SOLUTIONS

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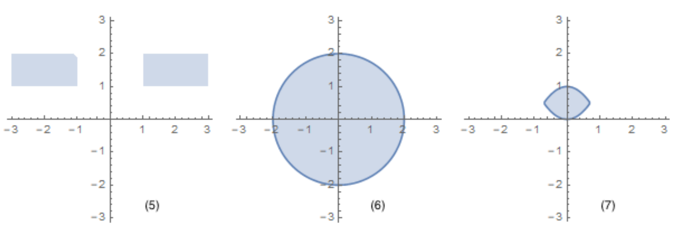
Exercise 1.

(1) if $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. This means that $x \in A$ or x belongs to both B and C. So, $x \in A \cup B$ and $x \in A \cup C$.
(2) if $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. This means that $x \in A$ and x belongs to B or C. So, $x \in A \cap B$ or $x \in A \cap C$.
(3) if $x \in A \setminus (B \cup C)$, then $x \in A$ but $x \notin B \cup C$. This means that $x \in A$ but x does not belong to B or C. So, $x \in A \setminus B$ and $x \in A \setminus C$.

Exercise 2.

(1) (2)

(3) (4)



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Exercise 3.

(1) Bijective. Inverse is $f^{-1}(x) = (x-1)^{1/3}$. The graph is a cubic with a zero at $x = -1$.
(2) Not injective and not surjective. The graph is a parabola which is convex (positive 2nd derivative) and zeros at $x = 0$ and $x = 1$.
(3) Injective, not surjective. Its inverse is $f^{-1}(x) = x^2 - 1$ defined for $x > 1$.
(4) Injective, not surjective. Its inverse is $f^{-1}(x) = \frac{1+x}{1-x}$ defined for $x < 1$. The graph is a hyperbola with a vertical asymptote at $x = -1$, horizontal asymptote at $y = 1$ and zero at $x = 1$.
(5) Injective, not surjective. Its inverse $f^{-1}(x) = -\log(x/2)$ defined for $x > 0$.
(6) Injective, not surjective. Its inverse $f^{-1}(x) = \sqrt{e^x - 1}$ defined for $x > 0$.

Exercise 4.

(1) $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} (-1)^n \frac{n+1}{2} & n \text{ is odd} \\ (-1)^n \frac{n-2}{2} & n \text{ is even} \end{cases}$$

(2) $f : \mathbb{R} \to \{x \in \mathbb{R} \colon -1 < x < 1\}$ defined by $f(x) = \frac{2}{\pi} \arctan(x)$

Exercise 5.

(1) **(a):**
- $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \geq 0$ is obvious.
- $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0$ iff $x_1 = y_1$ and $x_2 = y_2$, i.e., $x = y$.
- Obvious
- In any given triangle, the length of one side is always less of equal to the sum of the lengths of the other sides.

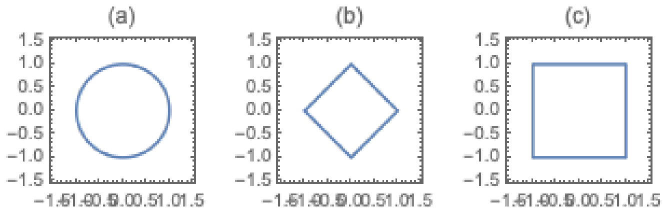
(b):
- $d(x, y) = |x_1 - y_1| + |x_2 - y_2| \geq 0$ is obvious.
- $|x_1 - y_1| + |x_2 - y_2| = 0$ iff $x_1 = y_1$ and $x_2 = y_2$, i.e., $x = y$.
- Obvious
-

$$\begin{aligned} d(x, z) &= |x_1 - z_1| + |x_2 - z_2| \\ &= |x_1 - y_1 - (z_1 - y_1)| + |x_2 - y_2 - (z_2 - y_2)| \\ &\leq |x_1 - y_1| + |z_1 - y_1| + |x_2 - y_2| + |z_2 - y_2| \\ &= d(x, y) + d(y, z) \end{aligned}$$

(c):
- $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} geq 0$ is obvious.
- $\max\{|x_1 - y_1|, |x_2 - y_2|\} = 0$ iff $x_1 = y_1$ and $x_2 = y_2$, i.e., $x = y$.
- Obvious

- $$\begin{aligned} d(x,z) &= \max\{|x_1 - z_1|, |x_2 - z_2|\} \\ &= \max\{|x_1 - y_1 - (z_1 - y_1)|, |x_2 - y_2 - (z_2 - y_2)|\} \\ &\leq \max\{|x_1 - y_1| + |z_1 - y_1|, |x_2 - y_2| + |z_2 - y_2|\} \\ &\leq \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|z_1 - y_1|, |z_2 - y_2|\} \\ &= d(x,y) + d(y,z) \end{aligned}$$

(2) The plots of $\{x \in \mathbb{R}^2 \colon d(x,0) = 1\}$ for each distance d:



Exercise 6. A point x belongs to $\bigcap_{n\in\mathbb{N}}]-1/n, 1/n[$ if and only if it belongs to all intervals $]-1/n, 1/n[$, $n \in \mathbb{N}$. Clearly, $0 \in]-1/n, 1/n[$ for every $n \in \mathbb{N}$. This shows that $\{0\} \subset \bigcap_{n\in\mathbb{N}}]-1/n, 1/n[$. To show the other direction of the inclusion, take $x \in \bigcap_{n\in\mathbb{N}}]-1/n, 1/n[$. Then, $|x| < 1/n$ for every $n \in \mathbb{N}$. This shows that $x = 0$.

Exercise 7.

(1) closed, unbounded
(2) compact
(3) compact
(4) open, unbounded
(5) open, unbounded
(6) open, unbounded
(7) closed, unbounded
(8) closed, unbounded
(9) bounded
(10) compact

Exercise 8.

(1) Any finite union of closed sets is closed. Moreover, any finite union of bounded sets is bounded. Because compact sets and closed and bounded, it follows from any finite union of compact sets is compact.
(2) If A is bounded then it is contained in an open ball B. This implies that $\overline{A} \subset \overline{B}$ Because $\overline{B}$ is a closed ball,$\overline{A}$ is bounded. Because it is closed by definition, it follows that $\overline{A}$ is compact. Now, if $\overline{A}$ is compact, then it is contained in an open ball B. But $A \subset \overline{A}$ which implies that A is bounded too.

Exercise 9. (1) $A_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$

(2) The proof is by induction. The set A_1 has 2^1 intervals. Suppose that A_n has 2^n intervals. Since A_{n+1} is obtained by extracting from each interval of A_n two subintervals, we conclude that A_{n+1} has $2 \times 2^n = 2^{n+1}$ intervals.

(3) A_n is a finite union of closed intervals, thus closed.

(4) It is clear that $C = \left(\bigcup_{n\geq 1} A_n^c\right)^c$. Since each A_n^c is open, the union will also be open. Thus, the complement of the union, which is C, is closed. As C is contained in $[0,1]$ we conclude that C is bounded. Thus C is compact.

Exercise 10. Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. Then

$$x - y = \lim_{n\to\infty} x_n - \lim_{n\to\infty} x_n = \lim_{n\to\infty} (x_n - x_n) = 0$$

This shows that $x = y$. So the limit of convergent sequences is unique.

Exercise 11. Given $x \in \mathbb{R}$, let $\{x_n\}_{n\in\mathbb{N}}$ be a converging sequence in $\mathbb{R}$ and x be its limit, i.e., $x = \lim_{n\to\infty} x_n$. By continuity of g at x,

$$g(x) = g(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} g(x_n).$$

Using the continuity of f at $g(x)$,

$$f(g(x)) = f(\lim_{n\to\infty} g(x_n)) = \lim_{n\to\infty} f(g(x_n)).$$

This shows that $f \circ g$ is continuous at x.

Exercise 12.

(1) $f(x) = x^2$ continuous in $D = [-1,1]$ because it is a polynomial function. Since D is a compact interval, by Weierstrass it has a maximum and minimum.

(2) $f(x) = x^3 - x^2 + 2 - 1$ continuous in $D = [-2,-1] \cup [1,2]$ because it is a polynomial function. Since D is compact (union of two compact intervals), by Weierstrass it has a maximum and minimum.

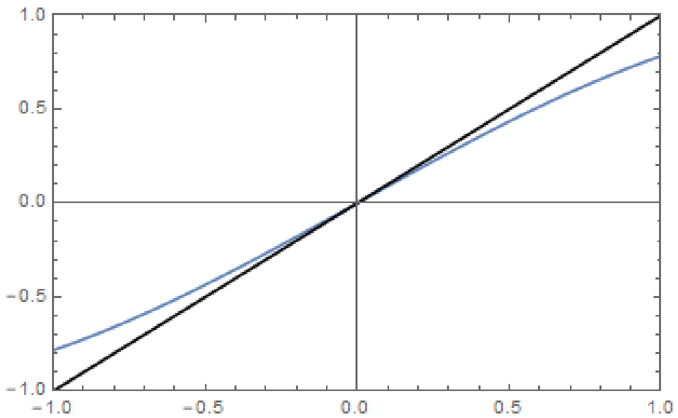
(3) $f(x) = x\cos^2(1/x)$ continuous in $D = \{(-1)^n/(2\pi n) \colon n \in \mathbb{N}\} \cup \{0\}$ because it is continuous in $\{(-1)^n/(2\pi n) \colon n \in \mathbb{N}\}$ (since both x and $\cos^2(1/x)$ are) and continuous at $x = 0$, i.e., given $x_n \to 0$ we have $f(0) = \lim_{n\to\infty} x_n \cos^2(1/x_n) = 0$. Since D is compact (close because it contains all its accumulation points), by Weierstrass it has a maximum and minimum.

(4) $f(x,y) = xy$ continuous in $D = [-1,1]^2$ because it is a polynomial function. Since D is a compact interval, by Weierstrass it has a maximum and minimum.

(5) $f(x,y) = x\log(y)$ is continuous in $D =]0,1]^2$ because $\log(y)$ is continuous for $y > 0$. The domain is not compact, therefore we cannot apply Weierstrass theorem. It does not have a maximum and minimum in D.

(6) $f(x, y) = e^{-x^2-y^2}$ is continuous in $D = \mathbb{R}^2$ because it is a composition of continuous functions. The domain is not compact, therefore we cannot apply Weierstrass theorem. However, it has a maximum in D at $(x, y) = (0, 0)$.

Exercise 13. The fixed point is the intersection of the graph of the function with the bisectrix $y = x$.



Exercise 14. Given $f :]0, 1/4[\to]0, 1/4,\ f(x) = x^2$ we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \le \frac{1}{2}|x - y|$$

because $x, y \in]0, 1/4[$. So f is a Lipschitz contraction. By the Banach fixed point theorem we cannot conclude that f has a fixed point because $]0, 1/4[$ is open. In fact, f has no fixed point in its domain of definition.

Exercise 15.

(1) $f'(x) = \frac{1}{4}(1 - 3x^2)$. Since $|1 - 3x^2|$ takes maximum value 2 in $[-1, 1]$ we have $|f'(x)| \le \frac{1}{2} < 1$. Thus f is Lipschitz with contraction $\lambda = 1/2$. Fixed points: $x = 0$.
(2) $f'(x) = \frac{1}{2}\frac{1}{1+x^2/4}$. Since $|\frac{1}{1+x^2/4}|$ takes maximum value 1 in $\mathbb{R}$ we get $|f'(x)| \le \frac{1}{2} < 1$. Thus f is Lipschitz with contraction $\lambda = 1/2$. Fixed points: $x = 0$.
(3) $f'(x) = \frac{3}{4}x^2 \cos x^3$. Since $|x^2 \cos x^3|$ takes maximum value 1 in $[-1, 1]$ we get $|f'(x)| \le \frac{3}{4} < 1$. Thus f is Lipschitz with contraction $\lambda = 3/4$. Fixed points: $x = 0$.
(4) Solved in Exercise 16.
(5) $\frac{\partial f_1}{\partial x} = 1/2$, $\frac{\partial f_1}{\partial y} = 0$, $\frac{\partial f_2}{\partial x} = 1/3$, $\frac{\partial f_2}{\partial y} = 0$. Thus $\max_{i,j}\left|\frac{\partial f_i}{\partial x_j}\right| = 1/2 < 1$. So f is Lipschitz with contraction $\lambda = 1/2$. Fixed point: $(x, y) = (10, -3/2)$.

Exercise 16. Let $f(x)=\sqrt{1+x}$ defined for $x\geq 0$. Given $y\geq x$ we have,

$$f(y)-f(x)=\int_x^y f'(u)\,du=\frac{1}{2}\int_x^y \frac{1}{\sqrt{1+u}}\,du\leq\frac{1}{2}(y-x)$$

Because $\frac{1}{\sqrt{1+u}}\leq 1$ for every $u\geq 0$. This shows that

$$|f(y)-f(x)|\leq\frac{1}{2}|y-x|$$

which means that f is a Lipschitz contraction. By the Banach fixed point theorem, it has a unique fixed point ρ in the closed interval $[0,+\infty[$. So, as in the proof of the theorem, the sequence $x_{n+1}=\sqrt{1+x_n}$ with $x_n=0$, converges to ρ. To compute ρ we notice that

$$\rho=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\sqrt{1+x_n}=\sqrt{1+\lim_{n\to\infty}x_n}=\sqrt{1+\rho}$$

Solving the equation $\rho=\sqrt{1+\rho}$ we find $\rho=\frac{1}{2}(1+\sqrt{5})$.

Exercise 17.

(1) convex
(2) not convex
(3) convex
(4) convex
(5) convex

Exercise 18. If $0<x<1$, then $0<\frac{1}{2}(x+1)<1$. This shows that $f(]0,1[)\subset]0,1[$. f has no fixed point in $]0,1[$. The Brouwer fixed point theorem does not apply because $]0,1[$ is not compact.

Exercise 19.

(1) Let $(x,y,z)\in\mathbb{R}^3_+$ such that $x+y+z=1$. Then

$$\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix}=\begin{pmatrix}0&1/2&1\\1&0&0\\0&1/2&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}y/2+z\\x\\y/2\end{pmatrix}$$

So

$$\bar{x}+\bar{y}+\bar{z}=y/2+z+x+y/2=x+y+z=1$$

This shows that $f(\Delta^2)\subset\Delta^2$.

(2) Because Δ^2 is compact and convex, $f:\Delta^2\to\Delta^2$ is continuous, then we can apply the Brouwer fixed point theorem and conclude that f has a fixed point in Δ^2.

(3) The fixed point satisfies the equations

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1/2&1\\1&0&0\\0&1/2&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Hence, $x=y$ and $z=y/2$. But $1=x+y+z=y+y+y/2$ which gives $y=2/5$, $x=2/5$ and $z=1/5$.

Exercise 20. $f(x) = 1/2(x+1)$. No.

Exercise 21. $f(x) = x$

Exercise 22.

(1) Yes
(2) The fixed points are $x = 0$ and $x = 2/3$
(3) 2^n

Exercise 23. The aggregate excess demand of commodity 1 is

$$\begin{aligned} g_1(p_1,p_2) &= x_{1,1}(p_1,p_2) - w_{1,1} + x_{2,1}(p_1,p_2) - w_{2,1} \\ &= \frac{\alpha(p_1+2p_2)}{p_1} - 1 + \frac{\alpha(2p_1+p_2)}{p_1} - 2 \\ &= 3\alpha - 3 + \frac{3\alpha p_2}{p_1} \end{aligned}$$

Similarly,

$$g_2(p_1,p_2) = 3(1-\alpha) - 3 + \frac{3(1-\alpha)p_1)}{p_2}$$

Thus

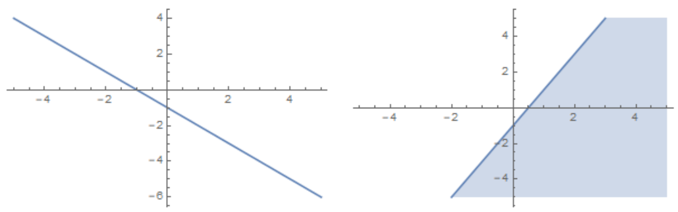
$$p_1 g_1(p_1,p_2) + p_2 g_2(p_1,p_2) = p_1\left(3\alpha - 3 + \frac{3\alpha p_2}{p_1}\right) + p_2\left(3(1-\alpha) - 3 + \frac{3(1-\alpha)p_1)}{p_2}\right) = 0$$

So, this economy satisfies the Walras's law. The equilibrium price is determined by

$$\begin{cases} g_1(p_1,p_2) = 0 \\ g_2(p_1,p_2) = 0 \end{cases} \Leftrightarrow \begin{cases} 3\alpha - 3 + \frac{3\alpha p_2}{p_1} = 0 \\ 3(1-\alpha) - 3 + \frac{3(1-\alpha)p_1)}{p_2} = 0 \end{cases}$$

which has solution $p_2 = (1-\alpha)p_1/\alpha$. But since prices are relative, i.e., $p_1 + p_2 = 1$ we get the equilibrium prices $p_1^* = \alpha$ and $p_2^* = 1-\alpha$.

Exercise 24. (1) and (2)



Exercise 25.

(1) $p = (1,0,\ldots,0)$ and $c = a$
(2) $H((1,0,0),-2)$
(3) $H((1,0,1),1)$

Exercise 26.

(1) yes, A and B are disjoint and convex

(2) no, A and B are not disjoint
(3) no, A is not convex

Exercise 27.

(1) F is not u.h.c. at $x = 1/2$ and does not have the closed graph property.
(2) F is u.h.c. and has the closed graph property.
(3) F is not u.h.c. at $x = 1/2$ and does not have the closed graph property.
(4) F is u.h.c. and does not have the closed graph property.

Exercise 28.

(1) The hypothesis of the theorem are satisfied. The fixed points are $\{1, 2\}$
(2) The hypothesis of the theorem are satisfied. The fixed points are $\{5, 7\}$
(3) The hypothesis of the theorem are satisfied. The fixed points are $\{7\}$
(4) The hypothesis of the theorem are not satisfied. $F(7)$ is not convex. There are no fixed points.

Exercise 29.

(1) Critical point $(0, 0, 0)$ is a local minimizer.
(2) Critical points are $(-1, -1, 2, 3)$ (saddle) and $(5/3, 5/3, 2, 3)$ (local maximizer)
(3) Critical points are $(0, 0, \pm 1/e)$ and $C = \{(x, y, z) \in \mathbb{R}^3 \colon z = 0,\ x^2 + y^2 = 1\}$. The point $(0, 0, 1/e)$ is a local minimizer and $(0, 0, -1/e)$ a local maximizer. The points in C are saddles.

Exercise 30. Since $Df(x, y) = \begin{bmatrix} 2x(1+y)^3 & 3(1+y)^2x^2 + 2y \end{bmatrix}$ we have that $(0, 0)$ is the unique critical point. Computing the Hessian of f one shows that $(0, 0)$ is a local minimizer. It is not a global minimizer because $\lim_{y \to -\infty} f(x, y) = -\infty$.

Exercise 31.

(1) neither convex nor concave
(2) concave when $a + b < 1$ and $a, b > 0$

Exercise 32. The largest domain where f is concave is $\{(x, y) \in \mathbb{R}^2 : x \geq 5/12\}$.

Exercise 33.

(1) The critical points (x, y, λ) of the Lagrangian are
$$\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{\sqrt{5}}{2}\right), \quad \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{\sqrt{5}}{2}\right)$$
The first point is a minimizer and the second a maximizer. So $(x, y) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ solves the maximization problem.

(2) The critical points (x, y, λ) of the Lagrangian are

$$(\pm\sqrt{2}, \pm\sqrt{2}, -8)$$

All four critical points are minimizers, so each point solves the minimization problem.

(3) The critical points $(x, y, z, \lambda_1, \lambda_2)$ of the Lagrangian are

$$\left((-\frac{1}{3}, -\frac{16}{3}, \frac{11}{3}, -\frac{6}{7}, \frac{3}{14}\right), \quad \left(\frac{1}{3}, \frac{16}{3}, -\frac{11}{3}, -\frac{6}{7}, -\frac{3}{14}\right)$$

The first point is a minimizer and the second point a maximizer. Thus $(x, y, z) = \left(\frac{1}{3}, \frac{16}{3}, -\frac{11}{3}\right)$ solves the maximization problem.

(4) The critical points $(x, y, z, \lambda_1, \lambda_2)$ of the Lagrangian are

$$\left((-\frac{3}{5}, \frac{4}{5}, \frac{17}{5}, -4, -\frac{5}{2}\right), \quad \left(\frac{3}{5}, -\frac{4}{5}, \frac{3}{5}, -4, \frac{5}{2}\right)$$

The first point is a maximizer and the second point a minimizer. Thus $(x, y, z) = \left(\frac{3}{5}, -\frac{4}{5}, \frac{3}{5}\right)$ solves the minimization problem.

Exercise 34.

(1) The critical points (x, y, λ) of the Lagrangian are

$$(\sqrt{2}, \sqrt{2}, 1), \quad (-\sqrt{2}, -\sqrt{2}, 1)$$

Since $B_2 = -1$ for both points we conclude that both critical points are local minimizers of f on D.

(2) The critical points (x, y, λ) of the Lagrangian are

$$(4, 4, -1/4), \quad (-4, -4, 1/4)$$

We have $B_2(x, y, \lambda) = 2xy\lambda$. So, $B_2(4, 4, -1/4) = -8$ and $B_2(-4, -4, 1/4) = 8$. We conclude that $(4, 4)$ is a local minimizer and $(-4, -4)$ a local maximizer.

(3) The Lagrangian has a single critical point

$$\left(\frac{2a}{3}, \frac{2a}{3} - b, -\frac{a}{3}, -\frac{2a}{3}, 0\right)$$

Since $B_3 = -6$ we conclude that the critical point in a local maximizer of f on D

Exercise 35.

(1) The solutions $(x, y, \lambda_1, \lambda_2)$ of the Kuhn-Tucker conditions are

$$(2, 1, 1, 0), \quad (-2, 1, 1, 0)$$
$$\left(0, \sqrt{5}, \frac{1}{\sqrt{5}}, 0\right), \quad (0, 0, 0, -2)$$
$$\left(-\sqrt{5}, 0, 1, -2\right), \quad \left(\sqrt{5}, 0, 1, -2\right)$$

Since $L(x, y, 1, 0)$ is concave, the points $(\pm 2, 1)$ are maximizers, thus solve the maximization problem. The points $(0, \sqrt{5})$ and $(\pm\sqrt{5}, 0)$ are saddles and $(0, 0)$ is a minimizer because $L(x, y, 0, -2)$ is convex.

(2) The solutions $(x, y, z, \lambda_1, \lambda_2)$ of the Kuhn-Tucker conditions are

$$\left(0, 1, 0, 1, -\frac{1}{2}\right), \quad \left(\log 2, \frac{1}{2} + \log 2, 0, 1, 0\right)$$

Because $L(x, y, z, 1, 0)$ is concave and both multipliers are ≥ 0 we conclude that $\left(\log 2, \frac{1}{2} + \log 2, 0\right)$ solves the maximization problem.

(3) The solutions $(x, y, \lambda_1, \lambda_2, \lambda_3)$ of the Kuhn-Tucker conditions are

$$\left(0, \frac{11}{2}, \frac{33}{2}, \frac{33}{2}, 0\right), \quad (3, 4, 12, 0, 0)$$
$$(11, 0, 44, 0, 88), \quad (0, 0, 0, 0, 0)$$

Since $L(x, y, \lambda_1, \lambda_2, \lambda_3)$ is convex in (x, y) we conclude that $(0, 0)$ is the minimizer, solving the minimization problem.

Exercise 36. The problem we want to solve is

$$\begin{aligned} \text{maximize} \quad & ax + by + cz \\ \text{subject to} \quad & \alpha x^2 + \beta y^2 + \gamma z^2 \leq L \\ & x \geq 0 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

The solution is

$$(x, y, z) = \left(\frac{a}{2\alpha\lambda}, \frac{b}{2\beta\lambda}, \frac{c}{2\gamma\lambda}\right)$$

where

$$\lambda = \sqrt{\frac{1}{L}\left(\frac{a^2}{4\alpha} + \frac{b^2}{4\beta} + \frac{c^2}{4\gamma}\right)}$$

Exercise 37.

(1) 1st order, linear, non-autonomous
(2) 2nd order, linear, non-autonomous
(3) 1st order, non-autonomous

Exercise 38.

(1) $x(t) = 4 + e^{-2t}c$
(2) $x(t) = \frac{e^t}{4} + e^{-3t}c$
(3) $x(t) = e^{-t^2}t + e^{-t^2}c$
(4) $x(t) = 2 + e^{-t^2}c$

Exercise 39.

(1) $x(t) = 4 - 4e^{-2t}$
(2) $x(t) = \frac{e^t}{4} - \frac{5}{4}e^{-3t}$
(3) $x(t) = e^{-t^2}t + e^{-t^2}$
(4) $x(t) = 2 - 4e^{-t^2}$

Exercise 40.

(1) $K' = \alpha\sigma K + H(t)$ with solution

$$K(t) = \frac{e^{\alpha\sigma t}\left(H_0\left(1 - e^{t(\mu-\alpha\sigma)}\right) + K_0(\alpha\sigma - \mu)\right)}{\alpha\sigma - \mu}$$

(2)

$$x(t) = \frac{X(t)}{N(t)} = \frac{\sigma}{N_0}K(t)e^{-\rho t}$$

(3)

$$\lim_{t\to+\infty} x(t) = \frac{H_0\sigma}{N_0(\rho - \alpha\sigma)}$$

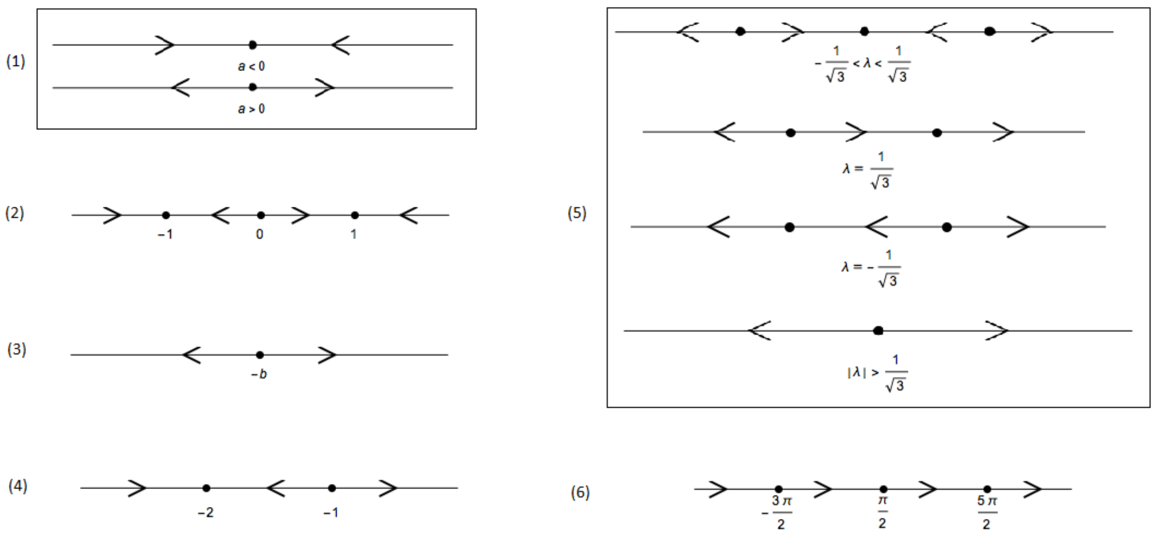
Exercise 41.

(1) $x(t) = e^{-t}t$
(2) $x(t) = \sqrt{t^2 - 1}$
(3) $x(t) = \frac{1-e^{2t}}{1+e^{2t}}$

Exercise 42.

(1) $x(t; x_0) = \log(t + e^{x_0})$ and $I_{x_0} =]-e^{x_0}, +\infty[$
(2) $x(t; x_0) = \begin{cases} \sqrt{t + x_0^2}, & x_0 \geq 0 \\ -\sqrt{t + x_0^2}, & x_0 < 0 \end{cases}$ and $I_{x_0} = [-x_0^2, +\infty[$

Exercise 43.



Exercise 44.

$$\begin{aligned} x(t) &= e^{at}x_0 + \frac{b}{a-d}(e^{at} - e^{dt})y_0 \\ y(t) &= e^{dt}y_0 \end{aligned}$$

Exercise 45.

(1) $J = \begin{bmatrix} \frac{1}{2}\left(1-\sqrt{5}\right) & 0 \\ 0 & \frac{1}{2}\left(1+\sqrt{5}\right) \end{bmatrix}$

(2) $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(3) $J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

(4) $J = \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

(5) $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(6) $J = \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix}$

Exercise 46.

(1)

$$X(t) = \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^t}{2} & -\frac{e^{-t}}{2} + \frac{e^t}{2} \\ -\frac{e^{-t}}{2} + \frac{e^t}{2} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix} X_0$$

(2)

$$X(t) = \begin{bmatrix} e^{t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3}} & \frac{2e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3}} \\ -\frac{2e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3}} & e^{t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3}} \end{bmatrix} X_0$$

(3)

$$X(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix} X_0$$

(4)

$$X(t) = \begin{bmatrix} e^{2t} & 0 \\ 0 & 1 \end{bmatrix} X_0$$

(5)

$$X(t) = \begin{bmatrix} e^{2t}(1-t) & e^{2t}t \\ -e^{2t}t & e^{2t}(t+1) \end{bmatrix} X_0$$

Exercise 47.

(1) $x(t) = \frac{5e^{-2t}}{4} + \frac{3e^{2t}}{4}$
(2) $x(t) = \sin(t)$
(3) $x(t) = \frac{e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3}} + e^{t/2}\cos\left(\frac{\sqrt{3}t}{2}\right)$

Exercise 48.

(1) $x(t) = -\frac{2e^{-2t}}{3} + \frac{e^{-t}}{2} + \frac{13e^{t}}{6} - 1$ and $y(t) = \frac{e^{-2t}}{3} - \frac{e^{-t}}{2} + \frac{13e^{t}}{6}$

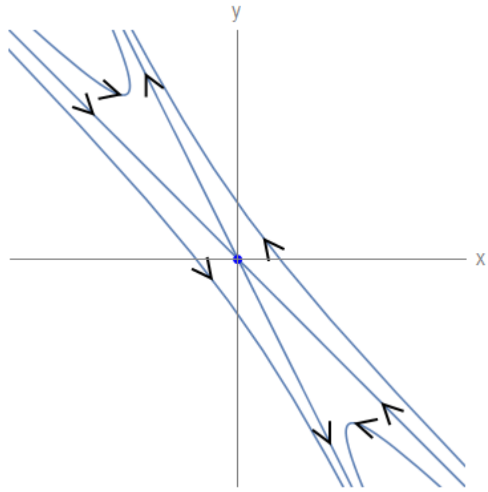
(2) $x(t) = -\frac{7e^{-3t}}{15} - \frac{e^{2t}}{5} - \frac{1}{3}$

Exercise 49.

(i):

$$J = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

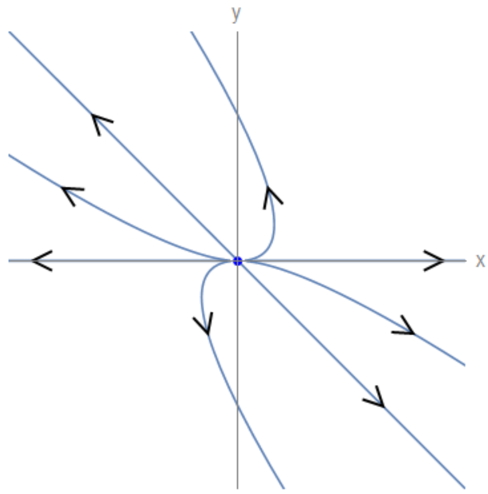
$$X(t) = \begin{bmatrix} 2e^{-3t} - e^{2t} & e^{-3t} - e^{2t} \\ -2e^{-3t} + 2e^{2t} & -e^{-3t} + 2e^{2t} \end{bmatrix} X_0$$



(ii):

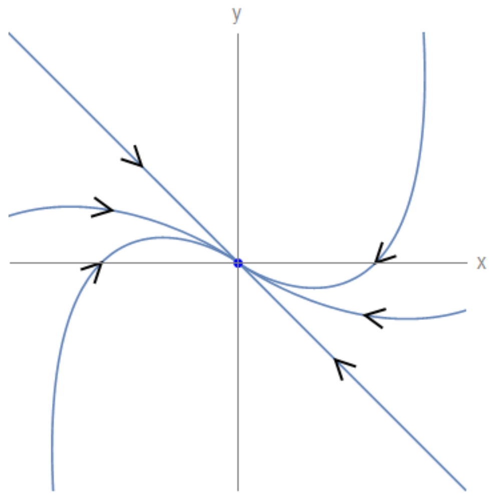
$$J = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} e^{t/2} & -e^{t/2}\left(-1 + e^{t/2}\right) \\ 0 & e^{t} \end{bmatrix} X_0$$



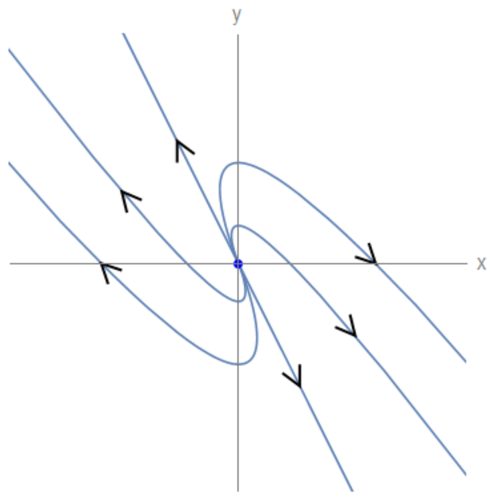
(iii):

$$J = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$
$$X(t) = \begin{bmatrix} e^{-2t}(t+1) & e^{-2t}t \\ -e^{-2t}t & e^{-2t}(1-t) \end{bmatrix} X_0$$



(iv):

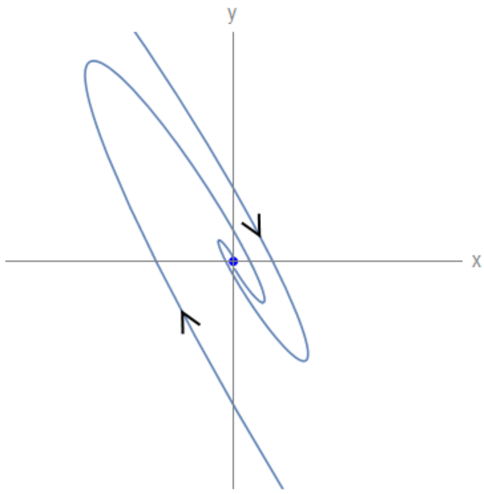
$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}$$
$$X(t) = \begin{bmatrix} e^{2t}(2t+1) & e^{2t}t \\ -4e^{2t}t & e^{2t}(1-2t) \end{bmatrix} X_0$$



(v):

$$J = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} -3 & -1 \\ 5 & 0 \end{bmatrix}$$

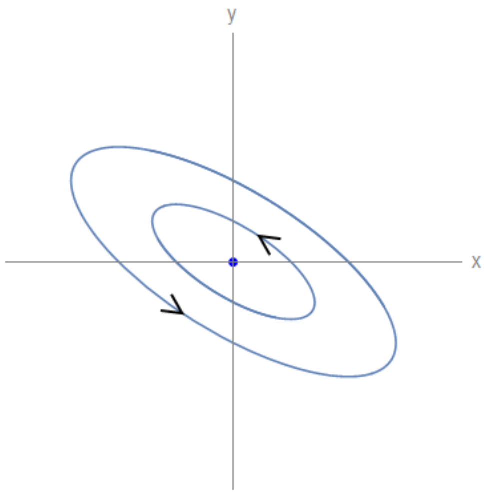
$$X(t) = \begin{bmatrix} e^{-t}\cos(2t) + 3e^{-t}\sin(2t) & 2e^{-t}\sin(2t) \\ -5e^{-t}\sin(2t) & e^{-t}\cos(2t) - 3e^{-t}\sin(2t) \end{bmatrix} X_0$$



(vi):

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} \cos(t) - \sin(t) & -2\sin(t) \\ \sin(t) & \cos(t) + \sin(t) \end{bmatrix} X_0$$



Exercise 50.

(1) The 2nd order ODE can be written as

$$X' = AX, \quad A = \begin{bmatrix} 0 & 1 \\ -b & 0 \end{bmatrix}$$

where $X(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$ and the initial condition is $X_0 = \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$.

The solution is

$$X(t) = \begin{bmatrix} \cos\left(\sqrt{b}t\right) & \frac{\sin\left(\sqrt{b}t\right)}{\sqrt{b}} \\ -\sqrt{b}\sin\left(\sqrt{b}t\right) & \cos\left(\sqrt{b}t\right) \end{bmatrix} X_0$$

So,

$$x(t) = \cos\left(\sqrt{b}t\right) x_0 + \frac{\sin\left(\sqrt{b}t\right)}{\sqrt{b}} x_0'$$

The phase portrait of the equation is a center.

(2) The 2nd order ODE can be written as

$$X' = AX, \quad A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

where $X(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$ and the initial condition is $X_0 = \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$.

Since $\mathrm{tr}(A) = -a$ and $\det(A) = b$ we conclude by the trace-determinant stability plane that

- when $a^2 < 4b$ the system is a stable focus (because $\mathrm{tr}(A) < 0$).
- when $a^2 > 4b$ the system is sink (because $\det(A) > 0$ and $\mathrm{tr}(A) < 0$).
- when $a^2 = 4b$ the system is a stable node.

The solutions for each case can be easily computed. For instance, when $a^2 < 4b$ we have

$$X(t) = e^{-\frac{a}{2}t} \begin{bmatrix} \cos(\omega t) + \frac{a}{2\omega}\sin(\omega t) & \frac{\sin(\omega t)}{\omega} \\ -\frac{b}{\omega}\sin(\omega t) & \cos(\omega t) - \frac{a}{2\omega}\sin(\omega t) \end{bmatrix} X_0$$

where

$$\omega = \frac{1}{2}\sqrt{4b - a^2}$$