Lévy Processes and Applications - Part 4

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Martingales

- Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space with filtration $(\mathcal{F}_t, t \geq 0)$.
- A stochastic process $X = (X(t), t \geq 0)$ is adapted to the $(\mathcal{F}_t, t \geq 0)$ if each $X(t)$ is $\mathcal{F}_t$-measurable
- An adapted Lévy process is a Markov process.

Definition

The process X is a martingale if X is adapted to $(\mathcal{F}_t, t \geq 0)$, $E[|X(t)|] < \infty$ for all $t \geq 0$ and

$$E[X(t) | \mathcal{F}_s] = X_s \quad \text{a.s} \quad \text{for all } s < t.$$

Theorem

An adapted Lévy process with finite first moment and zero mean is a martingale (with respect to its natural filtration)

Proof: X adapted, $E[|X(t)|] < \infty$ for all $t \geq 0$ and

$$\begin{aligned} E[X(t) | \mathcal{F}_s] &= E[X(s) + X(t) - X(s) | \mathcal{F}_s] \\ &= X(s) + E[X(t) - X(s)] = X(s). \end{aligned}$$

Martingales

Examples of Lévy processes that are also martingales:

- $\sigma B(t)$, $B(t)$ d-dim. BM and σ an $r \times d$ matrix.
- $\widetilde{N}(t) = N(t) - \lambda t$ - compensated Poisson process

Examples of martingales associated to Lévy processes:

- $\exp\{iuX(t) - t\eta(u)\}$ where $u \in \mathbb{R}$ is fixed and X is a Lévy process with Lévy symbol η.
- $\left[\widetilde{N}(t)\right]^2 - \lambda t$
- Exercise: Show that $\exp\{iuX(t) - t\eta(u)\}$ is a martingale.

Cádlàg paths

- $f : \mathbb{R}^+ \to \mathbb{R}$ is a càdlàg function if is "continue à droite et limité à gauche" - right continuous with left limits.
- Notation: $f(t-) := \lim_{s \uparrow t} f(s)$ and $\Delta f(t) := f(t) - f(t-)$.
- Every Lévy process has a càdlàg modification which is itself a Lévy process (proof: theorem 2.1.8, pag 87 - Applebaum).
- Note: given two processes $(X(t), t \geq 0)$ and $(Y(t), t \geq 0)$, we say that Y is a modification of X if, for each $t \geq 0$, $P[X(t) \neq Y(t)] = 0$. As a consequence, X and Y have the same finite dimensional distributions.

Assumptions

From now on, we will always assume that:

- $(\Omega, \mathcal{F}, P)$ will be a fixed filtered probability space with a filtration $(\mathcal{F}_t, t \geq 0)$.
- Every Lévy process X will be assumed to be $\mathcal{F}_t$-adapted and with càdlàg sample paths.
- $X(t) - X(s)$ is independent of $\mathcal{F}_s$ for all $s < t$.

The jumps of a Lévy process

- The jump process ΔX associated to X is defined by

$$\Delta X(t) = X(t) - X(t-).$$

Lemma

If X is a Lévy process, then for fixed $t > 0$, $\Delta X(t) = 0$ a.s.

Proof:

- Let $(t(n); n \in N)$ be a sequence in $\mathbb{R}^+$ with $t(n) \uparrow t$ as $n \to \infty$.
- X has càdlàg paths $\Longrightarrow \lim_{n\to\infty} X(t(n)) = X(t-)$.
- By the stochastic continuity condition (in the Lévy process definition) $\Longrightarrow X(t(n))$ converges in probability to $X(t)$, and so has a subsequence which converges a.s to $X(t)$. Then, by the uniqueness of the limits $X(t) = X(t-)$ (a.s.) and $\Delta X(t) = 0$ (a.s.).■

Poisson random measures

- Analytic difficulty in manipulating Lévy processes has to do with the fact that is possible to have:

$$\sum_{0\leq s\leq t} |\Delta X(s)| = \infty \quad \text{a.s.}$$

However, we always have that:

$$\sum_{0\leq s\leq t} |\Delta X(s)|^2 < \infty \quad \text{a.s.}$$

- In order to count jumps of specified size, define (for a set $A \in \mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$):

$$\begin{aligned} N(t,A) &= \#\left\{0 \leq s \leq t : \Delta X\left(s\right) \in A\right\} \\ &= \sum_{0\leq s\leq t} \mathbf{1}_A\left(\Delta X(s)\right) \end{aligned}$$

- For each $\omega \in \Omega$, $t \geq 0$, the map $A \to N(t,A)$ is a counting measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$. (Note: $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ is the $\sigma-$algebra of Borelian measurable sets in $\mathbb{R}^d - \{0\}$).

Poisson random measures

- Then
$$E[N(t,A)] = \int N(t,A)(\omega)\, dP(\omega)$$
is a measure on $\mathcal{B}(\mathbb{R}^d - \{0\})$.
- Notation: $\mu(\cdot) = E[N(1,\cdot)]$ is a measure on $\mathcal{B}(\mathbb{R}^d - \{0\})$ called the intensity measure (considers the mean number of jumps until time 1).
- We call $N(t,\cdot)$ a Poisson random measure.
- We say that $A \in \mathcal{B}(\mathbb{R}^d)$ is bounded below if $0 \notin \overline{A}$ (note: $\overline{A}$ is the closure of A =all points in A plus the limit points of A).

Lemma

If A is bounded below then $N(t,A) < \infty$ a.s. for all $t \geq 0$.

Proof: See Applebaum, Lemma 2.3.4 - page 101.

Poisson random measures

- If A fails to be bounded below, the Lemma may no longer hold (accumulation of large numbers of small jumps).

Theorem

1. If A is bounded below, then the process $(N(t, A), t \geq 0)$ *is a Poisson process with intensity* $\mu(A)$.
2. If $A_1, \ldots A_m \in \mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ *are disjoint then the r.v.* $N(t, A_1), \ldots, N(t, A_m)$ *are independent.*

Proof: pages 101-103 of Applebaum.

Poisson random measures

- Consequence: $\mu(A) < \infty$ whenever A is bounded below.
- Main properties of N:
  1. For each t and $\omega \in \Omega$, $N(t, \cdot)(\omega)$ is a counting measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$.
  2. For each A bounded below, $(N(t, A), t \geq 0)$ is a Poisson process with intensity $\mu(A) = E[N(1, A)]$.
  3. The compensated process $\widetilde{N}(t, A) = N(t, A) - t\mu(A)$ is a martingale.

Poisson integration

- Let f be a measurable function from $\mathbb{R}^d$ to $\mathbb{R}^d$ and let A be bounded below. Then we may define the Poisson integral of f as the random finite sum

$$\int_A f(x)\, N(t, dx)(\omega) = \sum_{x \in A} f(x)\, N(t, \{x\})(\omega),$$

where $\{x\}$ are the jump sizes of the process (in A), i.e. $N(t, \{x\}) \neq 0 \iff \Delta X(u) = x$ for some $0 \leq u \leq t$.

- We can also write

$$\int_A f(x)\, N(t, dx) = \sum_{0 \leq u \leq t} f(\Delta X(u))\, \mathbf{1}_A(\Delta X(u)).$$

Poisson integration

Theorem

Let A be bounded below. Then:

1. $\left(\int_A f(x)\, N(t, dx), t \geq 0\right)$ *is a compound Poisson process with characteristic function*

$$\exp\left(t \int_{\mathbb{R}^d} \left(e^{i(u, f(x))} - 1\right) \mu(dx)\right).$$

2. If $f \in L^1(A, \mu)$ *then*

$$\mathbb{E}\left[\int_A f(x)\, N(t, dx)\right] = t \int_A f(x)\, \mu(dx).$$

3. If $f \in L^2(A, \mu)$ *then*

$$\mathrm{Var}\left(\left|\int_A f(x)\, N(t, dx)\right|\right) = t \int_A |f(x)|^2\, \mu(dx).$$

Poisson integration

Sketch of the proof: 1. Assume that f is a simple function: $f = \sum_{j=1}^{n} c_j \mathbf{1}_{A_j}$ (with the A_j's disjoint). Then, by part 2 of the previous theorem, we have that

$$E\left[\exp\left\{i\left(u, \int_A f(x)\, N(t, dx)\right)\right\}\right] = \prod_{j=1}^{n} E\left[\exp\left\{i\left(u, \int_A c_j N(t, A_j)\right)\right\}\right]$$

$$= \prod_{j=1}^{n} \exp\left\{t\left(e^{i\left(u, c_j\right)} - 1\right)\mu(A_j)\right\} = \exp\left\{t\int_A \left(e^{i(u, f(x))} - 1\right)\mu(dx)\right\}.$$

Parts 2. and 3. follow from 1. by differentiation (moments from characteristic function: $E\left[X^k\right] = (-i)^k \Phi^{(k)}(0)$)■

Poisson integration

- For $f \in L^1(A, \mu)$, we define the compensated Poisson integral by

$$\int_A f(x)\, \widetilde{N}(t, dx) = \int_A f(x)\, N(t, dx) - t \int_A f(x)\, \mu(dx).$$

- The process $\left(\int_A f(x)\, \widetilde{N}(t, dx), t \geq 0 \right)$ is a martingale.
- If $f \in L^2(A, \mu)$ then

$$E\left[\left| \int_A f(x)\, \widetilde{N}(t, dx) \right|^2 \right] = t \int_A |f(x)|^2\, \mu(dx).$$

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