and these values, we obtain

$$price = 2.352.$$

4. .If $r = 5\%$, then the risk-neutral probability for an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 0.8928}{1.12 - 0.8928} = 0.6975.$$

Binomial tree values: 10; 11.2,8.928; 12.544, 10, 7.9709; 14.0493,11.2, 8.928, 7.116436

Payoff function of the derivative (call + put):

$$Payoff = \begin{cases} 8.5 - S_T & \text{if } S_T < 8.5 \\ 0 & \text{if } 8.5 \leq S_T \leq 12 \\ S_T - 12 & if S_T > 12 \end{cases}.$$

Payoff of the derivative: $C_3(u^3) = 14.0493 - 12 = 2.0493$, $C_3(u^2d) = 0$, $C_3(ud^2) = 0$, $C_3(d^3) = 8.5 - 7.116436 = 1.383564$

Using the usual backward procedure with $r = 0.05$ and $q = 0.6975$:

At time 2: $C_2\left(u^2\right)=\exp(-r)\left[qC_3\left(u^3\right)+(1-q)C_3\left(u^2d\right)\right]=1.3597$,
$C_2\left(ud\right)=\exp(-r)\left[qC_3\left(udu\right)+(1-q)C_3\left(ud^2\right)\right]=0$,
$C_2\left(d^2\right)=\exp(-r)\left[qC_3\left(d^2u\right)+(1-q)C_3\left(d^3\right)\right]=0.3981$.
At time 1: $C_1(u)=\exp(-r)\left[qC_2\left(u^2\right)+(1-q)C_2\left(ud\right)\right]=0.9021$, $C_1(d)=\exp(-r)\left[qC_2\left(du\right)+(1-q)C_2\left(d^2\right)\right]=0.1146$.
The Final price (at time 0) is$C_0=\exp(-r)\left[qC_1\left(u\right)+(1-q)C_1\left(d\right)\right]=0.6315$.

5. .

(a)

In order to have a porfolio with zero delta, $\Delta_p\times N+\Delta_S\times$number of shares= 0. Since $\Delta_p=-0.25$ and $\Delta_S=1$, we have

$$N=\frac{50000}{0.25}=200000.$$

(b) We have $\Delta_X=0.3$, $\Delta_Y=0.4$, $\Gamma_X=0.15$, $\Gamma_Y=0.25$. Let N_X be the number of derivatives X and N_Y be the number of derivatives Y in the portfolio. In order to have a zero delta and a zero gamma portfolio:

$$\begin{cases} 0.3N_X+0.4N_Y=0 \\ 200000\times 0.1+0.15N_X+0.25N_Y=0 \end{cases}$$

It is easy to solve this linear system os 2 equations. The solution is

$$\begin{cases} N_X=533333, \\ N_Y=-400000. \end{cases}$$

6. .

(a) The Vasicek model has the dynamics, under the risk-neutral measure Q:

$$dr(t)=\alpha(\mu-r(t))dt+\sigma d\widetilde{W}(t)$$

where $\widetilde{W}$ is a standard Brownian motion under Q.
The Cox-Ingersoll-Ross (CIR) model has the dynamics under Q:

$$dr(t)=\alpha(\mu-r(t))dt+\sigma\sqrt{r(t)}d\widetilde{W}(t).$$

Both models are one-factor models and are time homogeneous with three parameters. The critical difference between the two models occurs in the volatility, which is increasing in line with the square-root of $r(t)$ for the CIR model and it is constant for the Vasicek model. In the CIR model, since $\sqrt{r(t)}$ diminishes to zero as $r(t)$ approaches zero, and provided σ^2 is not too large ($r(t)$ will never hit zero provided $\sigma^2 \leq 2\alpha\mu$), we can guarantee that $r(t)$ will not hit zero. Consequently all other interest rates will also remain strictly positive. On the other hand, in the Vasicek model, there is some probability that the interest rates can be negative (and in some cases, very negative), since the solution of the Vasicek model has a normal distribution.

(b) Solve the SDE for the Vasicek model and deduce the form of the distribution of the zero-coupon bond price for this model

$$dr_t = \alpha\left(\mu - r_t\right)dt + \sigma d\widetilde{W}_t$$

$\alpha, \sigma > 0$ and $\mu \in \mathbb{R}$.
Solution of the associated ODE $dx_t = -\alpha x_t dt$ is $x_t = xe^{-\alpha t}$.
Consider the variable change $r_t = Y_t e^{-\alpha t}$ or $Y_t = r_t e^{\alpha t}$.
By the Itô formula applied to $f\left(t, x\right) = xe^{\alpha t}$ (wich is a $C^{1,2}$ function), we obtain

$$dY_t = \alpha r_r e^{\alpha t} dt + e^{\alpha t} dr_t + \frac{1}{2} \times 0.$$

Replacing the equation of dr_t and integrating, we have

$$Y_t = x + \mu\left(e^{\alpha t} - 1\right) + \sigma \int_0^t e^{\alpha s} dB_s.$$

Therefore

$$r_t = \mu + \left(x - \mu\right)e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dB_s,$$

Since $e^{\alpha s}$ is a deterministic function (square integrable deterministic function), then $\int_0^t e^{\alpha s} dB_s$ has Gaussian distribution and all the other factors are deterministic. Therefore, r_t has a normal distribution and calculating the expected value and the variance (using the Itô isometry for the variance and the property of zero expected value for the stochastic integral), we obtain the following distribution for r_t :

$$N\left[\mu + \left(x - \mu\right)e^{-\alpha t}, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right].$$

We obtain the invariant stationary distribution, by calculating the limit when $t \to \infty$, which is

$$N\left(\mu, \frac{\sigma^2}{2\alpha}\right).$$