Probability Theory and Stochastic Processes

Solutions

Jan 15, 2016

1. b) 0

2. 2

3. e^{-1}

4.

$$E(X|Y)(\omega) = \begin{cases} \omega + \frac{1}{4}, & \omega < \frac{1}{2} \\ \omega - \frac{1}{4}, & \omega \geq \frac{1}{2} \end{cases}$$

5. a) Recurrent non-null, period 1

b) Unique stationary distribution $(\frac{1}{a}, \ldots, \frac{1}{a})$, mean recurrence time a for all states.

6. Yes

Feb 1, 2016

1. a) 0

b) $\{\emptyset, \Omega, X^{-1}(\{a\}), X^{-1}(\{b\})\}$

c) We don't know

2. 0

3. a) states 1, 2: transient period=2; states 3,4: recurrent positive period=2

b) $(0, 0, 1/2, 1/2)$, $(+\infty, +\infty, 2, 2)$

4. a) not a martingale

b) $-\infty$

Jan 18, 2017

1. a)

$$F(x) = \begin{cases} 1, & x \geq \sqrt{2} \\ 0, & x < \sqrt{2} \end{cases}$$

$\phi(t) = e^{it\sqrt{2}}$. The distribution is the Dirac measure on $\mathbb{R}$ at $\sqrt{2}$.

b) Any that is equal to X a.e. Ex: $Y(x) = \sqrt{2}$.

2. Dirac distribution at 0.

3.

a) 1,2,3 transient; 4 positive recurrent

b) 1

c) $\pi = (0, 0, 0, 1)$, $\mu = (+\infty, +\infty, +\infty, 1)$

4.

a) not a martingale

b) $-\infty$

Feb 3, 2017

1. a)

$$F(x) = \begin{cases} 1, & x \geq 2 \\ x/2, & 0 \leq x < 2 \\ 0, & x < 0 \end{cases}$$

$\phi(t) = (e^{2it}-1)/(2it)$, $t \neq 0$, $\phi(0) = 1$. The distribution is the Lebesgue measure on $[0,2]$.

b) Any that is equal to X a.e.

2. 1/2

3. a) 1 transient, 2,3,4,5 positive recurrent

b) Per(1)=1, Per(2)=Per(3)=Per(4)=Per(5)=2

c) (0,1/6,1/6,1/3,1/3), $(+\infty, 6, 6, 3, 3)$

4. a) Yes

b) 1, 4/7, 4/7

Jan 17, 2018

1. a)

$$F(x) = \begin{cases} 1, & x \geq 0 \\ x+1, & -1 \leq x < 0 \\ 0, & x < -1 \end{cases}$$

$\phi(t) = (1-e^{-it})/(it)$, $t \neq 0$, $\phi(0) = 1$. The distribution is the Lebesgue measure on $[-1,0]$.

b) Any that is equal to X a.e.

2. b) 3/4

3. a) 1 positive recurrent, 2,3,4 transient. Per(1)=1=Per(4), there are no periods for 2 and 3.

b) (1,0,0,0), $(1, +\infty, +\infty, +\infty)$

c) 1

4. $E(X_1)$

Feb 2, 2018

1. a) No. E.g. $\Omega \notin \mathcal{A}$.

b) $\sigma(\mathcal{A}) = \{A \subset \Omega \colon A \text{ is countable or } A^c \text{ is countable}\}$

3. a) 2,3 transient, 1,4 positive recurrent, Per(1)=Per(2)=Per(3)=Per(4)=1

b) Stationary distributions: $(a, 0, 0, 1-a)$ for any $0 \leq a \leq 1$; mean recurrence times: $(1, +\infty, +\infty, 1)$.

c) 1

4. $E(X_1)$

Jan 21, 2019

1. a) True. Let $A_n = \{f - g \geq 1/n\} \in \mathcal{F}$ verifying $A_n \subset A_{n+1}$ and the inequality $\mu(A_n) \leq n \int_{A_n} (f-g)\, d\mu = 0$. Then, $\mu(\{f - g > 0\}) = \mu(\cup_n A_n) = \lim \mu(A_n) = 0$. Same idea for $\mu(\{f - g < 0\}) = 0$, so that $f = g$ a.e.

b) False. E.g. μ probability measure, $\mathcal{A} = \{\emptyset, \Omega\}$ and $\mathcal{F} = \sigma(\{C\})$ with $C \notin \mathcal{A}$ and $\mu(C) = 1/2$. For $f = 2\mathcal{X}_C - 1$ we have $\int_A f\, d\mu = 0$, $A \in \mathcal{A}$, but $f \neq 0$ a.e.

2. a) 1/4

b)

$$P(Y \leq y) = \begin{cases} 0, & x < 0 \\ y^2/2, & 0 \leq y < 1 \\ 1/2, & 1 \leq y < \sqrt{2} \\ 1, & x \geq \sqrt{2} \end{cases}$$

3. a) $\phi_{S_n}(t) = (pe^{-it} + (1-p)e^{it})^n$, $\phi_{S_n/n}(t) = (pe^{-it/n} + (1-p)e^{it/n})^n$. By the weak law of large numbers, the limit dist is δ_{1-2p}.

b) Martingale iff $p = 1/2$.

c) $E(\tau) = +\infty$.

d) $p^2(1-p)^2$.

4. $T = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$

Feb 6, 2019

2. a) 5/12, 11/4, 31/16

b)

$$P(Y \leq y) = \begin{cases} 0, & x < 0 \\ y^2/6, & 0 \leq y < \sqrt{3} \\ 1/2, & \sqrt{3} \leq y < 2 \\ 1, & x \geq 2 \end{cases}$$

c) $3\sqrt{3}/5 + 4$, $27/8 + 32 - (3\sqrt{3}/5 + 4)^2$

3. 8/3

4. a) No

b) 0, $10/(1-p)$

Jan 9, 2020

1.

b) 0

2.

a) $E(Y_n) = n(\mu - 1)$, $E(2^{Y_n}) = e^{n(\mu - \log 2)}$, $P(Y_2 = 1|X_1 = 0) = \mu^3/3!\, e^{-\mu}$

b) Y_n is a martingale iff $\mu = 1$. 2^{Y_n} is a martingale iff $\mu = \log 2$

c) Notice that $P(\tau = +\infty) = P(\cap_n\{\tau > n\}) = \lim_{n\to+\infty} P(\tau > n)$. Moreover,

$$\begin{aligned}
P(\tau > n) &\leq P(-1 < Y_i < 2, 0 \leq i \leq n) \\
&\leq P(X_1 \in \{0,1,2\}, \ldots, X_n \in \{0,1,2\}) \\
&= \prod_{i=1}^{n} P(X_i \in \{0,1,2\}) \\
&= [P(X_1 = 0) + P(X_1 = 1) + P(X_1 = 2)]^n = \left(\frac{5}{2e}\right)^n \to 0.
\end{aligned}$$

So $P(\tau < +\infty) = 1 - P(\tau = +\infty) = 1$. Since Y_n is a martingale and $|Y_{\tau\wedge n}| \leq 2$ (dominated), by the optional stopping theorem $E(Y_\tau) = E(Y_1) = E(X_1) - 1 = 0$.

3.

a) Aperiodic iff $a \neq 1$ or $b \neq 1$. There is an absorving state iff $a = 0$ or $b = 0$.

b) There are stationary distributions for any a, b. There is a unique stationary distribution iff $a \neq 0$ or $b \neq 0$.

4. All states are connected, i.e. between any two states there is a path connecting them. Ignore all loops. All states are still connected. There is a path between any two states. This path has N distinct states and distinct arrows. There are at most $N - 1$ arrows.

Feb 4, 2020

1. $f(x) = 0$

2. $\lim X_n = \mathcal{X}_{[0,1]\setminus\mathbb{Q}}$, $\lim E(X_n) = E(\lim X_n) = 1$

4.

a) $R = 2, 3$, $T = 1, 4$, all aperiodic

b) unique stationary distribution $(0, 1/2, 1/2, 0)$, mean recurrence times $(+\infty, 2, 2, +\infty)$

c) 0