LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Regular Period Exam - January 10, 2020

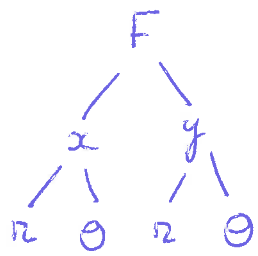
Duration: 1h15

Name:	**Number:**

1. [2,0 points] Consider a two-variable function F such that $\frac{\partial F}{\partial x}(1,1) = \frac{\partial^2 F}{\partial y^2}(1,1) = 1$ and $\frac{\partial F}{\partial y}(1,1) = \frac{\partial^2 F}{\partial x^2}(1,1) = 0$.

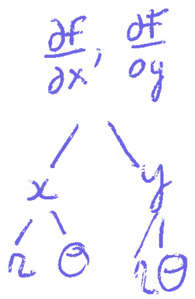
Setting $g(r,\theta) = F(r\cos(\theta), r\sin(\theta))$, compute $\frac{\partial^2 g}{\partial\theta\partial r}\left(\sqrt{2}, \frac{\pi}{4}\right)$.

Putting $x = r\cos\theta$ and $y = r\sin\theta$,



$$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial F}{\partial x}(x,y)\cos\theta + \frac{\partial F}{\partial y}\sin\theta$$



$$\frac{\partial^2 g}{\partial\theta\partial r} = -\sin\theta\frac{\partial F}{\partial x}(x,y) + \cos\theta\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 F}{\partial y\partial x}\frac{\partial y}{\partial\theta}\right)$$

$$+\cos\theta\frac{\partial F}{\partial y} + \sin\theta\left(\frac{\partial^2 F}{\partial y\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial^2 F}{\partial y^2}\frac{\partial y}{\partial\theta}\right)$$

$$= -\sin\theta\frac{\partial F}{\partial x} + \cos\theta\frac{\partial F}{\partial y} + \cos\theta\left(\frac{\partial^2 F}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 F}{\partial y\partial x}r\cos\theta\right) + \sin\theta\left(-\frac{\partial^2 F}{\partial x\partial y}r\sin\theta + \frac{\partial^2 F}{\partial y^2}r\cos\theta\right)$$

$$= -\sin\theta\frac{\partial F}{\partial x} + \cos\theta\frac{\partial F}{\partial y} + r\cos\theta\sin\theta\left(\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2}\right) + \cdots$$

$$\frac{\partial^2 g}{\partial\theta\partial r}\left(\sqrt{2}, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\frac{\partial f}{\partial x}(1,1) + \frac{\sqrt{2}}{2}\frac{\partial F}{\partial y}(1,1) + \sqrt{2}\left(\frac{\sqrt{2}}{2}\right)^2\left[\frac{\partial^2 F}{\partial y^2}(1,1) - \frac{\partial^2 F}{\partial x^2}(1,1)\right] + \cdots$$

$$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$$

2. **[2,0 points]** Show that the function defined in $\mathbb{R}^2$ by $f(x,y) = e^{-(x^2+y^2+2x)}$ has one single critical point and classify it.

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0 \iff \begin{cases} (-2x-2)\, e^{-(x^2+y^2+2x)} = 0 \\ -2y\, e^{-(x^2+y^2+2x)} = 0 \end{cases} \iff \begin{cases} x = -1 \\ y = 0 \end{cases}$$

One single critical point: $(-1,0)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -2e^{-(x^2+y^2+2x)} + (-2x-2)^2 e^{-(x^2+y^2+2x)} \quad : \quad \frac{\partial^2 f}{\partial x^2}(-1,0) = -2e$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -2e^{-(x^2+y^2+2x)} + 4y^2 e^{-(x^2+y^2+2x)} \quad : \quad \frac{\partial^2 f}{\partial y^2}(-1,0) = -2e$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -2y(-2x-2)\, e^{-(x^2+y^2+2x)} \quad : \quad \frac{\partial^2 f}{\partial x \partial y}(-1,0) = 0$$

$$H = \operatorname{Hess}_{(-1,0)} f = \begin{bmatrix} -2e & 0 \\ 0 & -2e \end{bmatrix}$$

Two negative eigenvalues. H is Negative Defined

$$\boxed{(-1,0) \text{ is a local maximum}}$$

3. [2,0 points] Consider the function $f(x,y,z) = \sin(x)\sin(y)\sin(z)$, $x,y,z \geq 0$. Show that if $x+y+z = \frac{\pi}{2}$, then

$$f(x,y,z) \leq \frac{1}{8}.$$

Hint: Apply the Lagrange multipliers method to f in a convenient set.

Note that $x,y,z \geq 0$, hence $x,y,z \in [0, \frac{\pi}{2}]$

We look for the maximum of f in the set $\Pi = \{x,y,z : \underbrace{x+y+z = \frac{\pi}{2}}_{g(x,y,z)}\}$

$$\begin{cases} \nabla f = \lambda \nabla g \\ x+y+z = \pi/2 \end{cases} \Leftrightarrow \begin{cases} \cos(x)\sin(y)\sin(z) = \lambda \\ \sin(x)\cos(y)\sin(z) = \lambda \\ \sin(x)\sin(y)\cos(z) = \lambda \\ x+y+z = \pi/2 \end{cases} \Leftrightarrow \begin{cases} \cos(x)\sin(y)\sin(z) = \sin(x)\cos(y)\sin(z) \\ \sin(x)\cos(y)\sin(z) = \lambda \\ \sin(x)\sin(y)\cos(z) = \lambda \\ x+y+z = \frac{\pi}{2} \end{cases}$$

1° case $\sin z = 0 \Leftrightarrow z = 0$

$$\begin{cases} 0 = 0 \\ \lambda = 0 \\ \sin(x)\sin(y)\underbrace{\cos(z)}_{\neq 0} = 0 \\ x+y+z = \pi/2 \end{cases} \Leftrightarrow \begin{cases} \left(\begin{matrix} x = 0 \\ \vee \\ y = 0 \end{matrix} \right) \\ x+y+z = \pi/2 \end{cases}$$

Points $A(0, \frac{\pi}{2}, 0)$ $B(\frac{\pi}{2}, 0, 0)$

2° case: $\sin z \neq 0$

$$\begin{cases} \cos x \sin y = \sin x \cos y \\ \sin x \sin y \cos z = \sin x \cos y \sin z \\ x+y+z = \pi/2 \end{cases}$$

First subcase: $\sin x = 0$

Then $\sin y = 0$. Point $C(0, 0, \frac{\pi}{2})$

Second sub case: $\sin x \neq 0$

$$\begin{cases} \cos x \sin y = \sin x \cos y \\ \sin y \cos z = \cos y \sin z \\ x+y+z = \pi/2 \end{cases} \quad \begin{cases} \tan x = \tan y \\ \tan y = \tan z \\ x+y+z = \pi/2 \end{cases} \quad \begin{cases} x = y = z \\ x+y+z = \pi/2 \end{cases}$$

Point $D(\pi/6, \pi/6, \pi/6)$

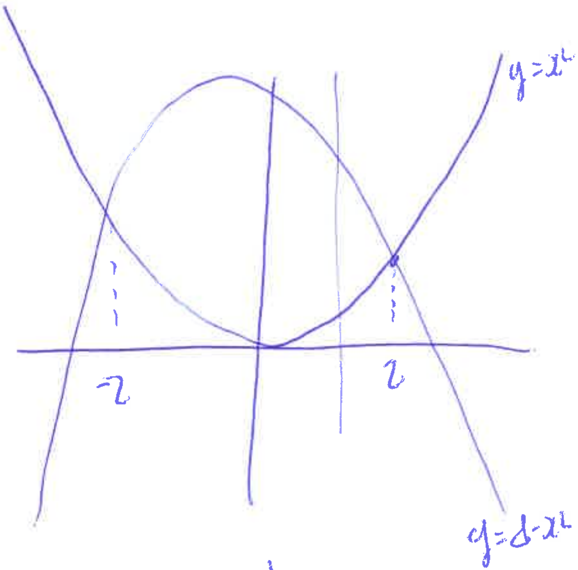
Minimum: $f(A) = f(B) = f(C) = 0$

Maximum $f(\frac{\pi}{6}, \pi/6, \pi/6) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

4. [2,0 points] Compute the double integral

$$\iint_{\mathcal{R}} xe^y dxdy$$

where $\mathcal{R}$ is the region bounded by the lines of equation $y = x^2$ and $y = 8 - x^2$.



$x^2 = 8 - x^2 \iff x = 2 \vee x = -2$

$$I = \int_{-2}^{2} \int_{x^2}^{8-x^2} xe^y \, dy \, dx = \int_{-2}^{2} \left[xe^y \right]_{x^2}^{8-x^2} dx$$

$$= \int_{-2}^{2} e^{8-x^2} . x - \int_{-2}^{2} xe^{x^2} dx$$

$$= -2e^{8} \left[e^{8-x^2} \right]_{-2}^{2} - \frac{1}{2} \left[e^{x^2} \right]_{-2}^{2} = 0.$$

5. [2,0 points] Find a function $y : x \in \mathbb{R}^+ \to y(x) \in \mathbb{R}$ such that

$$\begin{cases} \dfrac{e^y}{\sqrt{x^2+1}} y' - 2x = 0 \\ y(0) = 0. \end{cases}$$

This is a separable equation:

$$\frac{e^y}{\sqrt{x^2+1}} \frac{dy}{dx} = 2x \iff e^y dy = 2x(x^2+1)^{1/2} dx$$

Integrating:

$$\int e^y dy = \int 2x(x^2+1)^{1/2} dx \iff e^y = \frac{2}{3}(x^2+1)^{3/2} + C$$

$$y \qquad y = \ln\left(\frac{2}{3}(x^2+1)^{3/2} + C\right)$$

$$y(0) = 0 \iff \frac{2}{3} + C = 1 \; : \; C = \frac{1}{3}$$

$$\boxed{y = \ln\left(\frac{2}{3}(x^2+1)^{3/2} + \frac{1}{3}\right)}$$